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On the Second Homology of Crossed Modules

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Abstract

In this article, we present a new description of the integral second homology of crossed modules of groups and generalize two basic results on the integral second homology of groups for crossed modules. Using these, we strengthen some consequences on covering pairs and the universal relative central extensions of pairs of finite groups.

Keywords Crossed module · Exterior product · Second homology · Covering pair

Mathematics Subject Classification 18G05 · 18G99 · 20J99

1 Introduction

Several definitions for the homology of crossed modules have been given during the last years: Ellis [7] and Baues [2] introduced the homology of a crossed module to be the homology of its classifying space. Grandjeán and Ladra [10] defined the second homology crossed module by means of a Hopf formula applied to a particular kind of presentations called ε-projective. Also, associated with an extension of crossed modules, they [16] gave the construction of a five-term exact sequence for the homology of crossed modules. In continuation, Pirashvili [21] presented the notion of the tensor product of two abelian crossed modules, and he used it to construct the Ganea map, that is, extended the above five-term exact sequence one term further. Carrasco et al. [6], using the general theory of cotriple homology of Barr and Beck, defined the integral homology crossed modules of a crossed module as the simplicial derived functors of the abelianization functor from the category of crossed modules to the category of abelian crossed modules and generalized some classical results of the homology of

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groups. Considering projective presentations introduced in [6] instead of ε-projective presentations, they also obtained the results given in [10,16].

Recently, authors [22] introduced the notions of the non-abelian tensor and exterior products of two normal crossed submodules of some crossed module of groups, which are generalizations of the works of Brown and Loday [4,5] and Pirashvili [21].

In this article, we give a new description of the second homology of crossed modules; in fact, we describe the second homology of crossed modules as the central crossed submodules of their exterior products, which generalizes a result of Miller [18] for groups. Also, we show that the second homology of the direct product of two crossed modules is isomorphic to the direct product of the second homology of the factors and the tensor product of the two crossed modules abelianized. Finally, we give some applications to covering pairs and the universal central extensions of pairs of finite groups.

2 Preliminaries on Crossed Modules

In this section, we briefly recall some basic definitions in the category of crossed modules, which will be needed in the sequel.

A *crossed module* $\mathbf{T} = (T, G, \partial)$ is a group homomorphism $\partial : T \longrightarrow G$ together with an action of G on T, written ${}^{g}t$ for $t \in T$ and $g \in G$, satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(t)}t' = tt't^{-1}$, for all $t, t' \in T$, $g \in G$. It is worth noting that for any crossed module (T, G, ∂), $\mathrm{Im}\partial$ is a normal subgroup of G and $\ker \partial$ is a G-invariant subgroup in the center of T. Clearly, for any normal subgroup N of a group G, (N, G, i) is a crossed module, where i is the inclusion and G acts on N by conjugation. In this way, every group G can be seen as a crossed module in two obvious ways: $(1, G, i)$ or (G, G, id).

A *morphism of crossed modules* $(\gamma_1, \gamma_2) : (T, G, \partial) \longrightarrow (T', G', \partial')$ is a pair of homomorphisms $\gamma_1 : T \longrightarrow T'$ and $\gamma_2 : G \longrightarrow G'$ such that $\partial'\gamma_1 = \gamma_2\partial$ and $\gamma_1({}^{g}t) = {}^{\gamma_2(g)}\gamma_1(t)$ for all $g \in G, t \in T$.

Taking objects and morphisms as defined above, we obtain the category $\mathfrak{CM}$ of crossed modules. We refer the reader to [10,20] for obtaining more information on this category.

Let $\mathbf{T} = (T, G, \partial)$ be a crossed module with normal crossed submodules $\mathbf{S} = (S, H, \partial)$ and $\mathbf{L} = (L, K, \partial)$. The following is a list of notations which will be used:

- $Z(\mathbf{T}) = (T^G, Z(G) \cap st_G(T), \partial)$ is the *center* of $\mathbf{T}$, where $Z(G)$ denotes the center of the group G, $T^G = \{t \in T \mid {}^{g}t = t \text{ for all } g \in G\}$ and $st_G(T) = \{g \in G \mid {}^{g}t = t \text{ for all } t \in T\}$.
- $\mathbf{T}' = ([G, T], G', \partial)$ is the *commutator crossed submodule* of $\mathbf{T}$, where $G' = [G, G]$ and $[G, T] = \langle {}^{g}tt^{-1} \mid t \in T, g \in G\rangle$ is the displacement subgroup of T relative to G.
- $[\mathbf{S}, \mathbf{L}]$ is the normal crossed submodule $([K, S][H, L], [H, K], \partial)$ of $\mathbf{T}$.
- $\mathbf{T}_{ab} = (T/[G, T], G_{ab}, \overline{\partial})$ is the *abelianization* of $\mathbf{T}$, where $G_{ab} = G/G'$ and $\overline{\partial}$ is induced by ∂.

A crossed module (T, G, ∂) is *perfect* if it coincides with its commutator crossed submodule and is *abelian* if it coincides with its center.

In [6], it is proved that the category of crossed modules is an algebraic category, that is, there is a tripleable forgetful functor from the category $\mathfrak{CM}$ to the category of sets $\mathfrak{Set}$ and is deduced that every crossed module admits a projective presentation. Also, it is shown in [1] that if (Y, F, μ) is a projective crossed module, then it can be assumed that μ is inclusion and the groups F, F/Y are free objects in the category of groups. Now, let $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$ be a projective presentation of the crossed module (T, G, ∂). It is proved in [6] that the second homology crossed module of (T, G, ∂) is, up to isomorphism, the abelian crossed module

$$\left(\frac{V \cap [F, Y]}{[R, Y][F, V]}, \frac{R \cap F'}{[F, R]}, \bar{\mu}\right).$$

Considering a group G as a crossed module in the two usual ways, we have $H_2(1, G, i) = (1, H_2(G), i)$ or $H_2(G, G, id) = (H_2(G), H_2(G), id)$, which gives the Hopf's formula [13]. Note that in the above projective presentation of (T, G, ∂), if ∂ is injective, then $V = R \cap Y$.

A central extension $e : (A, B, \partial^*) \rightarrowtail (T^*, G^*, \partial^*) \twoheadrightarrow (T, G, \partial)$ is called a *stem extension* of (T, G, ∂) if $(A, B, \partial^*) \subseteq (T^*, G^*, \partial^*)'$. If, in addition, $(A, B, \partial^*) \cong H_2(T, G, \partial)$, then e is called a *stem cover*. In this case, (T^*, G^*, ∂^*) is said to be a *covering crossed module* of (T, G, ∂). In [19], it is proved that any crossed module (T, G, ∂) admits at least one covering crossed module and is determined the structure of all stem covers of a crossed module whose second homology is finite.

Finally, we recall from [22] that the *non-abelian tensor and exterior products* of two normal crossed submodules (S, H, ∂) and (L, K, ∂) of a given crossed module are defined, respectively, as

$$\begin{aligned}(S, H, \partial) \otimes (L, K, \partial) &= (\mathrm{coker}\alpha, H \otimes K, \delta),\\ (S, H, \partial) \wedge (L, K, \partial) &= \left(\frac{\mathrm{coker}\alpha}{I}, H \wedge K, \bar{\delta}\right),\end{aligned}$$

in which $\alpha = (id_S \otimes \partial, (\partial \otimes id_L)^{-1})$ and the map δ is induced on cokerα by the homomorphism $\beta = \langle \partial \otimes id_K, id_H \otimes \partial \rangle$:

$$\begin{array}{ccccc} S \otimes L & \xrightarrow{\alpha} & (S \otimes K) \rtimes (H \otimes L) & \xrightarrow{nat.} & \mathrm{coker}\alpha. \\ & & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\delta} & \\ & & H \otimes K & & \end{array}$$

Also, I is a normal subgroup of cokerα generated by the elements $(x \otimes y, (y \otimes x)(\partial(z) \otimes z))\mathrm{Im}\alpha$ for all $x, z \in S \cap L$, $y \in H \cap K$. In the case of abelian crossed modules, the definition of the tensor product holds for any two abelian crossed modules (see [21]).

The proof of the following lemma is straightforward, and it will be left to the reader.

Lemma 2.1 *Let* $\mathbf{A_1} = (A_1, B_1, \partial_1)$*,* $\mathbf{A_2} = (A_2, B_2, \partial_2)$ *and* $\mathbf{A} = (A, B, \partial)$ *be abelian crossed modules. Then*

(i) *If* $f = (f_1, f_2) : \mathbf{A_1} \longrightarrow \mathbf{A}$ *and* $g = (g_1, g_2) : \mathbf{A_2} \longrightarrow \mathbf{A}$ *are morphisms of crossed modules, then* $f * g = (f_1 * g_1, f_2 * g_2) : \mathbf{A_1} \times \mathbf{A_2} \longrightarrow \mathbf{A}$ *is a morphism of crossed modules, where* $f_1 * g_1(a_1, a_2) = f_1(a_1)g_1(a_2)$ *and* $f_2 * g_2(b_1, b_2) = f_2(b_1)g_2(b_2)$*, for all* $a_i \in A_i, b_i \in B_i, i = 1, 2$.

(ii) *There exists an isomorphism* $\nu = (\nu_1, \nu_2) : \mathbf{A_1} \otimes \mathbf{A_2} \longrightarrow \mathbf{A_2} \otimes \mathbf{A_1}$ *defined by*

$$\nu_1((a_1 \otimes b_2, b_1 \otimes a_2)Im\alpha) \\ = \big((a_2 \otimes b_1)^{-1}, (b_2 \otimes a_1)^{-1}\big)Im\alpha' \text{ and } \nu_2(b_1 \otimes b_2) = (b_2 \otimes b_1)^{-1}.$$

The next proposition provides some properties of the above notions, which is useful in our investigation.

Proposition 2.2 [22] *Let* $\mathbf{T} = (T, G, \partial)$ *be a crossed module. Then*

(i) *There is a morphism of crossed modules* $(\tau_1, \tau_2): \mathbf{T} \otimes \mathbf{T} \longrightarrow \mathbf{T}$*, where* $Im(\tau_1, \tau_2) = \mathbf{T}'$ *and* $J_2(\mathbf{T}) := \ker(\tau_1, \tau_2)$ *is abelian.*

(ii) *There is an isomorphism of crossed modules* $(\widetilde{\varphi}, id): \mathbf{T} \wedge \mathbf{T} \longrightarrow (G \wedge T, G \wedge G, id \wedge \partial)$*, where* $\widetilde{\varphi}$ *is induced by* $\mu : (T \otimes G) \rtimes (G \otimes T) \longrightarrow G \otimes T$*,* $\mu(x, y) = \theta(x)y$*, in which* $\theta: T \otimes G \longrightarrow G \otimes T$ *is defined by* $\theta(t \otimes g) = (g \otimes t)^{-1}$.

(iii) *Let* $\mathbf{S}$ *and* $\mathbf{L}$ *be normal crossed submodules of* $\mathbf{T}$*. If* $[\mathbf{S}, \mathbf{L}] = 1$*, then* $\mathbf{S} \otimes \mathbf{L} \cong \mathbf{S}_{ab} \otimes \mathbf{L}_{ab}$.

(iv) *If* $\mathbf{T}$ *is simply connected (that is,* ∂ *is onto), then there is a natural exact sequence*

$$\Gamma(\mathbf{T}_{ab}) \longrightarrow \mathbf{T} \otimes \mathbf{T} \twoheadrightarrow \mathbf{T} \wedge \mathbf{T},$$

in which $\Gamma(\mathbf{T}_{ab})$ *is a generalized version of Whitehead's universal quadratic functor (see* [21] *for more information).*

3 Main Results

Miller [18] proves that, for any group G, the second homology of G is isomorphic to the kernel of the commutator map $G \wedge G \xrightarrow{[\,,\,]} G$. Using this result, he determines the behavior of the functor $H_2(-)$ with respect to the direct product of groups and, also, Ellis [8] gets a six-term exact sequence in homology

$$H_2(G, N) \longrightarrow H_2(G) \longrightarrow H_2(Q) \longrightarrow N/[G, N] \longrightarrow H_1(G) \twoheadrightarrow H_1(Q),$$

from a short exact sequence of groups $N \rightarrowtail G \twoheadrightarrow Q$. Here, $H_2(G, N)$ denotes the second relative Chevally–Eilenberg homology of the pair (G, N), which is isomorphic to $\ker(G \wedge N \longrightarrow G)$ (see [5]). In this section, we generalize these results to crossed modules. In fact, we prove

Theorem 3.1 *(i) For arbitrary crossed module* $\mathbf{T} = (T, G, \partial)$*,* $H_2(\mathbf{T}) \cong \ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T})$*. In particular,* $H_2(\mathbf{T}) \cong (\ker(G \wedge T \longrightarrow T), \ker(G \wedge G \longrightarrow G), id \wedge \partial)$*.*

(ii) If $e : \mathbf{S} \rightarrowtail \mathbf{T} \twoheadrightarrow \mathbf{M}$ *is an extension of crossed modules, then there is a natural exact sequence*

$$\ker(\mathbf{T} \wedge \mathbf{S} \longrightarrow \mathbf{T}) \longrightarrow H_2(\mathbf{T}) \longrightarrow H_2(\mathbf{M}) \longrightarrow \frac{\mathbf{S}}{[\mathbf{S}, \mathbf{T}]} \longrightarrow H_1(\mathbf{T}) \twoheadrightarrow H_1(\mathbf{M}),$$

where $H_1(-)$ *denotes the first homology of crossed modules* [11].

Theorem 3.2 *For crossed modules* $\mathbf{T_1} = (T_1, G_1, \partial_1)$ *and* $\mathbf{T_2} = (T_2, G_2, \partial_2)$*, there are isomorphisms*

$$J_2(\mathbf{T_1} \times \mathbf{T_2}) \cong J_2(\mathbf{T_1}) \times J_2(\mathbf{T_2}) \times (\mathbf{T_1}_{ab} \otimes \mathbf{T_2}_{ab}) \times (\mathbf{T_2}_{ab} \otimes \mathbf{T_1}_{ab}),$$
$$H_2(\mathbf{T_1} \times \mathbf{T_2}) \cong H_2(\mathbf{T_1}) \times H_2(\mathbf{T_2}) \times (\mathbf{T_1}_{ab} \otimes \mathbf{T_2}_{ab}).$$

In order to prove these theorems, we need the following two propositions, which are generalizations of [8, Proposition 1] and [3, Proposition 10], respectively.

Proposition 3.3 *Let* $\mathbf{S} = (S, H, \partial)$ *be a normal crossed submodule of a crossed module* $\mathbf{T} = (T, G, \partial)$*. Then there is an exact sequence of crossed modules*

$$\mathbf{T} \wedge \mathbf{S} \xrightarrow{\eta} \mathbf{T} \wedge \mathbf{T} \overset{\pi}{\twoheadrightarrow} \frac{\mathbf{T}}{\mathbf{S}} \wedge \frac{\mathbf{T}}{\mathbf{S}}.$$

Proof By the definition of the exterior product of crossed modules, we set

$$\mathbf{T} \wedge \mathbf{S} = (\mathrm{coker}\alpha_1/I_1, G \wedge H, \bar{\delta}_1)\ ,\ \mathbf{T} \wedge \mathbf{T} = (\mathrm{coker}\alpha/I, G \wedge G, \bar{\delta}),$$
$$\frac{\mathbf{T}}{\mathbf{S}} \wedge \frac{\mathbf{T}}{\mathbf{S}} = (\mathrm{coker}\bar{\alpha}/\bar{I}, \frac{G}{H} \wedge \frac{G}{H}, \widetilde{\delta}).$$

It is easy to see that the natural epimorphisms $T \longrightarrow T/S$ and $G \longrightarrow G/H$ induce the homomorphisms $\pi_1 : \mathrm{coker}\alpha/I \longrightarrow \mathrm{coker}\bar{\alpha}/\bar{I}$ and $\pi_2 : G \wedge G \longrightarrow G/H \wedge G/H$ such that $\pi = (\pi_1, \pi_2)$ is a surjective morphism. Also, the functional homomorphisms $\beta_1 : T \otimes H \longrightarrow T \otimes G$ and $\beta_2 : G \otimes S \longrightarrow G \otimes T$ give rise to the homomorphism $\varphi : (T \otimes H) \rtimes (G \otimes S) \longrightarrow (T \otimes G) \rtimes (G \otimes T)$ defined by $\varphi(\nu, \omega) = (\beta_1(\nu), \beta_2(\omega))$. Since $\varphi(Im\alpha_1) \subseteq Im\alpha$, we can obtain the homomorphism $\widetilde{\varphi} : \mathrm{coker}\alpha_1 \longrightarrow \mathrm{coker}\alpha$ induced by φ. Certainly, $\widetilde{\varphi}(I_1) \subseteq I$ and so $\widetilde{\varphi}$ induces a homomorphism $\eta_1 : \mathrm{coker}\alpha_1/I_1 \longrightarrow \mathrm{coker}\alpha/I$. It is straightforward that $\eta = (\eta_1, \eta_2)$ is a morphism of crossed modules in which $\eta_2 : G \wedge H \longrightarrow G \wedge G$ is the functional homomorphism. To complete the proof, we need to indicate that $\mathrm{Im}\eta = \ker \pi$. We have $\mathrm{Im}\eta_2 = \ker \pi_2$, thanks to Ellis [8, Proposition 1]. It is easily verified that $\mathrm{Im}\eta_1$ is a normal subgroup of $\mathrm{coker}\alpha/I$ contained in $\ker \pi_1$. We now show that this inclusion is an equality by constructing an isomorphism $\widetilde{\kappa} : \mathrm{coker}\bar{\alpha}/\bar{I} \longrightarrow \mathrm{coker}\eta_1$. Consider the maps $e_1 : T/S \times G/H \longrightarrow \mathrm{coker}\eta_1$ and $e_2 : G/H \times T/S \longrightarrow \mathrm{coker}\eta_1$ defined by $e_1(tS, gH) = \overline{(t \otimes g, 1)}Im\eta_1$ and $e_2(gH, tS) = \overline{(1, g \otimes t)}\mathrm{Im}\eta_1$, respectively. For $t_1, t_2 \in T$, $g_1, g_2 \in G$, $s \in S$, $h \in H$, if $t_1 = t_2 s$, $g_1 = g_2 h$, then

$$
\begin{aligned}
e_1(t_1S, g_1H) &= \overline{(t_2s \otimes g_2h, 1)}Im\eta_1 \\
&= \overline{(s \otimes g_2, 1)(s \otimes h, 1)(t_2 \otimes g_2, 1)(t_2 \otimes h, 1)}Im\eta_1 \\
&= \overline{(s \otimes g_2, 1)(t_2 \otimes g_2, 1)}Im\eta_1,
\end{aligned}
$$

because $\overline{(s \otimes h, 1)}, \overline{(t_2 \otimes h, 1)} \in \mathrm{Im}\eta_1$. On the other hand, we have $\overline{(s \otimes g_2, 1)} = \overline{(1, g_2 \otimes s)}^{-1} \in Im\eta_1$, since $\overline{(s \otimes g_2, g_2 \otimes s)} = 1$ in cokerα/I. It is therefore inferred that $e_1(t_1S, g_1H) = \overline{(t_2 \otimes g_2, 1)}Im\eta_1 = e_1(t_2S, g_2H)$. So, e_1 and, similarly, e_2 are well-defined. It is easy to verify that e_1 and e_2 are crossed pairings, and the universal property of the tensor product thus yields the homomorphisms $\bar{e}_1 : T/S \otimes G/H \longrightarrow$ cokerη_1 and $\bar{e}_2 : G/H \otimes T/S \longrightarrow$ cokerη_1. Now, the map

$$\kappa : (T/S \otimes G/H) \rtimes (G/H \otimes T/S) \longrightarrow \mathrm{coker}\eta_1$$

defined by $\kappa(x, y) = \bar{e}_1(x)\bar{e}_2(y)$ is a homomorphism that annihilates $Im\bar{\alpha}$. So κ induces the homomorphism $\bar{\kappa}$: coker$\bar{\alpha} \longrightarrow$ cokerη_1 with $\bar{\kappa}(\bar{I}) = 0$, which in turn induces a homomorphism $\widetilde{\kappa}$: coker$\bar{\alpha}/\bar{I} \longrightarrow$ cokerη_1. It is routine to check that $\widetilde{\kappa}$ is an isomorphism with inverse induced by π_2. The proof is complete. □

Proposition 3.4 *Let* $\mathbf{M} = (M, P, \partial_1)$ *and* $\mathbf{N} = (N, Q, \partial_2)$ *be two crossed modules and* $\mathbf{T} = (T, G, \partial_1 \times \partial_2)$ *be a normal crossed submodule of* $\mathbf{M} \times \mathbf{N}$. *Then*

$$(\mathbf{M} \times \mathbf{N}) \otimes \mathbf{T} \cong (\mathbf{M} \otimes \mathbf{T}) \times (\mathbf{N} \otimes \mathbf{T}).$$

Proof Using the definition of the tensor product of crossed modules, we suppose that $\mathbf{M} \otimes \mathbf{T} = (\mathrm{coker}\alpha_1, P \otimes G, \delta_1)$, $\mathbf{N} \otimes \mathbf{T} = (\mathrm{coker}\alpha_2, Q \otimes G, \delta_2)$ and $(\mathbf{M} \times \mathbf{N}) \otimes \mathbf{T} = (\mathrm{coker}\alpha, (P \times Q) \otimes G, \delta)$, where $\alpha_1 = (id_M \otimes (\partial_1 \times \partial_2), (\partial_1 \otimes id_T)^{-1})$, $\alpha_2 = (id_N \otimes (\partial_1 \times \partial_2), (\partial_2 \otimes id_T)^{-1})$ and $\alpha = (id_{M\times N} \otimes (\partial_1 \times \partial_2), ((\partial_1 \times \partial_2) \otimes id_T)^{-1})$. We only need to define an isomorphism (ψ_1, ψ_2)

$$
\begin{array}{ccc}
\mathrm{coker}\alpha & \xrightarrow{\psi_1} & \mathrm{coker}\alpha_1 \times \mathrm{coker}\alpha_2 \\
\Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta_1 \times \delta_2} \\
(P \times Q) \otimes G & \xrightarrow{\psi_2} & (P \otimes G) \times (Q \otimes G).
\end{array}
$$

The second component ψ_2 is the isomorphism given in [3, Proposition 10], which is defined on generators by $\psi_2((p, q) \otimes g) = (p \otimes g, q \otimes g)$. We now construct ψ_1, which will be induced on cokerα by a homomorphism $\langle \phi_1, \phi_2 \rangle$

$$
\begin{array}{ccc}
((M \times N) \otimes G) \rtimes (P \times Q) \otimes T & \xrightarrow{nat.} & \mathrm{coker}\alpha. \\
\Big\downarrow{\scriptstyle \langle \phi_1, \phi_2 \rangle} & \swarrow{\scriptstyle \psi_1} & \\
\mathrm{coker}\alpha_1 \times \mathrm{coker}\alpha_2 & &
\end{array}
$$

Let us define ϕ_1 and ϕ_2 on generators as follows:

$$\phi_1((m,n)\otimes g) = ((m\otimes g,1)\mathrm{Im}\alpha_1, (n\otimes g,1)\mathrm{Im}\alpha_2),$$
$$\phi_2((p,q)\otimes t) = ((1,p\otimes t)\mathrm{Im}\alpha_1, (1,q\otimes t)\mathrm{Im}\alpha_2).$$

Since the action of M on groups N and Q is trivial, for all $m\in M$, $n\in N$ and $g=(g_1,g_2)\in G$, we have

$$\begin{aligned} {}^{m}(n\otimes g) &= \big(m\otimes {}^{n}gg^{-1}\big)(n\otimes g) = \big(m\otimes {}^{(1,n)}(g_1,g_2)\big(g_1^{-1},g_2^{-1}\big)\big)(n\otimes g) \\ &= \big(m\otimes {}^{n}g_2g_2^{-1}\big)(n\otimes g) = {}^{m}(n\otimes g_2)(n\otimes g_2)^{-1}(n\otimes g) = n\otimes g. \end{aligned}$$

So, M on $N\otimes G$ and similarly, N on $M\otimes G$ act trivially. Using these results and after a long calculations, one can see that ϕ_1 and ϕ_2 preserve the defining relations of the tensor product of groups and are then homomorphisms. We now claim that $\phi_1({}^{b}a) = {}^{\phi_2(b)}\phi_1(a)$ for all $a\in(M\times N)\otimes G$ and $b\in(P\times Q)\otimes T$. Without loss of generality, we may assume that $a=(m,n)\otimes g$ and $b=(p,q)\otimes(t_1,t_2)$, where $(t_1,t_2)\in T$. Then

$$\begin{aligned} \phi_1({}^{b}a) &= \phi_1\big({}^{(p,q)\otimes(t_1,t_2)}((m,n)\otimes g)\big) = \phi_1\big({}^{({}^{p}t_1t_1^{-1},\ {}^{q}t_2t_2^{-1})}((m,n)\otimes g)\big) \\ &= (\overline{({}^{{}^{p}t_1t_1^{-1}}m\otimes {}^{({}^{p}t_1t_1^{-1},{}^{q}t_2t_2^{-1})}g,1)}, \overline{({}^{{}^{q}t_2t_2^{-1}}n\otimes {}^{({}^{p}t_1t_1^{-1},{}^{q}t_2t_2^{-1})}g,1)}) \\ &= (\overline{({}^{{}^{q}t_2t_2^{-1}}({}^{{}^{p}t_1t_1^{-1}}(m\otimes g)),1)}, \overline{({}^{{}^{p}t_1t_1^{-1}}({}^{{}^{q}t_2t_2^{-1}}(n\otimes g)),1)}) \\ &= (\overline{({}^{p\otimes t}(m\otimes g),1)}, \overline{({}^{q\otimes t}(n\otimes g),1)}) \\ &={}^{(\overline{(1,p\otimes t)},\overline{(1,q\otimes t)})}\,(\overline{(m\otimes g,1)}, \overline{(n\otimes g,1)}) \\ &= {}^{\phi_2((p,q)\otimes t)}\phi_1((m,n)\otimes g). \end{aligned}$$

Note that the forth equality follows from the fact that N acts trivially on $M\otimes G$. It, therefore, follows from Lemma 2.1(i) that the map $\phi=\langle\phi_1,\phi_2\rangle : ((M\times N)\otimes G)\rtimes((P\times Q)\otimes T)\longrightarrow \mathrm{coker}\alpha_1\times\mathrm{coker}\alpha_2$ defined by $\phi(a,b)=(\phi_1(a),\phi_2(b))$ is a homomorphism. Because ϕ annihilates $\mathrm{Im}\alpha$, ϕ induces the homomorphism ψ_1. We prove that ψ_1 is an isomorphism by giving an inverse for it. Consider the canonical homomorphisms $\eta_1:\mathrm{coker}\alpha_1\longrightarrow\mathrm{coker}\alpha$, $(m\otimes g, p\otimes t)\mathrm{Im}\alpha_1\longmapsto((m,1)\otimes g,(p,1)\otimes t)\mathrm{Im}\alpha$, and $\eta_2:\mathrm{coker}\alpha_2\longrightarrow\mathrm{coker}\alpha$, $(n\otimes g,q\otimes t)\mathrm{Im}\alpha_2\longmapsto((1,n)\otimes g,(1,q)\otimes t)\mathrm{Im}\alpha$. Then the map $\eta=\langle\eta_1,\eta_2\rangle:\mathrm{coker}\alpha_1\times\mathrm{coker}\alpha_2\longrightarrow\mathrm{coker}\alpha$ given by $\eta(x,y)=\eta_1(x)\eta_2(y)$ is an inverse for ψ_1. One easily sees that (ψ_1,ψ_2) is a morphism of crossed modules. □

Proof of Theorem 3.1 (i) Let $(V,R,\mu)\rightarrowtail(Y,F,\mu)\twoheadrightarrow(T,G,\partial)$ be a projective presentation of the crossed module (T,G,∂). It is sufficient to prove

$$(T,G,\partial)\wedge(T,G,\partial)\cong\frac{(Y,F,\mu)'}{[(Y,F,\mu),(V,R,\mu)]}.$$

Since (Y, F, μ) is a projective crossed module, F/Y is a free group. It follows from [8, Theorem 6] that the kernel of the epimorphism $\lambda_Y : F \wedge Y \longrightarrow [F, Y]$ is trivial. Also, the homomorphism $\lambda_F : F \wedge F \longrightarrow F'$ is an isomorphism thanks to Ellis [8, Proposition 2]. Now, it is readily seen that $(\lambda_Y, \lambda_F) : (F \wedge Y, F \wedge F, id \wedge \mu) \longrightarrow ([F, Y], F', \mu)$ is an isomorphism of crossed modules and

$$(Y, F, \mu) \wedge (Y, F, \mu) \cong ([F, Y], F', \mu).$$

We consider the diagram

$$\begin{array}{ccccc}
(V, R, \mu) \wedge (Y, F, \mu) & \xrightarrow{(\eta_1, \eta_2)} & (Y, F, \mu) \wedge (Y, F, \mu) & \twoheadrightarrow & (T, G, \partial) \wedge (T, G, \partial) \\
\big\downarrow = & & \big\downarrow (\widetilde{\varphi}, id) & & \big\downarrow = \\
(V, R, \mu) \wedge (Y, F, \mu) & \xrightarrow{\psi} & (F \wedge Y, F \wedge F, id \wedge \mu) & \twoheadrightarrow & (T, G, \partial) \wedge (T, G, \partial),
\end{array}$$

where (η_1, η_2) and $(\widetilde{\varphi}, id)$ are the morphisms given in Propositions 2.2(ii) and 3.3, respectively, and $\psi = (\psi_1, \psi_2)$ is the composition of $(\widetilde{\varphi}, id)$ and (η_1, η_2). As $\mathrm{Im}\psi_1 = \langle f \wedge v, r \wedge y \mid f \in F, v \in V, r \in R, y \in Y \rangle$ and $\mathrm{Im}\psi_2 = \langle r \wedge f \mid r \in R, f \in F \rangle$, for the morphism (λ_Y, λ_F) given above, we have

$$(\lambda_Y, \lambda_F)(\mathrm{Im}\psi_1, \mathrm{Im}\psi_2, id \wedge \mu) = ([F, V][R, Y], [R, F], \mu),$$

and therefore,

$$(T, G, \partial) \wedge (T, G, \partial) \cong \frac{(F \wedge Y, F \wedge F, id \wedge \mu)}{\mathrm{Im}\psi} \cong \frac{(Y, F, \mu)'}{[(Y, F, \mu), (V, R, \mu)]}.$$

The proof is complete.

(ii) There is the following commutative diagram of crossed modules:

$$\begin{array}{ccccc}
\ker(\mathbf{T} \wedge \mathbf{S} \longrightarrow \mathbf{T}) & \rightarrowtail & \mathbf{T} \wedge \mathbf{S} & \longrightarrow & \mathbf{T}' \cap \mathbf{S} \\
 & & \big\downarrow & & \big\downarrow \\
\ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T}) & \rightarrowtail & \mathbf{T} \wedge \mathbf{T} & \twoheadrightarrow & \mathbf{T}' \\
 & & \big\downarrow & & \big\downarrow \\
\ker(\mathbf{M} \wedge \mathbf{M} \longrightarrow \mathbf{M}) & \rightarrowtail & \mathbf{M} \wedge \mathbf{M} & \twoheadrightarrow & \mathbf{M}'
\end{array}$$

where the rows and, invoking Proposition 3.3, columns are exact. Then, by the snake lemma, we get the following exact sequence

$$\ker(\mathbf{T} \wedge \mathbf{S} \longrightarrow \mathbf{T}) \longrightarrow \ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T}) \longrightarrow \ker(\mathbf{M} \wedge \mathbf{M} \longrightarrow \mathbf{M}),$$

and the result now follows from part (i) and [6, Theorem 12(i)]. □

Proof of Theorem 3.2 By Proposition 2.2(i) and Theorem 3.1(i), we only need to prove that

$$\begin{aligned}(\mathbf{T_1}\times\mathbf{T_2})\otimes(\mathbf{T_1}\times\mathbf{T_2}) &\cong (\mathbf{T_1}\otimes\mathbf{T_1})\times(\mathbf{T_2}\otimes\mathbf{T_2})\times(\mathbf{T_1}_{ab}\\ &\qquad\otimes\mathbf{T_2}_{ab})\times(\mathbf{T_2}_{ab}\otimes\mathbf{T_1}_{ab}), \qquad (1)\\ (\mathbf{T_1}\times\mathbf{T_2})\wedge(\mathbf{T_1}\times\mathbf{T_2}) &\cong (\mathbf{T_1}\wedge\mathbf{T_1})\times(\mathbf{T_2}\wedge\mathbf{T_2})\times(\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab}). \qquad (2)\end{aligned}$$

Identifying $\mathbf{T_1}$ and $\mathbf{T_2}$ with their images in the crossed module $\mathbf{T_1}\times\mathbf{T_2}$, $[\mathbf{T_1},\mathbf{T_2}]=1$ and so, according to Proposition 2.2(iii), $\mathbf{T_1}\otimes\mathbf{T_2}\cong\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab}$ and $\mathbf{T_2}\otimes\mathbf{T_1}\cong\mathbf{T_2}_{ab}\otimes\mathbf{T_1}_{ab}$. We therefore obtain the isomorphism (1) by applying Proposition 3.4 twice.

To prove the isomorphism (2), we consider the following diagram

$$\begin{array}{ccc}
(\mathbf{T_1}\times\mathbf{T_2})\otimes(\mathbf{T_1}\times\mathbf{T_2}) & \xrightarrow{\ nat.\ } & (\mathbf{T_1}\times\mathbf{T_2})\wedge(\mathbf{T_1}\times\mathbf{T_2})\\
\psi\uparrow\ \downarrow\varphi & & \\
(\mathbf{T_1}\otimes\mathbf{T_1})\times(\mathbf{T_2}\otimes\mathbf{T_2})\times(\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab})\times(\mathbf{T_2}_{ab}\otimes\mathbf{T_1}_{ab}) & & \theta\uparrow\ \downarrow\eta\\
i\uparrow\ \downarrow\lambda=id_{\mathbf{T_1}\otimes\mathbf{T_1}}\times id_{\mathbf{T_2}\otimes\mathbf{T_2}}\times\left(id_{\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab}}*\nu\right) & & \\
(\mathbf{T_1}\otimes\mathbf{T_1})\times(\mathbf{T_2}\otimes\mathbf{T_2})\times(\mathbf{T_{1ab}}\otimes\mathbf{T_{2ab}}) & \xrightarrow{\ nat.\ } & (\mathbf{T_1}\wedge\mathbf{T_1})\times(\mathbf{T_2}\wedge\mathbf{T_2})\times(\mathbf{T_{1ab}}\otimes\mathbf{T_{2ab}}),
\end{array}$$

where ψ,φ are isomorphisms obtained in the proof of Proposition 3.4, i is the inclusion morphism, and $id_{\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab}}*\nu:(\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab})\times(\mathbf{T_2}_{ab}\otimes\mathbf{T_1}_{ab})\longrightarrow\mathbf{T_1}_{ab}\otimes\mathbf{T_2}_{ab}$ is the surjective morphism given in Lemma 2.1(i). An easy calculation shows that

$$\begin{aligned}\lambda\varphi((\mathbf{T}_1\times\mathbf{T}_2)\Box(\mathbf{T}_1\times\mathbf{T}_2)) &\subseteq (\mathbf{T}_1\Box\mathbf{T}_1)\times(\mathbf{T}_2\Box\mathbf{T}_2),\\ \psi i((\mathbf{T}_1\Box\mathbf{T}_1)\times(\mathbf{T}_2\Box\mathbf{T}_2)) &\subseteq (\mathbf{T}_1\times\mathbf{T}_2)\Box(\mathbf{T}_1\times\mathbf{T}_2),\end{aligned}$$

from which we get the induced morphisms η and θ. Since η is surjective, we prove that η is isomorphism by showing that θ is a left inverse of η. But this follows from the following observations. For any $t_i\in T_i$, $g_i\in G_i$ $(i=1,2)$, we have

$$\begin{aligned}(t_1,1)\otimes(g_1,g_2) &= ((t_1,1)\otimes(1,g_2))^{(1,g_1)}((t_1,1)\otimes(g_1,1))\\ &= ((t_1,1)\otimes(1,g_2))((t_1,1)\otimes(g_1,1)),\end{aligned}$$

and similarly, $(g_1,1)\otimes(t_1,t_2)=((g_1,1)\otimes(t_1,1))((g_1,1)\otimes(1,t_2))$. The proof is complete. □

We have the following consequences of the above theorems.

Corollary 3.5 *Let (N,G,i) be the inclusion crossed module. Then* $H_2(G,N)\cong\dfrac{R\cap[F,Y]}{[R,Y][F,R\cap Y]}$, *where* $(R\cap Y,R,\mu)\rightarrowtail(Y,F,\mu)\twoheadrightarrow(N,G,i)$ *is a projective presentation of* (N,G,i).

Proof It follows from Theorem 3.1(i) and Hopf formula for $H_2(N,G,i)$. □

Corollary 3.6 *Let* (N_1, G_1, i_1) *and* (N_2, G_2, i_2) *be inclusion crossed modules. Then*

$$H_2(G_1 \times G_2, N_1 \times N_2) \cong H_2(G_1, N_1) \times H_2(G_2, N_2) \times \left(\frac{(\overline{N}_1 \otimes \overline{G}_2) \times (\overline{G}_1 \otimes \overline{N}_2)}{\langle (\overline{n_1} \otimes \overline{i_2(n_2)}, (\overline{i_1(n_1)} \otimes \overline{n_2})^{-1}) | n_1 \in N_1, n_2 \in N_2 \rangle} \right),$$

$$H_2(G_1 \times G_2) \cong H_2(G_1) \times H_2(G_2) \times (\overline{G}_1 \otimes \overline{G}_2).$$

Proof It follows from Theorem 3.2 and its proof. □

Corollary 3.7 *For any simply connected* $\mathbf{T} = (T, G, \partial)$*, there is a commutative diagram of crossed modules with exact rows and columns.*

$$\begin{array}{ccccc}
\Gamma(\mathbf{T}_{ab}) & \longrightarrow & J_2(\mathbf{T}) & \twoheadrightarrow & H_2(\mathbf{T}) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(\mathbf{T}_{ab}) & \longrightarrow & \mathbf{T} \otimes \mathbf{T} & \twoheadrightarrow & \mathbf{T} \wedge \mathbf{T} \\
& & \downarrow & & \downarrow \\
& & \mathbf{T}' & \longrightarrow & \mathbf{T}'
\end{array}$$

Proof It follows from Proposition 2.2(iv) and Theorem 3.1(i). □

Corollary 3.8 *For any finite perfect crossed module* $\mathbf{T} = (T, G, \partial)$*, the second homology of its covers is trivial.*

Proof In view of [20, Theorem 2.68], the extension

$$\ker(\tau_1, \tau_2) \rightarrowtail (G \otimes T, G \otimes G, id \otimes \partial) \overset{(\tau_1,\tau_2)}{\twoheadrightarrow} \mathbf{T} \tag{3}$$

is the universal central extension of $\mathbf{T}$, in which the commutator morphism (τ_1, τ_2) is given by $\tau_1(g_1 \otimes t) = {}^{g_1}tt^{-1}$ and $\tau_2(g_1 \otimes g_2) = [g_1, g_2]$, for all $t \in T, g_1, g_2 \in G$. Due to [24, Proposition 5 and Corollary 2], the extension (3) is, up to isomorphism, the only stem cover of $\mathbf{T}$. Applying Theorem 3.1(ii) to the extension (3), we deduce that $H_2(G \otimes T, G \otimes G, id \otimes \partial)$ is a homomorphic image of $(G \otimes T, G \otimes G, id \otimes \partial) \wedge \ker(\tau_1, \tau_2)$. But the perfectness of the crossed module $\mathbf{T}$, together with Proposition 2.1(iii) yields that

$$(G \otimes T, G \otimes G, id \otimes \partial) \wedge \ker(\tau_1, \tau_2) = (G \otimes T, G \otimes G, id \otimes \partial) \otimes \ker(\tau_1, \tau_2) = 1,$$

and then the result holds. □

4 Applications to the Pair of Groups

Loday [17] and Ellis [9], respectively, introduced the notions of relative central extension and cover for pairs of groups and used them as useful tools to develop the theory of capability, Schur multiplier, and central series of groups to a theory for pairs of groups. This section is devoted to structural results on these notions.

Let (G, N) be a pair of groups, in which N is a normal subgroup of G. We recall from [9,17] that

- A crossed module (M, G, δ) is called a *relative central extension* of (G, N) if $\delta(M) = N$ and $\ker\delta \subseteq M^G$. A morphism between two relative central extensions $\mathbf{M} = (M, G, \delta)$ and $\mathbf{M}' = (M', G, \delta')$ of (G, N) is a crossed module morphism $(f, id) : \mathbf{M} \longrightarrow \mathbf{M}'$. In particular, if f is surjective, then $\mathbf{M}'$ is called a homomorphic image of $\mathbf{M}$.
- A relative central extension (M, G, δ) of the pair (G, N) is called *universal* if there exists a unique morphism from it to any relative central extension of (G, N). It is proved in [17] that the pair (G, N) has a universal relative central extension if and only if N is G-perfect (that is, $[G, N] = N$).
- A relative central extension (M, G, δ) of (G, N) is called a *relative stem extension* if $\ker\delta \subseteq [G, M]$. If, in addition, $ker\delta \cong H_2(G, N)$ then (M, G, δ) is said to be a *covering pair* for (G, N). It is established in [19] that any pair of groups admits at least one covering pair.

We begin with the following key lemma, which deals with the connection between the stem covers and the universal central extensions of crossed modules with corresponding concepts for pairs of groups.

Lemma 4.1 *Let (G, N) be a pair of groups and $e\colon (A, B, \delta) \rightarrowtail (M, H, \delta) \overset{(\psi_1,\psi_2)}{\twoheadrightarrow} (N, G, i)$ be a stem cover (respectively, universal central extension) of the inclusion crossed module (N, G, i). Then the composite homomorphism $i\psi_1 : M \longrightarrow G$ with the action of G on M given by ${}^g m ={}^h m$, in which h is any element in the pre-image of g via ψ_2, is a covering pair (respectively, universal relative central extension) of (G, N).*

Proof The case of stem cover follows from [19, Corollary 3.2]. We so assume that e is a universal central extension of (N, G, i), and (M_1, G, δ_1) is an arbitrary relative central extension of the pair (G, N). Considering the central extension $(ker\delta_1, 1, 1) \rightarrowtail (M_1, G, \delta_1) \overset{(\delta_1,id)}{\twoheadrightarrow} (N, G, i)$, we get a unique morphism $(\beta_1, \psi_2) : (M, H, \delta) \longrightarrow (M_1, G, \delta_1)$ such that $(\delta_1, id)(\beta_1, \psi_2) = (\psi_1, \psi_2)$. It is straightforward to see that $(\beta_1, id) : (M, G, i\psi_1) \longrightarrow (M_1, G, \delta_1)$ is a morphism of crossed modules. This morphism is unique, because if $(\beta_1', id) : (M, G, i\psi_1) \longrightarrow (M_1, G, \delta_1)$ is another morphism, then $(\beta_1', \psi_2) : (M, H, \delta) \longrightarrow (M_1, G, \delta_1)$ is a morphism satisfying $(\delta_1, id)(\beta_1', \psi_2) = (\psi_1, \psi_2)$. But this result yields that $\beta_1 = \beta_1'$, as desired. □

The following corollary is a consequence of the above lemma.

Corollary 4.2 *Let $(R\cap Y, R, \mu) \rightarrowtail (Y, F, \mu) \overset{(\pi_1,\pi_2)}{\twoheadrightarrow} (N, G, i)$ be a projective presentation of a perfect inclusion crossed module (N, G, i). Then $([F, Y]/([R, Y][F, R\cap Y], G, i\bar{\pi}_1)$ is a covering pair as well as the universal relative central extension of the pair (G, N), where $\bar{\pi}_1$ is induced by π_1.*

Proof This follows from Lemma 4.1, [23, Proposition 4.3] and [6, Page 171]. □

In the following theorem, we see how Theorem 3.1(i) together with [19, Theorem 3.6] can be used to determine the structure of covering pairs.

Theorem 4.3 *Let (G, N) be a pair of finite groups, and $(R\cap Y, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (N, G, i)$ be a projective presentation of the crossed module (N, G, i). Then the crossed module (M, G, δ) is a covering pair of (G, N) if and only if there is a normal subgroup U of F with $M \cong Y/U$, $\ker\delta \cong (R\cap Y)/U$, and*

$$\frac{R\cap Y}{[R, Y][F, R\cap Y]} = H_2(G, N) \times \frac{U}{[R, Y][F, R\cap Y]}. \tag{4}$$

Proof Let (M, G, δ) be a covering pair of (G, N). Then $(\ker\delta, 1, 1) \rightarrowtail (M, G, \delta) \twoheadrightarrow (N, G, i)$ is a stem extension of the crossed module (N, G, i) and so, according to Mohammadzadeh et al. [19, Lemma 3.3], we find an epimorphism $\beta : Y/([R, Y][F, R\cap Y]) \longrightarrow M$ such that $\beta(H_2(G, N)) = \ker\delta$. Set $\ker\beta = U/([R, Y][F, R\cap Y])$ for some normal subgroup U of Y, then $M \cong Y/U$ and $\ker\delta \cong (R\cap Y)/U$. Also, the finiteness of $H_2(G, N)$ ensures that $H_2(G, N)\cap\ker\beta = 1$. As the kernel of the restriction of β_1 to $(R\cap Y)/([R, Y][F, R\cap Y])$ is $\ker\beta$ and the image of this restriction is $\ker\delta$, we conclude that U satisfies the condition (4).

Conversely, let U be a normal subgroup of F satisfying (4), and $\bar{\mu} : Y/([R, Y][F, R\cap Y]) \longrightarrow F/[F, R]$ be the crossed module induced by μ. In view of [15, Theorems 2.1.4(i) and 2.4.6(iv)], there is a normal subgroup S of F such that $R/[F, R] = H_2(G) \times (S/[F, R])$. One easily sees that $\bar{\mu}(U/([R, Y][F, R\cap Y])) \subseteq S/[F, R]$. Using these results and Theorem 3.1(i), we obtain

$$\left(\frac{R\cap Y}{[R, Y][F, R\cap Y]}, \frac{R}{[F, R]}, \bar{\mu}\right) = H_2(N, G, i) \times \left(\frac{U}{[R, Y][F, R\cap Y]}, \frac{S}{[F, R]}, \bar{\mu}\right).$$

It, therefore, follows from [19, Proposition 3.1(i)] that $(Y/U, F/S, \bar{\mu})$ is a covering crossed module of (N, G, i) and so, the composite homomorphism $Y/U \longrightarrow Y/(R\cap Y) \xrightarrow{\cong} N \xrightarrow{i} G$ is a covering pair of the pair (G, N), thanks to Lemma 4.1. The proof is complete. □

The following corollaries are generalizations of the works of Jones and Wiegold [14], and Yamazaki [25].

Corollary 4.4 *Let (M_i, G, δ_i) $i = 1, 2$, be two covering pairs of a pair (G, N) of finite groups. Then*

(i) M_1 and M_2 are isoclinic.

(ii) $Z(M_1)/\ker\delta_1 \cong Z(M_2)/\ker\delta_2$.

Proof Suppose $(R\cap Y, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (N, G, i)$ is a projective presentation of the inclusion crossed module (N, G, i). From Theorem 4.3 and its proof, we have an epimorphism $\beta : Y/([R, Y][F, R\cap Y]) \longrightarrow M_1$ such that $(R\cap Y)/([R, Y][F, R\cap Y]) = H_2(G, N) \times \ker\beta$. But this implies that

$$\left(\frac{Y}{[R, Y][F, R\cap Y]}\right)' \cap \ker\beta \subseteq \frac{[F, Y]}{[R, Y][F, R\cap Y]} \cap \ker\beta = 1.$$

So, by Hall [12, Page 134], M_1 and similarly, M_2 are isoclinic to $Y/([R,Y][F,R\cap Y])$, which proves (i).

To prove (ii), we only need to show that the factor $Z(M_1)/\ker\delta_1$ is determined uniquely by the free presentation $N\cong Y/(R\cap Y)$. Put

$$\ker\beta=\frac{U}{[R,Y][F,R\cap Y]}\quad\text{and}\quad Z\left(\frac{Y}{[R,Y][F,R\cap Y]}\right)=\frac{W}{[R,Y][F,R\cap Y]}$$

for some normal subgroups U and W of Y. Then $[Y,W]\subseteq U$ and so $W/U\subseteq Z(Y/U)$. On the other hand, if $xU\in Z(Y/U)$ then $[x,y]\in U\cap[Y,F]=[R,Y][F,R\cap Y]$ for all $y\in Y$. Hence

$$x[R,Y][F,R\cap Y]\in Z\left(\frac{Y}{[R,Y][F,R\cap Y]}\right)=\frac{W}{[R,Y][F,R\cap Y]},$$

which implies that $Z(Y/U)=W/U$. It, therefore, follows that $Z(M_1)/\ker\delta_1\cong W/(R\cap Y)$, as required. □

Corollary 4.5 *Any relative stem extension of a pair of finite groups is a homomorphic image of one of its covering pairs.*

Proof Let (M,G,δ) be an arbitrary relative stem extension of a pair (G,N) of finite groups, and let $(R\cap Y,R,\mu)\rightarrowtail(Y,F,\mu)\overset{(\pi_1,\pi_2)}{\twoheadrightarrow}(N,G,i)$ be a projective presentation of the inclusion crossed module (N,G,i). Due to Mohammadzadeh et al. [19, Lemma 3.3], there is the following commutative diagram with exact rows and surjective columns.

$$\begin{array}{ccccc}
\frac{R\cap Y}{[R,Y][F,R\cap Y]} & \overset{\subseteq}{\rightarrowtail} & \frac{Y}{[R,Y][F,R\cap Y]} & \overset{\bar{\pi}_1}{\twoheadrightarrow} & N\\
\downarrow\beta| & & \downarrow\beta & & \downarrow id_N\\
\ker\delta & \overset{\subseteq}{\rightarrowtail} & M & \overset{\delta}{\twoheadrightarrow} & N,
\end{array}$$

where $\beta|$ is the restriction of β. Put $\ker\beta|=\ker\beta=T/([R,Y][F,R\cap Y])$ for some normal subgroup T of Y. It can be seen in the proof of Theorem 4.3 that

$$\frac{R\cap Y}{T}\cong\ker\delta\cong\frac{R\cap[F,Y]}{T\cap R\cap[F,Y]}\cong\frac{T(R\cap[F,Y])}{T},$$

which, because of the finiteness of $\ker\delta$, implies that $T(R\cap[F,Y])=R\cap Y$. Also, the triviality of the second homology of F/Y yields that $Y\cap F'=[F,Y]$. So, we have

$$\frac{T}{T\cap[F,Y]}=\frac{T}{T\cap R\cap[F,Y]}\cong\frac{T(R\cap[F,Y])}{R\cap[F,Y]}=\frac{R\cap Y}{R\cap Y\cap F'}\cong\frac{(R\cap Y)F'}{F'}\leq\frac{F}{F'}.$$

Thus the following exact sequence of abelian groups splits

$$\frac{T\cap[F,Y]}{[R,Y][F,R\cap Y]}\rightarrowtail\frac{T}{[R,Y][F,R\cap Y]}\twoheadrightarrow\frac{T}{T\cap[F,Y]},$$

and hence

$$\frac{T}{[R,Y][F,R\cap Y]} = \frac{T\cap[F,Y]}{[R,Y][F,R\cap Y]} \times \frac{U}{[R,Y][F,R\cap Y]},$$

where $U/([R,Y][F,R\cap Y]) \cong T/(T\cap[F,Y])$. Using these results, we have

$$U(R\cap[F,Y]) = R\cap Y \quad \text{and} \quad U\cap(R\cap[F,Y]) = [R,Y][F,R\cap Y],$$

which show that U satisfies (4). Accordingly, owing to Theorem 4.3, the map $\delta : Y/U \longrightarrow G$ induced by β is a covering pair of (G,N) and, moreover, M is a homomorphic image of Y/U, which completes the proof. □

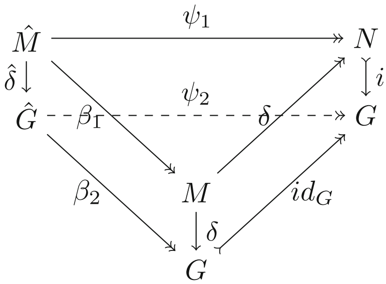
Given a pair (G,N) of groups, it is shown in Lemma 4.1 that any stem cover of the inclusion crossed module (N,G,i) gives a covering pair for (G,N). Now, we prove the converse in the case where G is finite.

Proposition 4.6 *Let (G,N) be a pair of finite groups. Then any covering pair of (G,N) can be lifted to a stem cover of the inclusion crossed module (N,G,i).*

Proof Let (M,G,δ) be a covering pair of (G,N). By Mohammadzadeh et al. [19, Theorem 3.4], there is a stem cover

$$(\hat{A},\hat{B},\hat{\delta}) \rightarrowtail (\hat{M},\hat{G},\hat{\delta}) \overset{(\psi_1,\psi_2)}{\twoheadrightarrow} (N,G,i),$$

and a surjective morphism $(\beta_1,\beta_2) : (\hat{M},\hat{G},\hat{\delta}) \longrightarrow (M,G,\delta)$ such that the diagram



is commutative. Invoking Lemma 4.1, $(\hat{M},G,i\psi_1)$ is a covering pair of (G,N). Since the groups M and $\hat{M}$ have the same order, it follows that β_1 is an isomorphism. Now, the composite homomorphism $\hat{\delta}\beta_1^{-1} : M \longrightarrow \hat{G}$, together with the action of $\hat{G}$ on M defined by ${}^{\hat{g}}m = {}^{\beta_2(\hat{g})}m$ is a crossed module. Because, if $m, m_1, m_2 \in M$, $\hat{g} \in \hat{G}$ and $\beta_1(\hat{m}) = m$ for some $\hat{m} \in \hat{M}$, then using the above diagram we have

$$\hat{\delta}\beta_1^{-1}\left({}^{\hat{g}}m\right) = \hat{\delta}\beta_1^{-1}\left({}^{\beta_2(\hat{g})}\beta_1(\hat{m})\right) = \hat{\delta}\beta_1^{-1}\left(\beta_1({}^{\hat{g}}\hat{m})\right) = \hat{\delta}\left({}^{\hat{g}}\hat{m}\right) = {}^{\hat{g}}\hat{\delta}\beta_1^{-1}(m),$$
$${}^{\hat{\delta}\beta_1^{-1}(m_1)}m_2 = {}^{\beta_2\hat{\delta}\beta_1^{-1}(m_1)}m_2 = {}^{\delta(m_1)}m_2 = {}^{m_1}m_2.$$

It is routine to verify that $\ker\delta \subseteq [\hat{G}, M] \cap M^{\hat{G}}$, $\hat{B} = \ker\beta_2 \subseteq Z(\hat{G}) \cap st_{\hat{G}}(M) \cap \hat{G}'$ and $(\beta_1, id) : (\hat{A}, \hat{B}, \hat{\delta}) \longrightarrow (\ker\delta, \ker\beta_2, \hat{\delta}\beta_1^{-1})$ is an isomorphism. We therefore conclude that the extension

$$\left(\ker\delta, \ker\beta_2, \hat{\delta}\beta_1^{-1}\right) \rightarrowtail \left(M, \hat{G}, \hat{\delta}\beta_1^{-1}\right) \overset{(\delta,\beta_2)}{\twoheadrightarrow} (N, G, i)$$

is a stem cover of (N, G, i). Moreover, (M, G, δ) and $(M, G, i\delta)$ are isomorphic. □

We end this article with the following interesting corollary.

Corollary 4.7 *Let G be a finite group with a normal subgroup N such that the inclusion crossed module (N, G, i) is perfect. Then*

(i) *All covering pairs of the pair (G, N) are isomorphic.*

(ii) *A relative central extension of (G, N) is universal if and only if it is a covering pair.*

(iii) *Every universal central extension of (G, N) can be lifted to a universal central extension of (N, G, i).*

Proof (i) Combine the above proposition with [24, Corollary 3.8].

(ii) In the proof of Corollary 3.8, it was shown that the perfect crossed module (N, G, i) admits a universal central extension, which is, up to isomorphism, its unique stem cover. The result now follows from these facts, Lemma 4.1, part (i), and the uniqueness of the universal relative central extension.

(iii) This follows from part (ii) and Proposition 4.6. □

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