The Perturbed Riemann Problem for a Macroscopic Production Model with Chaplygin Gas

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Abstract

The exact Riemann solutions for a macroscopic production model under the equation of state given by the Chaplygin gas are solved explicitly for all possible Riemann initial data. It is discovered interestingly that a composite hyperbolic wave is involved in Riemann solution under some specially designated initial conditions, which is made up of a rarefaction wave and a delta contact discontinuity attached on the wave front of the rarefaction wave. Furthermore, the constructions of global solutions to the perturbed Riemann problem for this system are also displayed in completely explicit forms when the initial data are taken to be three piecewise constant states under some suitable restrictive conditions by using the method of characteristics. During the process of constructing global solutions, the interactions of elementary waves are studied in detail. Moreover, it is proved rigorously that Riemann solutions are stable with respect to the specific small perturbations of Riemann initial data.

Keywords Riemann problem · Wave interaction · Chaplygin gas · Macroscopic production model

Mathematics Subject Classification 35L65 · 35L67 · 76N15

1 Introduction

In recent years, the macroscopic production models have become one of the important research topics in the study of production planning and control in the manufacturing

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industry. In the previous stage, the fluid-like continuous models [1,2] have been intro-
duced to model high-volume product flows, which can be used to depict the product
flow by using discrete event simulations in an accumulated way. The most widely
used model is a scalar conservation law under appropriate assumptions on the flux
function which is called as a clearing function for the product flow. In general, a given
clearing function describes the averaged sample data, but cannot illustrate the data
diffusion. In order to solve this problem, the following macroscopic production model
[3] consisting of two conservation laws is proposed in the form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \left(\rho u \left(1 + P(\rho)\right)\right)_t + \left(\rho u^2 \left(1 + P(\rho)\right)\right)_x = 0, \end{cases} \tag{1.1}$$

in which ρ and u are used to represent the product density and velocity, respectively,
which are often required to be non-negative. In addition, t and x express the time and
the production stage (or the degree of completion) to describe the work-in-progress
and the pressure term P refers to the anticipated factor in the production line which
is usually given by the equation of state $P = P(\rho)$.

It is of great interest to notice that some interesting nonlinear phenomena were
observed in [3] when the pressure term was taken as $P(\rho) = \rho$. Especially, it was
obtained in [4] that the delta standing wave was also involved in Riemann solu-
tions for system (1.1) under this pressure term $P(\rho) = \rho$. In fact, the singularity
of $\delta-$concentration was also investigated in [5,6] which can be used to illustrate some
extreme situations in the production line. Nowadays, it was known in [7–11] that delta
shock wave is involved in Riemann solutions for the isentropic Chaplygin gas dynam-
ical system. It is worthwhile to mention that the equation of state for the Chaplygin gas
is taken to be the negative pressure $P(\rho) = -\frac{1}{\rho}$, which often happens in the product
flow. Actually, this equation of state $P(\rho) = -\frac{1}{\rho}$ was also used in the AR traffic flow
model to describe some complicated traffic phenomena well [12]. It is known that the
behaviors of the macroscopic production model (1.1) are very similar to those of the
AR traffic flow model. Thus, it is natural to introduce the equation of state for the
Chaplygin gas $P(\rho) = -\frac{1}{\rho}$ into the macroscopic production model (1.1) in order to
discover some new interesting nonlinear phenomena. In the current paper, from the
viewpoint of hyperbolic systems of conservation laws, it is also of great interest to
consider system (1.1) under the equation of state $P(\rho) = -\frac{1}{\rho}$ given by the Chaplygin
gas, which is rewritten precisely as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u - u)_t + (\rho u^2 - u^2)_x = 0. \end{cases} \tag{1.2}$$

In order to develop wave analysis, we first draw our attention on system (1.2) with the
following Riemann-type initial condition

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), \ x < 0, \\ (\rho_+, u_+), \ x > 0. \end{cases} \tag{1.3}$$

The Riemann problem is a special initial value problem with the initial condition consisting of two constant states separated by the origin. It is one of the most fundamental problems in the field of nonlinear hyperbolic systems of conservation laws. The structures of Riemann solutions can be used well to describe the various and complicated nonlinear phenomena and are also regarded as the basic for numerical schemes, for the reason that Riemann solutions are served as the basic building blocks of constructing solutions to the general initial value problem by employing the random choice method. It is of great interest to develop numerical methods such as Godunov methods or even Riemann-free methods [13–18], which are based on the constructions of Riemann solutions in details. Thus, the constructions of Riemann solutions for different hyperbolic systems of conservation laws are very important and also have attracted the attention of researchers from analytical and numerical aspects.

It is not difficult to get that system (1.2) owns two real and different eigenvalues $\lambda_1 = u - \frac{u}{\rho - 1}$ and $\lambda_2 = u$. Obviously, the line $\rho = 1$ is a singular curve in the quarter (ρ, u) phase plane. In the present paper, we restrict ourselves into the region $\Omega = \{(\rho, u) | \rho > 1, u > 0\}$, such that $\lambda_1 < \lambda_2$ and then the system (1.2) is strictly hyperbolic in this region Ω. In fact, the other region $\{(\rho, u) | 0 < \rho < 1, u > 0\}$ can be dealt with similarly. It is worthwhile to notice that the shock curve shares the same expression formula with the rarefaction one in the quarter (ρ, u) phase plane, namely system (1.2) belongs to the so-called Temple class [19–21]. Furthermore, it is of great interest to observe that a composite wave is involved in Riemann solution of (1.2) and (1.3) under certain initial condition $0 < u_+ < \frac{(\rho_- - 1)u_-}{\rho_-}$, which is made up of a rarefaction wave and a delta contact discontinuity attached on the wave front of rarefaction wave. In the current paper, we only construct possible solution of the Riemann problem (1.2) and (1.3) for this certain initial condition and the uniqueness problem is still unsolved and left for the future work. In addition, it is well known that delta shock wave should be introduced in order to solve Riemann problems for some weak hyperbolic conservation systems and the theory of delta shock wave has been well established and developed in recent years, see [22–28] for examples. However, as far as we know, our constructed composite wave solution with the singularity of delta function cannot be found in any Riemann solution but only appears in the study [29–31] of wave interaction between a delta shock wave with a rarefaction wave for some hyperbolic systems of conservation laws.

It is well known that Riemann solutions of (1.2) and (1.3) cannot illustrate the dynamic pictures of system (1.2) in all the situations. In order to capture some more complicated nonlinear phenomena, it is necessary to investigate the perturbed (or double) Riemann problem for system (1.2) with the following constant states in three pieces

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & -\infty < x < 0, \\ (\rho_m, u_m), & 0 < x < \varepsilon, \\ (\rho_+, u_+), & \varepsilon < x < +\infty, \end{cases} \quad (1.4)$$

where ε is taken to be a adequately small positive number. In order to construct the global solutions to the perturbed Riemann problem (1.2) and (1.4), we need to deal with the problem of wave interactions carefully. Due to the fact that the uniqueness problem is unsolved when $0 < u_+ < \frac{(\rho_- - 1)u_-}{\rho_-}$, we restrict ourselves to only consider

the interactions of elementary waves including shock wave, rarefaction wave and contact discontinuity. For this purpose, it is required that the initial condition (1.4) should satisfy $u_m > \frac{(\rho_- - 1)u_-}{\rho_-}$ and $u_+ > \frac{(\rho_- - 1)u_-}{\rho_-}$ simultaneously. Thanks to system (1.2) with special structure of Temple class, the global solutions of (1.2) and (1.4) are constructed in fully explicit forms by using the method of characteristics. As a consequence, it is evident to show that the limits of perturbed Riemann solutions of (1.2) and (1.4) converge to the corresponding ones of (1.2) and (1.3) when the perturbation parameter ε approaches zero. That is to say, Riemann solutions of (1.2) and (1.3) are stable with respect to certain small perturbation of initial condition given by (1.4) where ε is regarded as the so-called perturbation parameter. It should be stressed that the initial condition taken in the form (1.4) has been intensively used to investigate the problem of wave interactions [32–38] for various hyperbolic systems of conservation laws.

The rest of this paper is organized in the following way. In Sect. 2, we investigate the general properties and elementary waves for system (1.2) and then solve exact Riemann solutions of (1.2) and (1.3) for all possible initial conditions. In Sect. 3, the perturbed Riemann solutions of (1.2) and (1.4) are constructed by studying all the occurring wave interactions under the two restrictive conditions $u_m > \frac{(\rho_- - 1)u_-}{\rho_-}$ and $u_+ > \frac{(\rho_- - 1)u_-}{\rho_-}$. Finally, the limit $\varepsilon \to 0$ is taken, which allows us to get stabilities of Riemann solutions with respect to this special perturbation of initial condition and then the conclusion is drawn in Sect. 4.

2 Riemann Solutions of (1.2) and (1.3)

In this section, we firstly present some basic characteristics for system (1.2). Then, we construct Riemann solutions of (1.2) and (1.3) when the initial condition (1.3) lies in the region $\Omega = \{(\rho, u) | \rho > 1, u > 0\}$. System (1.2) admits the following quasi-linear form

$$\begin{pmatrix} 1 & 0 \\ u & \rho - 1 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ u^2 & 2(\rho - 1)u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are represented as

$$\lambda_1 = u - \frac{u}{\rho - 1}, \qquad \lambda_2 = u. \tag{2.1}$$

It is evident that system (1.2) is strictly hyperbolic in the region Ω. The corresponding right eigenvectors for $\lambda_i (i = 1, 2)$ are given, respectively, by $\overrightarrow{r_1} = (\rho(\rho - 1), -u)^T$ and $\overrightarrow{r_2} = (1, 0)^T$. A simple calculation shows that $\nabla \lambda_1 \cdot \overrightarrow{r_1} = \frac{2u}{\rho - 1} \neq 0$ in the region Ω and $\nabla \lambda_2 \cdot \overrightarrow{r_2} \equiv 0$, in which $\nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial u})$. So, it is easily shown that the first characteristic field is genuinely nonlinear in the region Ω and the second characteristic field is always linearly degenerate. As a consequence, the wave of λ_1-family is either shock wave or rarefaction wave as well as the wave of λ_2-family is always contact

discontinuity. For convenience, we use the notations S, R and J, respectively, to stand for shock wave, rarefaction wave and contact discontinuity in the current paper. In addition, the Riemann invariants along the characteristic fields are chosen, respectively, as $u - \frac{u}{\rho}$ and u.

To solve self-similar Riemann solutions of the form $(\rho, u)(x, t) = (\rho, u)(\xi)$ with $\xi = \frac{x}{t}$, the Riemann problem (1.2) and (1.3) is reduced to

$$
\begin{cases}
-\xi \rho_\xi + (\rho u)_\xi = 0, \\
-\xi (\rho u - u)_\xi + (\rho u^2 - u^2)_\xi = 0,
\end{cases}
\tag{2.2}
$$

with the boundary conditions $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$. For smooth solutions, system (2.2) is rewritten as

$$
\begin{pmatrix} u - \xi & \rho \\ -\xi u + u^2 & -\xi(\rho - 1) + 2(\rho - 1)u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

It provides not only general solution (constant state), but also singular solution. It should be pointed out that a rarefaction wave is a piecewise smooth continuous solution of (2.2). Let the left state (ρ_-, u_-) be a fixed point in the region Ω, then we get the 1-rarefaction curve originating from (ρ_-, u_-) as follows:

$$
R(\rho_-, u_-) : \xi = \lambda_1 = u - \frac{u}{\rho - 1}, \qquad u - \frac{u}{\rho} = u_- - \frac{u_-}{\rho_-}, \qquad \rho > \rho_-, \quad u < u_-.
\tag{2.3}
$$

It is worth mentioning that if the given left state (ρ_-, u_-) lies in the region Ω, then the 1-rarefaction curve has the line $u = u_- - \frac{u_-}{\rho_-}$ as its asymptote. By differentiating ρ with respect to u in (2.3), we find that

$$
\frac{d\rho}{du} = \frac{-\rho_-(\rho_- - 1)u_-}{(\rho_- u - (\rho_- - 1)u_-)^2} < 0, \qquad \frac{d^2\rho}{du^2} = \frac{2\rho_-^2(\rho_- - 1)u_-}{(\rho_- u - (\rho_- - 1)u_-)^3} = \frac{2\rho^3(\rho_- - 1)u_-}{\rho_- u^3} > 0,
$$

which implies that the 1-rarefaction curve is convex in Ω.

The R-H conditions at a discontinuous curve $x = x(t)$ are given by

$$
\begin{cases}
\sigma[\rho] = [\rho u], \\
\sigma[\rho u - u] = [\rho u^2 - u^2],
\end{cases}
\tag{2.4}
$$

where $\sigma = \frac{dx}{dt}$ is the velocity of discontinuity and $[\rho] = \rho_r - \rho_l$ with $\rho_l = \rho(x(t) - 0, t)$ and $\rho_r = \rho(x(t) + 0, t)$ is the jump of ρ across the discontinuity, etc. On the one hand, if $\sigma = 0$, then we can only obtain a constant state solution. On the other hand, if $\sigma \neq 0$, then eliminating σ in (2.4) yields

$$
(\rho_r u_r - u_r - \rho_l u_l + u_l)(\rho_r u_r - \rho_l u_l) = (\rho_r - \rho_l)(\rho_r u_r^2 - u_r^2 - \rho_l u_l^2 + u_l^2),
$$

which is simplified into

$$(u_l - u_r)\Big(\rho_r u_l (\rho_l - 1) - \rho_l u_r (\rho_r - 1)\Big) = 0.$$

On the one hand, if $u_l \neq u_r$, then we have $u_l - \frac{u_l}{\rho_l} = u_r - \frac{u_r}{\rho_r}$. We then find $\rho_l > \rho_r$ by using the Lax entropy conditions. Therefore, let the left state $(\rho_l, u_l) = (\rho_-, u_-)$ be fixed in the region Ω, then the 1-shock curve originating from (ρ_-, u_-) is expressed as

$$S(\rho_-, u_-) : \sigma = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \qquad u - \frac{u}{\rho} = u_- - \frac{u_-}{\rho_-}, \qquad \rho < \rho_-, \quad u > u_-. \quad (2.5)$$

If the given left state (ρ_-, u_-) lies in Ω, then the 1-shock curve has the line $\rho = 1$ as its asymptote. On the other hand, if $u_l = u_r$, then one has $\sigma = u_l = u_r$. Therefore, the curve of contact discontinuity starting from the given left state (ρ_-, u_-) is represented as

$$J(\rho_-, u_-) : \tau = u = u_-. \qquad (2.6)$$

In summary, if the given left state (ρ_-, u_-) lies in the region Ω, then the set of states consist of 1-shock curve $S(\rho_-, u_-)$, 1-rarefaction curve $R(\rho_-, u_-)$ and 2-contact discontinuity curve $J(\rho_-, u_-)$ (see Fig. 1). Then the unique global Riemann solution of (1.2) and (1.3) can be constructed in terms of the right state (ρ_+, u_+) in the different parts I, II and III of the region Ω.

More specifically, if $u_- < u_+$, then the Riemann solution of (1.2) and (1.3) can be expressed by the symbol $S + J$ in the following form (see Fig. 2a)

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & \frac{x}{t} < \sigma, \\ (\rho_*, u_*), & \sigma < \frac{x}{t} < \tau, \\ (\rho_+, u_+), & \frac{x}{t} > \tau, \end{cases} \qquad (2.7)$$

in which the intermediate state (ρ_*, u_*) is given by

$$(\rho_*, u_*) = \left(\frac{\rho_- u_+}{\rho_- u_+ - (\rho_- - 1)u_-}, u_+ \right), \qquad (2.8)$$

and the wave speeds of S and J are computed, respectively, by

$$\sigma = \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-} = \frac{(\rho_- - 1)u_- - u_+}{\rho_- - 1} = u_- - \frac{u_+}{\rho_- - 1}, \qquad \tau = u_+. \quad (2.9)$$

If $\frac{(\rho_- - 1)u_-}{\rho_-} < u_+ < u_-$, then the Riemann solution of (1.2) and (1.3) can be expressed by the symbol $R + J$ in the following form (see Fig. 2b)

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & \frac{x}{t} < \lambda_1(\rho_-, u_-), \\ (\rho, u), & \lambda_1(\rho_-, u_-) < \frac{x}{t} < \lambda_1(\rho_*, u_*), \\ (\rho_*, u_*), & \lambda_1(\rho_*, u_*) < \frac{x}{t} < \tau, \\ (\rho_+, u_+), & \frac{x}{t} > \tau, \end{cases} \qquad (2.10)$$

Fig. 1 The (ρ, u) phase plane for the Riemann problem (1.2) and (1.3) is shown for the given left state (ρ_-, u_-) located in the region $\Omega = \{(\rho, u) | \rho > 1, u > 0\}$, in which the rarefaction curve $R(\rho_-, u_-)$ takes the line $u = u_- - \frac{u_-}{\rho_-}$ and the shock curve $S(\rho_-, u_-)$ takes the line $\rho = 1$ as their asymptotic lines, respectively
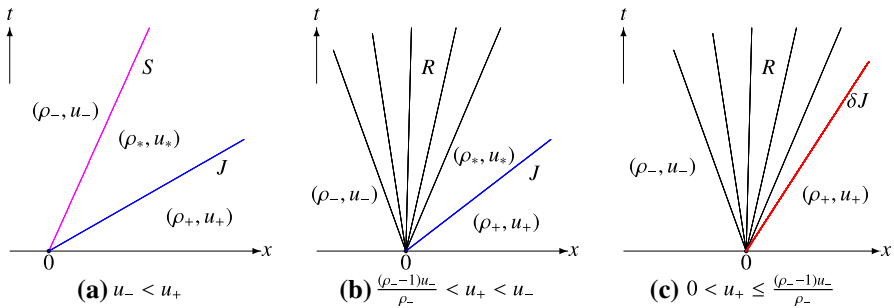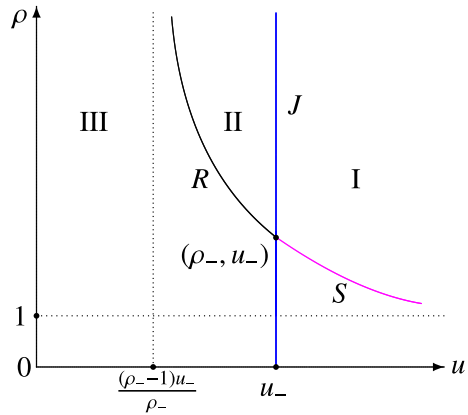
Fig. 2 The constructions of solutions to the Riemann problem (1.2) and (1.3) are displayed for all kinds of initial conditions

in which (ρ_*, u_*) is also given by (2.8) and the state (ρ, u) in the rarefaction wave fan follows from (2.3) that

$$(\rho, u) = \left(\frac{(\rho_- - 1)u_- + \sqrt{(\rho_- - 1)^2 u_-^2 - \frac{\rho_-(\rho_- - 1)u_- x}{t}}}{\sqrt{(\rho_- - 1)^2 u_-^2 - \frac{\rho_-(\rho_- - 1)u_- x}{t}}}, \frac{(\rho_- - 1)u_- + \sqrt{(\rho_- - 1)^2 u_-^2 - \frac{\rho_-(\rho_- - 1)u_- x}{t}}}{\rho_-} \right).$$

(2.11)

In the end, we need to consider the case $0 < u_+ < \frac{(\rho_- - 1)u_-}{\rho_-}$. Before solving it, we shall focus on the limit of Riemann solution (2.10) when $u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-} \right)_+$ is taken. In terms of $u_- > 0$ and $\rho_- > 1$, it follows from (2.8) that

$$\lim_{u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-} \right)_+} \rho_* = \lim_{u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-} \right)_+} \left(\frac{\rho_- u_+}{\rho_- u_+ - (\rho_- - 1)u_-} \right) = +\infty,$$

$$\lim_{u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-} \right)_+} u_* = \lim_{u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-} \right)_+} u_+ = \frac{(\rho_- - 1)u_-}{\rho_-},$$

which allows us to conclude that the singularity is formed for the state variable ρ in the limiting $u_+ \to \left(\frac{(\rho_- - 1)u_-}{\rho_-}\right)_+$ situation. It then follows from (2.7) combining with (2.8) that

$$\lim_{u_+ \to \left(\frac{(\rho_- -1)u_-}{\rho_-}\right)_+} \lambda_1(\rho_*, u_*) = \lim_{u_+ \to \left(\frac{(\rho_- -1)u_-}{\rho_-}\right)_+} \left(u_* - \frac{u_*}{\rho_* - 1}\right) = \frac{(\rho_- - 1)u_-}{\rho_-},$$

$$\lim_{u_+ \to \left(\frac{(\rho_- -1)u_-}{\rho_-}\right)_+} \tau = \lim_{u_+ \to \left(\frac{(\rho_- -1)u_-}{\rho_-}\right)_+} u_+ = \frac{(\rho_- - 1)u_-}{\rho_-}.$$

As a result, it is not difficult to see that the front of rarefaction wave R and the contact discontinuity J coincide with each other on the line $x = \left(\frac{(\rho_- - 1)u_-}{\rho_-}\right)t$ and the density between them tends to $+\infty$ in the limiting $u_+ \to \left(\frac{(\rho_- -1)u_-}{\rho_-}\right)_+$ situation. Thus, if the initial condition (1.3) satisfies the special requirement $u_+ = \frac{(\rho_- -1)u_-}{\rho_-}$, we can construct a composite wave $R\delta J$, which is made up of a rarefaction wave R and a delta contact discontinuity δJ attached on the wave front of R. More specifically, when $u_+ = \frac{(\rho_- -1)u_-}{\rho_-}$, the Riemann solution of (1.2) and (1.3) may be displayed in the form

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & \frac{x}{t} < \lambda_1(\rho_-, u_-), \\ (\rho, u), & \lambda_1(\rho_-, u_-) < \frac{x}{t} < u_+, \\ (+\infty, u_+), & \frac{x}{t} = u_+, \\ (\rho_+, u_+), & \frac{x}{t} > u_+, \end{cases} \tag{2.12}$$

in which the state (ρ, u) in the rarefaction wave fan R is also calculated by (2.11).

Motivated by the above detailed observation on the construction of Riemann solution for the special case $u_+ = \frac{(\rho_- -1)u_-}{\rho_-}$, we plan to construct possible Riemann solution of (1.2) and (1.3) when $0 < u_+ < \frac{(\rho_- -1)u_-}{\rho_-}$. It is not difficult to get that

$$\lambda_1(\rho_+, u_+) < \lambda_2(\rho_+, u_+) < \frac{(\rho_- - 1)u_-}{\rho_-},$$

which means that all the characteristic lines on the right-hand side of the contact discontinuity $x = \frac{(\rho_- - 1)u_- t}{\rho_-}$ will enter the line of this contact discontinuity. Thus, the delta contact discontinuity is formed whose mass only comes from the particles on the right-hand side. Different from delta shock wave, the left-hand side of contact discontinuity is rarefaction wave and thus the strength of delta contact discontinuity cannot be solved by the so-called generalized Rankine–Hugoniot relations of delta shock wave. But we notice that mass only comes from the particles on the right-hand side. Hence, we calculate the strength of delta contact discontinuity by using the density multiplying by the difference between the speed of the particles and the speed of delta contact discontinuity as

$$\beta(t) = \rho_+ \left(\frac{(\rho_- - 1)u_-}{\rho_-} - u_+\right)t.$$

Therefore, when $0 < u_+ < \frac{(\rho_- -1)u_-}{\rho_-}$, we can construct the possible Riemann solution of (1.2) and (1.3) by using the composite wave $R\delta J$, which is given by (see Fig. 2c)

$$
(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & \frac{x}{t} < \lambda_1(\rho_-, u_-), \\ (\rho, u), & \lambda_1(\rho_-, u_-) < \frac{x}{t} < \frac{(\rho_- -1)u_-}{\rho_-}, \\ (\beta(t)\delta(x - \frac{(\rho_- -1)u_- t}{\rho_-}), \frac{(\rho_- -1)u_-}{\rho_-}), & \frac{x}{t} = \frac{(\rho_- -1)u_-}{\rho_-}, \\ (\rho_+, u_+), & \frac{x}{t} > \frac{(\rho_- -1)u_-}{\rho_-}. \end{cases}
$$

(2.13)

It should be mentioned that the delta shock wave is an over-compressive shock wave in the sense that it is usually an isolated discontinuity to connect the left state and the right state directly for the Riemann problem. In other words, the delta shock wave is surrounded by the two constant states on both sides of it. By comparison, in the wave fan of $R\delta J$, the delta contact discontinuity δJ is attached on the wave front of R, namely δJ is surrounded by a rarefaction wave on the left-hand side and a constant state on the right-hand side. Different from the delta shock wave, it is impossible to use the generalized Rankine–Hugoniot relations to calculate the strength of δJ for the reason that the left-hand side of it is a rarefaction wave and the density ρ in the characteristic line of the rarefaction wave R tends to $+\infty$ if the characteristic line tends closer and closer to the line of δJ. In the current work, the strength $\beta(t)$ in (2.13) is obtained by virtue of the fact that all of the mass gathering together on the line of δJ only comes from the particles on the right-hand side of it. It is of great interest to notice that the solution consisting of delta shock wave surrounded by two rarefaction waves on both sides of delta shock line has been constructed in [39] recently, which is similar to the composite wave $R\delta J$ in this paper. In conclusion, the Riemann solutions of (1.2) and (1.3) are constructed fully for all the possible cases, namely $S + J$ for $u_- < u_+$, $R + J$ for $\frac{(\rho_- -1)u_-}{\rho_-} < u_+ < u_-$, as well as $R\delta J$ for $0 < u_+ < \frac{(\rho_- -1)u_-}{\rho_-}$.

3 Wave Interactions

It is remarked that system (1.2) is attributed to Temple class whose wave interactions have relatively simple structures. Due to the fact that we can only use the composite wave $R\delta J$ to construct possible Riemann solution of (1.2) and (1.3) when $0 < u_+ < \frac{(\rho_- -1)u_-}{\rho_-}$, whose uniqueness is still unsolved. Thus, we restrict ourselves to only consider the interactions of elementary waves including shock wave, rarefaction wave and contact discontinuity. For this purpose, we are dedicated to the special initial value problem for system (1.2) with the constant states in three pieces (1.4) which should satisfy $u_m > \frac{(\rho_- -1)u_-}{\rho_-}$ and $u_+ > \frac{(\rho_- -1)u_-}{\rho_-}$ simultaneously. In brief, our main purpose in this section is to construct global solutions of the perturbed Riemann problem (1.2) and (1.4) by using the method of characteristics when the composite wave $R\delta J$ is not involved. In this respect, there exist four cases according to the different combinations of waves originating from the two initial points $(0, 0)$ and $(\varepsilon, 0)$ as follows:

(1) $S + J$ and $S + J$, (2) $R + J$ and $R + J$, (3) $R + J$ and $S + J$, (4) $S + J$ and $R + J$.
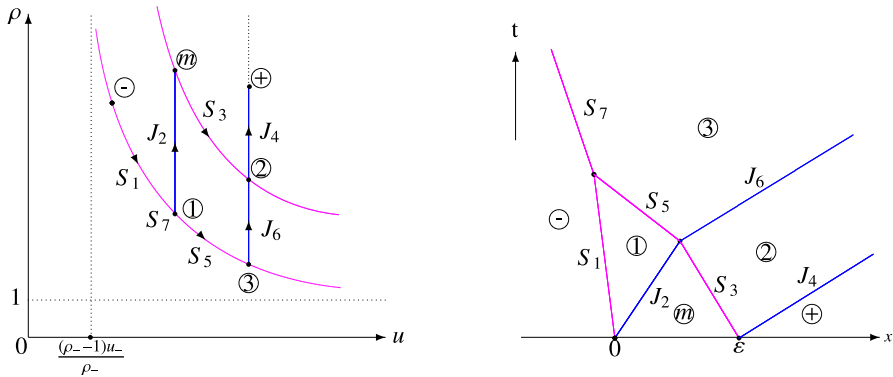
Fig. 3 The interaction between $S + J$ and $S + J$ is shown when $\frac{(\rho_- - 1)u_-}{\rho_-} < u_- < u_m < u_+$

In addition, by taking the initial condition (1.4), we can further consider whether the limits $\varepsilon \to 0$ of solutions of (1.2) and (1.4) are in accordance with the corresponding ones of (1.2) and (1.3) or not.

Case 1. $S + J$ and $S + J$

In this work, we begin by taking into account the case that there are a shock wave followed by a contact discontinuity emitting from the initial points $(0,0)$ and $(\varepsilon, 0)$, respectively. For the sake of convenience, we use S_1, J_2 and S_3, J_4 to denote them, respectively (see Fig. 3). In this case, when the time t is adequately small, the solution of (1.2) and (1.4) can be abbreviated by using the symbols as

$$(u_-, v_-) + S_1 + (u_1, v_1) + J_2 + (u_m, v_m) + S_3 + (u_2, v_2) + J_4 + (u_+, v_+),$$

where "$+$" stands for "followed by". Obviously, the occurrence of this case depends on the conditions $\frac{(\rho_- - 1)u_-}{\rho_-} < u_- < u_m < u_+$. By virtue of (2.8), the intermediate states (ρ_1, u_1) and (ρ_2, u_2) can be given, respectively, by

$$(\rho_1, u_1) = \left(\frac{\rho_- u_m}{\rho_- u_m - (\rho_- - 1)u_-}, u_m \right), \quad (\rho_2, u_2) = \left(\frac{\rho_m u_+}{\rho_m u_+ - (\rho_m - 1)u_m}, u_+ \right). \quad (3.1)$$

Proposition 3.1 *The contact discontinuity J_2 collides with the shock wave S_3 in finite time. Subsequently, the interaction between J_2 and S_3 gives rise to a new shock wave S_5 and a new contact discontinuity J_6, respectively. Finally, the two shock waves S_1 and S_5 coalesce into a new shock wave S_7 as well as the two contact discontinuities J_4 and J_6 are parallel with each other.*

Proof It is evident that the wave speeds of J_2 and S_3 are given, respectively, by

$$\tau_2 = u_m, \quad \sigma_3 = \frac{(\rho_m - 1)u_m - u_+}{\rho_m - 1} = u_m - \frac{u_+}{\rho_m - 1}. \quad (3.2)$$

Taking into account $\rho > 1$ and $u > 0$, we have

$$\tau_2 - \sigma_3 = \frac{u_+}{\rho_m - 1} > 0,$$

which implies that $\tau_2 > \sigma_3$. Therefore, J_2 collides with S_3 in finite time. The point of intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 = \tau_2 t_1 = u_m t_1, \\ x_1 - \varepsilon = \sigma_3 t_1 = \left(u_m - \dfrac{u_+}{\rho_m - 1}\right)t_1, \end{cases}$$

which yields

$$(x_1, t_1) = \left(\frac{(\rho_m - 1)u_m\varepsilon}{u_+}, \frac{(\rho_m - 1)\varepsilon}{u_+}\right). \tag{3.3}$$

Obviously, J_2 and S_3 intersect at a finite time when a new Riemann problem is formed. At the time $t = t_1$, we again have a Riemann problem for system (1.2) subject to the Riemann-type initial data (ρ_1, u_1) and (ρ_2, u_2). On account of the relations $\frac{(\rho_- - 1)u_-}{\rho_-} < u_- < u_1 = u_m < u_+ = u_2$, the Riemann solution at the point (x_1, t_1) is still a shock wave followed by a contact discontinuity, which are denoted with S_5 and J_6, respectively. Moreover, the intermediate state (ρ_3, u_3) between S_5 and J_6 can also be obtained by

$$(\rho_3, u_3) = \left(\frac{\rho_1 u_2}{\rho_1 u_2 - (\rho_1 - 1)u_1}, u_2\right) = \left(\frac{\rho_- u_+}{\rho_- u_+ - (\rho_- - 1)u_-}, u_+\right). \tag{3.4}$$

By using (3.1) and (3.4), it is not difficult to achieve the wave speeds of S_5 and J_6 as

$$\sigma_5 = \frac{\rho_1 u_1 - \rho_3 u_3}{\rho_1 - \rho_3} = u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-}, \qquad \tau_6 = u_+. \tag{3.5}$$

On the one hand, one has

$$\sigma_5 - \sigma_3 = u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-} - \left(u_m - \frac{u_+}{\rho_m - 1}\right) = \frac{\rho_m(\rho_- - 1)u_- - \rho_-(\rho_m - 1)u_m}{(\rho_m - 1)(\rho_- - 1)u_-}u_+$$

$$= \frac{\rho_m\rho_-\left(u_- - \frac{u_-}{\rho_-} - \left(u_m - \frac{u_m}{\rho_m}\right)\right)}{(\rho_m - 1)(\rho_- - 1)u_-} = \frac{\rho_m\rho_-\left(u_1 - \frac{u_1}{\rho_1} - \left(u_m - \frac{u_m}{\rho_m}\right)\right)}{(\rho_m - 1)(\rho_- - 1)u_-}$$

$$= \frac{\rho_m\rho_-\left(\frac{1}{\rho_m} - \frac{1}{\rho_1}\right)u_m}{(\rho_m - 1)(\rho_- - 1)u_-} < 0,$$

which permits us to see that the shock wave reduces the speed forwards (or add up the speed backwards) when across the contact discontinuity. On the other hand, it is evident that $0 < \tau_2 = u_m < u_+ = \tau_6$, which means that the contact discontinuity adds up the speed forwards when across the shock wave. Eventually, J_4 is parallel to J_6 due to the relation $\tau_4 = \tau_6 = u_+$.

In the following, we consider the interaction between S_1 and S_5. The wave speed of S_1 is given by $\sigma_1 = u_- - \dfrac{u_m}{\rho_- - 1}$. Comparing with σ_5 in (3.5), we have

$$
\begin{aligned}
\sigma_5 - \sigma_1 &= u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-} - \left(u_- - \frac{u_m}{\rho_- - 1}\right) = (u_m + u_+ - u_-) + \frac{u_m(u_- - \rho_- u_+)}{(\rho_- - 1)u_-} \\
&= \left(1 + \frac{u_- - \rho_- u_+}{(\rho_- - 1)u_-}\right)u_m + (u_+ - u_-) = \frac{\rho_- u_m(u_- - u_+)}{(\rho_- - 1)u_-} + (u_+ - u_-) \\
&= (u_- - u_+)\left(\frac{\rho_- u_m}{(\rho_- - 1)u_-} - 1\right) < 0.
\end{aligned}
$$

In other words, the shock wave S_1 catches up with the shock wave S_5 in finite time. The point of intersection (x_2, t_2) is determined by

$$
\begin{cases}
x_2 = \sigma_1 t_2 = \left(u_- - \dfrac{u_m}{\rho_- - 1}\right)t_2, \\
x_2 - x_1 = \sigma_5(t_2 - t_1) = \left(u_m + u_+ - \dfrac{\rho_- u_m u_+}{(\rho_- - 1)u_-}\right)(t_2 - t_1),
\end{cases}
$$

where (x_1, t_1) is given by (3.3). Then, we have

$$
(x_2, t_2) = \left(\frac{(\rho_m - 1)\big((\rho_- - 1)u_- - u_m\big)\varepsilon}{(\rho_- - 1)(u_+ - u_-)}, \ \frac{(\rho_m - 1)\varepsilon}{u_+ - u_-}\right). \tag{3.6}
$$

It follows from (3.4) that

$$
u_3 - \frac{u_3}{\rho_3} = u_+ - \frac{\rho_- u_+ - (\rho_- - 1)u_-}{\rho_-} = u_- - \frac{u_-}{\rho_-}.
$$

Hence, it is concluded that the two states (ρ_-, u_-) and (ρ_3, u_3) can be connected by a single shock wave S_7, whose wave speed is $\sigma_7 = u_- - \dfrac{u_+}{\rho_- - 1}$. That is to say, the two shock waves S_1 and S_5 coalesce into a new shock wave S_7. Comparing with σ_1 and σ_5, we have

$$
\sigma_1 - \sigma_7 = u_- - \frac{u_m}{\rho_- - 1} - \left(u_- - \frac{u_+}{\rho_- - 1}\right) = \frac{u_+ - u_m}{\rho_- - 1} > 0,
$$

and

$$
\begin{aligned}
\sigma_5 - \sigma_7 &= u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-} - \left(u_- - \frac{u_+}{\rho_- - 1}\right) = (u_m + u_+ - u_-) + \frac{u_+(u_- - \rho_- u_m)}{(\rho_- - 1)u_-} \\
&= \left(1 + \frac{u_- - \rho_- u_m}{(\rho_- - 1)u_-}\right)u_+ + (u_m - u_-) = \frac{\rho_- u_+(u_- - u_m)}{(\rho_- - 1)u_-} + (u_m - u_-) \\
&= \frac{(u_- - u_m)(\rho_-(u_+ - u_-) + u_-)}{(\rho_- - 1)u_-} < 0,
\end{aligned}
$$

in which the inequalities $u_- < u_m < u_+$ and $\rho_- > 1$ have been used. As a consequence, $\sigma_5 < \sigma_7 < \sigma_1$ can be established, namely the wave speed of S_7 is between
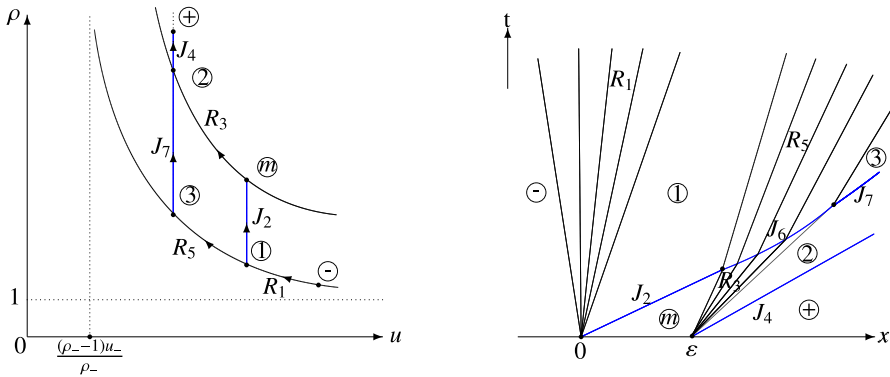
Fig. 4 The interaction between $R + J$ and $R + J$ is shown when $\frac{(\rho_- - 1)u_-}{\rho_-} < u_+ < u_m < u_-$

those of S_1 and S_5. With the help of the above calculations and discussions, we can draw Fig. 3 to illustrate this case in detail. The proof of this proposition is completed.

□

Case 2. $R + J$ and $R + J$

Secondly, we focus on the situation that there are a rarefaction wave followed by a contact discontinuity emitting from the initial points $(0,0)$ and $(\varepsilon, 0)$, respectively. For the sake of convenience, we use R_1, J_2 and R_3, J_4 to denote them, respectively (see Fig. 4). For t adequately small, the solution of (1.2) and (1.4) is abbreviated as

$$(u_-, v_-) + R_1 + (u_1, v_1) + J_2 + (u_m, v_m) + R_3 + (u_2, v_2) + J_4 + (u_+, v_+).$$

The occurrence of this case depends on the conditions $\frac{(\rho_- - 1)u_-}{\rho_-} < u_+ < u_m < u_-$. Remark that the intermediate states (ρ_1, u_1), (ρ_2, u_2) and (ρ_3, u_3) are the same as those in Case 1.

Proposition 3.2 *The contact discontinuity J_2 collides with the wave back of the rarefaction wave R_3 in finite time. Subsequently, the interaction between J_2 and R_3 gives rise to a new rarefaction wave R_5 and a new contact discontinuity J_7, respectively. In the end, the wave front of R_1 is parallel to the wave back of R_5 as well as J_4 is parallel to J_7, respectively.*

Proof It suffices to show that the wave speeds of J_2 and the wave back of R_3 are given, respectively, by

$$\tau_2 = u_m, \qquad \xi_3(\rho_m, u_m) = u_m - \frac{u_m}{\rho_m - 1}. \tag{3.7}$$

We then get

$$\tau_2 - \xi_3(\rho_m, u_m) = \frac{u_m}{\rho_m - 1} > 0,$$

which implies that J_2 collides with the wave back of R_3 in finite time. The point of intersection (x_1, t_1) is determined by

$$
\begin{cases}
x_1 = \tau_2 t_1 = u_m t_1, \\
x_1 - \varepsilon = \xi_3(\rho_m, u_m) t_1 = \left(u_m - \dfrac{u_m}{\rho_m - 1} \right) t_1,
\end{cases}
$$

which yields

$$
(x_1, t_1) = \left((\rho_m - 1)\varepsilon, \ \frac{(\rho_m - 1)\varepsilon}{u_m} \right). \tag{3.8}
$$

We find that J_2 and the wave back of R_3 intersect at a finite time. Subsequently, J_2 begins to penetrate the rarefaction wave R_3 after t_1, which is recorded as J_6 in the process of penetration. The expression of J_6 during the penetrating process is computed by

$$
\frac{dx}{dt} = u, \qquad \frac{x - \varepsilon}{t} = u - \frac{u}{\rho - 1}, \qquad u - \frac{u}{\rho} = u_m - \frac{u_m}{\rho_m}, \qquad x(t_1) = x_1. \tag{3.9}
$$

Differentiating the second equation in (3.9) with respect to t, we have

$$
u = \frac{dx}{dt} = \frac{(\rho - 2)u}{\rho - 1} + \frac{(\rho - 2)t}{\rho - 1} \frac{du}{dt} + \frac{ut}{(\rho - 1)^2} \frac{d\rho}{dt},
$$

which is simplified into

$$
u = (\rho - 2)t \frac{du}{dt} + \frac{ut}{\rho - 1} \frac{d\rho}{dt}. \tag{3.10}
$$

Differentiating the third equation in (3.9) with respect to t yields

$$
\frac{du}{dt} + \frac{u}{\rho(\rho - 1)} \frac{d\rho}{dt} = 0. \tag{3.11}
$$

Which, together with (3.10), gives

$$
\frac{du}{dt} + \frac{1}{\rho t} \left(u - (\rho - 2)t \frac{du}{dt} \right) = 0,
$$

such that one has

$$
\frac{du}{dt} = -\frac{u}{2t}.
$$

Hence, we can get

$$
\frac{dx}{dt} = u = u_m \sqrt{\frac{t_1}{t}}.
$$

As a result, the curve of J_6 is expressed as

$$x = 2u_m\sqrt{t_1 t} + x_1 - 2u_m t_1. \tag{3.12}$$

In the end, by combining (3.12) together with the line of wave back of R_3 given by

$$x_2 - \varepsilon = \left(u_2 - \frac{u_2}{\rho_2 - 1}\right)t_2.$$

The end point of J_6 passing through R_3 can be obtained as

$$(x_2, t_2) = \left(\frac{(\rho_m - 1)(\rho_m u_+ + 2u_m)\varepsilon}{2(\rho_m - 1)u_m - \rho_m u_+}, \frac{\rho_m^2(\rho_m - 1)u_m\varepsilon}{\left(2(\rho_m - 1)u_m - \rho_m u_+\right)^2}\right). \tag{3.13}$$

There is no doubt that the two states (ρ_2, u_2) and (ρ_3, u_3) can be connected by the contact discontinuity J_7 for the reason that $u_3 = u_2 = u_+$. In the end, it is evident that the two contact discontinuities J_4 and J_7 are parallel with each other due to the fact that the wave speeds of J_7 and J_4 are the same as $\tau_7 = \tau_4 = u_+$. The proof is finished. □

Case 3. $R + J$ and $S + J$

Thirdly, we are devoted to the case that there are a rarefaction wave followed by a contact discontinuity emitting from the origin (0,0) and a shock wave followed by a contact discontinuity emitting from the initial point $(\varepsilon, 0)$, respectively. As before, we use R_1, J_2 and S_3, J_4 to stand for them, respectively. For t adequately small, the solution of (1.2) and (1.4) can be symbolized as (see Figs. 5 and 6)

$$(u_-, v_-) + R_1 + (u_1, v_1) + J_2 + (u_m, v_m) + S_3 + (u_2, v_2) + J_4 + (u_+, v_+).$$

The occurrence of this case depends on the conditions $\frac{(\rho_- - 1)u_-}{\rho_-} < u_m < u_-$ and $\frac{(\rho_- - 1)u_-}{\rho_-} < u_m < u_+$. Similarly, the intermediate states (ρ_1, u_1), (ρ_2, u_2) and (ρ_3, u_3) and the point of intersection (x_1, t_1) are also calculated in the same formulae as before and then a new Riemann problem is formed at the time t_1.

Proposition 3.3 *The wave front of the rarefaction wave R_1 catches up with the shock wave S_5 in finite time. More specifically, S_5 is able to penetrate the whole R_1 thoroughly when $u_- < u_+$ or not when $u_- > u_+$. Moreover, the two contact discontinuities J_4 and J_6 are parallel with each other.*

Proof The wave speeds of the wave front of R_1 and S_5 are as follows:

$$\xi_1(\rho_1, u_1) = \frac{(\rho_1 - 2)u_1}{\rho_1 - 1} = 2u_m - \frac{\rho_- u_m^2}{(\rho_- - 1)u_-}, \qquad \sigma_5 = u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-}. \tag{3.14}$$
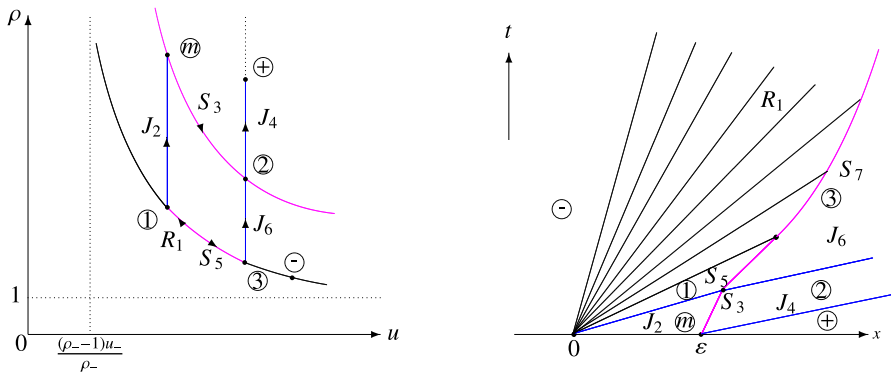
Fig. 5 The interaction between $R + J$ and $S + J$ is shown when $\frac{(\rho_- - 1)u_-}{\rho_-} < u_m < u_+ < u_-$, in which S_5 cannot penetrate the whole R_1 in finite time and ultimately has the line $x = \left(\frac{\left(2(\rho_- - 1)u_- - \rho_- u_+\right)u_+}{(\rho_- - 1)u_-} \right) t$ as its asymptote
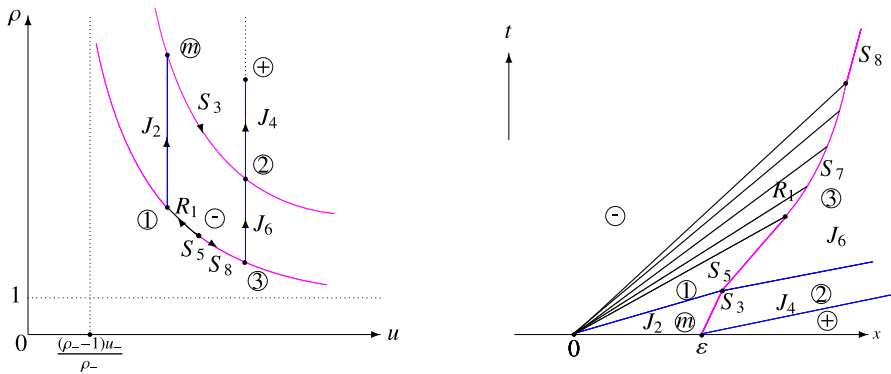
Fig. 6 The interaction between $R + J$ and $S + J$ is shown when $\frac{(\rho_- - 1)u_-}{\rho_-} < u_m < u_- < u_+$, in which S_5 is able to penetrate the whole R_1 completely

Then, we have

$$\xi_1(\rho_1, u_1) - \sigma_5 = 2u_m - \frac{\rho_- u_m^2}{(\rho_- - 1)u_-} - (u_m + u_+) + \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-}$$
$$= (u_m - u_+)\left(1 - \frac{\rho_- u_m}{(\rho_- - 1)u_-}\right) > 0,$$

which permits us to get $\xi_1(\rho_1, u_1) > \sigma_5$. Hence, the wave front of R_1 catches up with S_5 in finite time and then the point of intersection (x_2, t_2) is determined by

$$\begin{cases} x_2 = \xi_1(\rho_1, u_1)t_2 = \left(2u_m - \frac{\rho_- u_m^2}{(\rho_- - 1)u_-}\right)t_2, \\ x_2 - x_1 = \sigma_5(t_2 - t_1) = \left(u_m + u_+ - \frac{\rho_- u_m u_+}{(\rho_- - 1)u_-}\right)(t_2 - t_1), \end{cases}$$

which yields

$$(x_2, t_2) = \left(\frac{(\rho_m - 1)\left(2(\rho_- - 1)u_- - \rho_- u_m\right)u_m \varepsilon}{(\rho_- - 1)u_-(u_+ - u_m)}, \frac{(\rho_m - 1)\varepsilon}{u_+ - u_m} \right). \quad (3.15)$$

Analogously, S_5 begins to penetrate R_1 after t_2, which is recorded as S_7 in the process of penetration. The expression of S_7 during the penetrating process is able to be computed by

$$\frac{dx}{dt} = u - \frac{u_+}{\rho - 1}, \qquad \frac{x}{t} = u - \frac{u}{\rho - 1}, \qquad u - \frac{u}{\rho} = u_- - \frac{u_-}{\rho_-}, \qquad x(t_2) = x_2. \quad (3.16)$$

Differentiating the second equation in (3.16) with respect to t, we have

$$u - \frac{u_+}{\rho - 1} = \frac{dx}{dt} = \frac{(\rho - 2)u}{\rho - 1} + \frac{(\rho - 2)t}{\rho - 1}\frac{du}{dt} + \frac{ut}{(\rho - 1)^2}\frac{d\rho}{dt},$$

which is simplified into

$$u - u_+ = (\rho - 2)t\frac{du}{dt} + \frac{ut}{(\rho - 1)}\frac{d\rho}{dt}. \quad (3.17)$$

Differentiating the third equation in (3.16) with respect to t yields

$$\frac{du}{dt} + \frac{u}{\rho(\rho - 1)}\frac{d\rho}{dt} = 0. \quad (3.18)$$

Which, together with (3.17), gives

$$\frac{du}{dt} + \frac{1}{\rho t}\left(u - u_+ - (\rho - 2)t\frac{du}{dt}\right) = 0,$$

such that one has

$$\frac{du}{dt} = -\frac{u - u_+}{2t}.$$

Hence, we can get

$$\frac{dx}{dt} = u - \frac{u_+}{\rho - 1} = \frac{(\rho_- - 1)u_- - \rho_- u_+}{(\rho_- - 1)u_-}u + u_+ = \frac{(\rho_- - 1)u_- - \rho_- u_+}{(\rho_- - 1)u_-}\left((u_m - u_+)\sqrt{\frac{t_2}{t}} + u_+\right) + u_+.$$

As a result, the curve of S_7 is expressed as

$$x = \frac{\left(2(\rho_- - 1)u_- - \rho_- u_+\right)u_+ t}{(\rho_- - 1)u_-}$$
$$- \frac{2\left((\rho_- - 1)u_- - \rho_- u_+\right)\sqrt{(\rho_m - 1)(u_+ - u_m)\varepsilon t}}{(\rho_- - 1)u_-} + \frac{\rho_-(\rho_m - 1)(u_m - u_+)\varepsilon}{(\rho_- - 1)u_-}. \quad (3.19)$$

In the end, by combining (3.19) together with the line of wave back of R_3 given by

$$x_3 = \left(u_- - \frac{u_-}{\rho_- - 1} \right) t_3.$$

There are two situations in the process of penetration:

(1) If $u_- > u_+$, then S_7 cannot penetrate R_1 completely and ultimately has the line $x = \left(\frac{(2(\rho_- - 1)u_- - \rho_- u_+)u_+}{(\rho_- - 1)u_-} \right) t$ as its asymptote.

(2) If $u_- < u_+$, then S_7 has the ability to penetrate the whole rarefaction wave R_1 completely at a point (x_3, t_3) and then be denoted with a new shock wave S_8 after penetration, whose wave speed is given by $\sigma_8 = u_- - \frac{u_+}{\rho_- - 1}$. It suffices to get

$$(x_3, t_3) = \left(\frac{(\rho_- - 2)(\rho_m - 1)(u_+ - u_m)u_- \varepsilon}{(\rho_- - 1)(u_+ - u_-)^2}, \frac{(\rho_m - 1)(u_+ - u_m)\varepsilon}{(u_+ - u_-)^2} \right). \quad (3.20)$$

As before, the two contact discontinuities J_4 and J_6 are also parallel.   □

Case 4. $S + J$ and $R + J$

In the end, we shall focus our attention on the situation that there are a shock wave followed by a contact discontinuity emitting from the initial point $(0,0)$ and a rarefaction wave followed by a contact discontinuity emitting from the initial point $(\varepsilon, 0)$, respectively. Similarly, we use S_1, J_2 and R_3, J_4 to denote them, respectively. For t adequately small, the solution of (1.2) and (1.4) can be symbolized as

$$(u_-, v_-) + S_1 + (u_1, v_1) + J_2 + (u_m, v_m) + R_3 + (u_2, v_2) + J_4 + (u_+, v_+).$$

The occurrence of this case depends on the conditions $\frac{(\rho_- - 1)u_-}{\rho_-} < u_- < u_m$ and $\frac{(\rho_- - 1)u_-}{\rho_-} < u_+ < u_m$. Precisely, all the intermediate states are calculated in the same way as before. It is noted that the process of J_2 penetrating R_3 is the same as that in Case 2. Finally, we need to consider how does the shock wave S_1 to penetrate the non-centered rarefaction wave R_5. The analytic calculation is impossible due to the fact that the rarefaction wave R_5 is non-centered. But we notice that both S_1 and R_5 belong to the wave of λ_1-family. Thus, we can judge that S_1 is able to cancel R_5 when $u_- < u_+$ or cannot penetrate R_5 completely when $u_- > u_+$. In fact, this situation is similar to that in Case 3. In other words, the wave interaction in Case 4 is just the combination of those in Cases 2 and 3. Hence, we only describe the process and omit the details here.

4 Conclusion

The wave interaction problems for system (1.2) have been well investigated in fully explicit forms by using the method of characteristics, including shock wave, rarefaction wave and contact discontinuity. Based on the above results, we can construct the

global solutions to the perturbed Riemann problem (1.2) and (1.4) under the assumption $min(u_m, u_+) > \frac{(\rho_- - 1)u_-}{\rho_-}$ such that the composite wave $R\delta J$ is not involved. It is evident to see that the limits $\varepsilon \to 0$ of solutions of (1.2) and (1.4) tend to the corresponding ones of (1.2) and (1.3) by analyzing the above four cases in detail. As a consequence, it is verified strictly that the Riemann solutions of (1.2) and (1.3) are stable with respect to the specific small perturbations (1.4) of Riemann initial data under the assumption $min(u_m, u_+) > \frac{(\rho_- - 1)u_-}{\rho_-}$.

It should be stressed that the composite wave $R\delta J$ is involved in the Riemann solution of (1.2) and (1.3) when the initial condition (1.3) satisfies $0 < u_+ < \frac{(\rho_- - 1)u_-}{\rho_-}$. This is a new kind of singular hyperbolic wave which has not been paid enough attention in current literatures. There are still some interesting, but difficult problems needed to be considered. The first concern is that our constructed composite wave $R\delta J$ is the uniqueness Riemann solution of (1.2) and (1.3) or not when $0 < u_+ < \frac{(\rho_- - 1)u_-}{\rho_-}$. The second concern is that the composite wave $R\delta J$ is stable or not under some small perturbations of Riemann initial data (1.3) such as the perturbation (1.4). We plan to consider these interesting but difficult problems in our future work.

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