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Split Common Fixed Point Problem for Quasi-Pseudo-Contractive Mapping in Hilbert Spaces

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Abstract

In this paper, the split common fixed point problem for quasi-pseudo-contractive mappings is studied in Hilbert spaces. By using the hybrid projection method, a new algorithm and some strong convergence theorems are established under suitable assumptions. Our results not only improve and generalize some recent results but also give an affirmative answer to an open question.

Keywords Split common fixed point problem · Quasi-pseudo-contractive mapping · Demicontractive operator · Quasi-nonexpansive mapping · Directed operator · Firmly nonexpansive mapping · Strong convergence

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces and $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators. Denote the fixed point sets of S and T by $Fix(S)$ and $Fix(T)$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. The "so-called" *split common fixed point problem* is to find a point $x^* \in H_1$ such that

$$x^* \in Fix(S) \quad \text{and} \quad Ax^* \in Fix(T). \tag{1.1}$$

As well known, the split common fixed point problem (1.1) is a generalization of the split feasibility problem arising from signal processing and image restoration ([1–8]). Note that solving (1.1) can be translated to solve the following fixed point equation:

$$x^* = S(x^* - \tau A^*((I - T)Ax^*), \ \tau > 0. \tag{1.2}$$

In order to solve Eq. (1.2), Censor and Segal [9] proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem ([8,10–15]). In particular, in 2017, Wang [16] introduced the following new iterative algorithm for the split common fixed point problem for firmly nonexpansive mappings.

Algorithm 1.1 Choose an arbitrary initial guess $x_0 \in H_1$.

Step 1 Given x_n, compute the next iteration via the formula:

$$x_{n+1} = x_n - \rho_n[x_n - Sx_n + A^*(I - T)Ax_n], \quad n \geq 0. \tag{1.3}$$

Step 2 If the following equality holds

$$||x_{n+1} - Sx_{n+1} + A^*(I - T)Ax_{n+1}|| = 0, \tag{1.4}$$

then stop; otherwise, go to step 1.

Subsequently, he proved the following result.

Theorem 1.2 *(Wang [16]). Assume the following conditions are satisfied:*

(1) *A is a bounded linear operator;*
(2) *the solution set of problem (1.1), denoted by Ω, is nonempty;*
(3) *both S and T are firmly nonexpansive operators.*

If the sequence $\{\rho_n\}$ satisfies the conditions: $\sum_{n=0}^{\infty} \rho_n = \infty$ and $\sum_{n=0}^{\infty} \rho_n^2 < \infty$, then the sequence $\{x_n\}$ generated by Algorithm 1.1 converges weakly to a solution z of problem (1.1).

At the same time, Wang [16] gave the following remark.

Remark 1.3 It is easy to see that, in Algorithm 1.1, the selection of the step size $\{\rho_n\}$ does not depend on the operator norm $||A||$. It seems that the assumption (3) cannot weaken to directed operators.

Inspired by the works of [6–8,16–18], the main purpose of this paper is to introduce and analyze a new iterative method for solving the split common fixed point problem in Hilbert spaces. Using this method, we remove the assumptions imposed on the operator norm $||A||$. And the sequence generated by the algorithm converges strongly to a solution of problem (1.1). Our results not only give an affirmative answer to Remark 1.3 in [16] but also extend and improve the results in Yao et al. [6], Moudafi [7,8] and Wang [16,17] from the firmly nonexpansive operators, directed operators and demicontractive operators to a more general quasi-pseudo-contractive operators.

2 Preliminaries and Lemmas

Let H be a real Hilbert space, C be a nonempty closed and convex subset of H and $T : C \to C$ be a nonlinear mapping.

Definition 2.1 T is said to be

(i) Nonexpansive if $||Tx - Ty|| \leq ||x - y|| \quad \forall x, y \in C$;
(ii) Quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$||Tx - x^*|| \leq ||x - x^*|| \; \forall x \in C \quad \text{and } x^* \in Fix(T);$$

(iii) Firmly nonexpansive if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2 \quad \forall x, y \in C,$$

or equivalently

$$||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle \quad \forall x, y \in C.$$

(iv) Firmly quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 - ||(I - T)x||^2 \quad \forall x \in C \quad \text{and } x^* \in Fix(T).$$

(v) Directed if $Fix(T) \neq \emptyset$ and $\langle Tx - x^*, \; Tx - x\rangle \leq 0 \; \forall x \in C$ and $x^* \in Fix(T)$;
(vi) k-Demicontractive if $Fix(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + k||Tx - x||^2 \; \forall x \in C \quad \text{and } x^* \in Fix(T),$$

Remark 2.2 (1) It is easy to see that $T : C \to C$ is directed if and only if

$$||Tx - x^*||^2 \leq ||x - x^*||^2 - ||Tx - x||^2 \; \forall x \in C \quad \text{and } x^* \in F(T), \quad (2.1)$$

i.e., each firmly quasi-nonexpansive mapping is a directed mapping.

(2) $T : C \to C$ is a quasi-nonexpansive mapping if and only if

$$\langle x - Tx, x - x^* \rangle \geq \frac{1}{2}||x - Tx||^2, \ \forall x \in C, \ x^* \in Fix(T). \tag{2.2}$$

(3) $T : C \to C$ is a k-demicontractive mapping if and only if

$$\langle x - Tx, x - x^* \rangle \geq \frac{1-k}{2}||x - Tx||^2, \ \forall x \in C, \ x^* \in Fix(T). \tag{2.3}$$

Definition 2.3 An operator $T : C \to C$ is said to be *pseudo-contractive* if

$$\langle Tx - Ty, \ x - y \rangle \leq ||x - y||^2 \quad \forall x, y \in C.$$

The interest of pseudo-contractive operators lies in their connection with monotone mappings; namely, T is a pseudo-contraction if and only if $I - T$ is a monotone mapping. It is well known that T is pseudo-contractive if and only if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2 \quad \forall x; y \in C.$$

Definition 2.4 An operator $T : C \to C$ is said to be *quasi-pseudo-contractive* if $Fix(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||Tx - x||^2 \quad \forall x \in C \text{ and } x^* \in F(T) : \tag{2.4}$$

Remark 2.5 From the definitions above, we note that the class of quasi-pseudo-contractive mappings is more general and fundamental which includes many kinds of nonlinear mappings such as demicontractive mappings, directed mappings, quasi-nonexpansive mappings and quasi-firmly nonexpansive mappings as its special cases.

Example of quasi-pseudo-contractive mappings Let H be the closed interval $[0, 1]$ with the absolute value as norm. Let $T : H \to H$ be the mapping defined by:

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is clear that $Fix(T) = \{\frac{1}{2}\}$. Hence, for $x \in [0, \frac{1}{2}]$ we have

$$|Tx - \frac{1}{2}|^2 = 0 \leq |x - \frac{1}{2}|^2 + |x - Tx|^2;$$

Also, for $x \in (\frac{1}{2}, 1]$ we have

$$|Tx - \frac{1}{2}|^2 = |\frac{1}{2}|^2 \leq |x - \frac{1}{2}|^2 + |Tx - x|^2;$$

These show that for $x \in [0, 1]$ we have

$$|Tx - \frac{1}{2}|^2 \leq |x - \frac{1}{2}|^2 + |x - Tx|^2,$$

i.e., T is a quasi-pseudo-contractive mapping.

In what follows, we adopt the following notations: $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x; $x_n \to x$ means that $\{x_n\}$ converges strongly to x.

$\omega_w(x_n) := \{x : \exists \{x_{n_j}\} \subset \{x_n\} \text{ such that } x_{n_j} \rightharpoonup x\}$ is the set of weak cluster points of the sequence $\{x_n\}$.

Definition 2.6 Let C be a nonempty, closed and convex subset of a real Hilbert space H. It is well known that for each $x \in H$, there is unique point $P_C(x) \in C$ such that

$$||x - P_C(x)|| = \inf_{u \in C} ||x - u||. \tag{2.5}$$

The mapping $P_C : H \to C$ defined by (2.5) is called the metric projection of H onto C. Moreover, the following conclusions hold (see, for example, [19]):

$$\langle x - P_C(x), \; y - P_C(x) \rangle \leq 0 \; \forall x \in H, \quad y \in C; \tag{2.6}$$

$$||x - P_C(x)||^2 + ||y - P_C(x)||^2 \leq ||x - y||^2, \quad \forall x \in H \text{ and } y \in C. \tag{2.7}$$

Definition 2.7 A mapping $T : C \to C$ is said to be *demiclosed at 0* if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and $||x_n - T(x_n)|| \to 0$, then $T(x) = x$.

Lemma 2.8 ([20]) *Let H be a real Hilbert space and $\{x_n\}$ be a sequence in H. Then, the following statements hold:*

(i) *If $x_n \rightharpoonup x^*$ and $||x_n - x^*|| \to 0$ as $n \to \infty$, then $x_n \to x^*$ as $n \to \infty$;*
(ii) *If $x_n \rightharpoonup x^*$ as $n \to \infty$, then $||x^*|| \leq \liminf_{n \to \infty} ||x_n||$.*

Lemma 2.9 [21] *Let $\{a_n\}$ and $\{b_n\}$ be positive real sequences such that $\sum_{n=0}^{\infty} b_n < \infty$. If either $a_{n+1} \leq (1 + b_n)a_n$ or $a_{n+1} \leq a_n + b_n$, then the limit $\lim_{n \to \infty} a_n$ exists.*

Lemma 2.10 ([5]) *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote by*

$$K := (1 - \xi)I + \xi T((1 - \eta)I + \eta T), \tag{2.8}$$

If $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions hold:

(1) *$Fix(T) = Fix(T((1 - \eta)I + \eta T)) = Fix(K)$;*
(2) *If T is demiclosed at 0, then K is also demiclosed at 0;*
(3) *K is a L^2-Lipschitzian mapping;*
(4) *In addition, if $T : H \to H$ is quasi-pseudo-contractive, then the mapping K is quasi-nonexpansive, that is,*

$$||Kx - u^*|| \leq ||x - u^*|| \; \forall x \in H \quad \text{and } u^* \in Fix(T) = Fix(K). \tag{2.9}$$

3 Main Results

Throughout this section, we assume that the following conditions are satisfied:

(A1) H_1 and H_2 are two real Hilbert spaces;
(A2) $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two L-Lipschitz continuous and quasi-pseudo-contractive mappings with $L > 1$;
(A3) Let $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$. Define operators $U : H_1 \to H_1$ and $V : H_2 \to H_2$ by

$$\begin{cases} U := (1-\xi)I + \xi S((1-\eta)I + \eta S); \\ V := (1-\xi)I + \xi T((1-\eta)I + \eta T); \end{cases}$$

(A4) $A : H_1 \to H_2$ is a bounded linear operator, and A^* is its adjoint operator.
(A5) Denote by Ω the solution set of problem (1.1):

$$\Omega := \{z : z \in Fix(S) \quad \text{and} \quad Az \in Fix(T)\}.$$

Therefore, Ω is a closed and convex subset of H_1.

Algorithm 3.1 .Choose an arbitrary initial guess $x_0 \in H_1$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = Ux_n, \\ z_n = V(Ay_n), \\ C_n = \{z \in H_1 : ||y_n - z|| \le ||x_n - z||\} \bigcap \{z \in H_1 : ||z_n - Az|| \le ||Ay_n - Az||\}, \\ E_n = \{z \in H_1 : \langle z - x_n,\ x_0 - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap E_n}(x_0),\ n \ge 0 \end{cases} \tag{3.1}$$

We are now in a position to give the main results of the paper.

Theorem 3.2 *Let $H_1, H_2, S, T, U, V, A, \Omega$ be the same as above. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. If the following conditions are satisfied:*

(i) S, T are demiclosed at zero;
(ii) the solution set Ω of problem (1.1) is not empty,

then $\{x_n\}$ converges strongly to a solution x^* of problem (1.1).

Proof (I) First, we point out that by the assumptions (A1)–(A4) and conditions (i)–(ii), the following conclusions can be obtained from Lemma 2.10:

(Conclusion 1) U and V are two quasi-nonexpansive mappings and $Fix(U) = Fix(S)$; $Fix(V) = Fix(T)$. Therefore, the solution set Ω of problem (1.1) is:

$$\begin{aligned} \Omega &= \{z : z \in Fix(S) \quad \text{and } Az \in Fix(T)\} \\ &= \{z : z \in Fix(U) \text{ and } Az \in Fix(V)\} \ne \emptyset; \end{aligned}$$

(Conclusion 2) U and V are demiclosed at zero; (Conclusion 3) U and V are L^2-Lipschitz continuous mappings.

(II) Now, we prove that the sequence $\{x_n\}$ generated by Algorithm 3.1 is well defined.

First, we prove that for each $n \geq 0$, C_n and E_n are closed and convex subsets of H_1. In fact, the sets $C_n,\ D_n$ and E_n can be rewritten in the following forms:

$$\begin{cases} C_n = \{z \in H_1 : \langle x_n - y_n,\ z\rangle \leq \frac{1}{2}(||x_n||^2 - ||y_n||^2)\} \\ \qquad \bigcap \{z \in H_1 : \langle Ay_n - z_n,\ Az\rangle \leq \frac{1}{2}(||Ay_n||^2 - ||z_n||^2)\}, \\ E_n = \{z \in H_1 : \langle x_0 - x_n,\ z\rangle \leq \langle x_n,\ x_0 - x_n\rangle\}, \end{cases}$$

Since $\langle x_n - y_n,\ z\rangle$, $\langle Ay_n - z_n,\ Az\rangle$ and $\langle x_0 - x_n,\ z\rangle$ are continuous and convex functions in z, therefore for each $n \geq 0$, C_n and E_n are closed and convex subsets in H_1.

Next, we prove that $\Omega \subset C_n \bigcap E_n$ for all $n \geq 0$. Indeed, taking any point $p \in \Omega$, we have $p \in Fix(U) = Fix(S)$ and $Ap \in Fix(V) = Fix(T)$. Since U and V are quasi-nonexpansive, we have

$$||y_n - p|| = ||Ux_n - Up|| \leq ||x_n - p||,$$
$$||z_n - Ap|| = ||V(Ay_n) - V(Ap)|| \leq ||Ay_n - Ap||.$$

These imply that $\Omega \subset C_n\ \forall n \geq 0$.

Now, we show that $\Omega \subset E_n$ for all $n \geq 0$. In fact, since $E_0 = H_1$, $\Omega \subset E_0$. Now, suppose that $\Omega \subset E_n$ for some $n \geq 1$, therefore $\Omega \subset C_n \bigcap E_n$. Since $x_{n+1} = P_{C_n \bigcap E_n}(x_0),\ n \geq 0$, by (2.6) (the property of projection operator), we have

$$\langle z - x_{n+1}, x_0 - x_{n+1}\rangle \leq 0,\ \forall z \in C_n \bigcap E_n$$

Since $p \in C_n \bigcap E_n$, we have

$$\langle p - x_{n+1}, x_0 - x_{n+1}\rangle \leq 0.$$

This implies that $p \in E_{n+1}$. Thus by induction, we conclude that $\Omega \subset E_n,\ \forall n \geq 0$. Hence, $\Omega \subset C_n \bigcap E_n,\ \forall n \geq 0$. This implies that $C_n \bigcap E_n,\ n \geq 0$ is a nonempty closed and convex subset of H_1. Therefore, the sequence $\{x_n\}$ defined by (3.1) is well defined.

(III) Now, we prove that $\{x_n\}$ is a bounded sequence and

$$||x_{n+1} - x_n|| \to 0,\quad ||x_n - y_n|| \to 0\quad \text{and } ||z_n - Ay_n|| \to 0\ as\ n \to \infty \tag{3.2}$$

Let $x^* = P_\Omega(x_0)$. Hence, $x^* \in \Omega \subset E_n, \forall n \geq 0$. Further by the definition of E_n and the characterization of the metric projection (2.6), we have that $x_n = P_{E_n}(x_0)$. And so, we have

$$||x_n - x_0|| \le ||x^* - x_0||, \quad \forall n \ge 0. \tag{3.3}$$

This implies that the sequence $\{x_n\}$ is bounded.

Since $x_{n+1} \in E_n$ and $x_n = P_{E_n}(x_0)$, it follows from (2.7) that

$$||x_{n+1} - x_n||^2 \le ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2. \tag{3.4}$$

This shows that the sequence $\{||x_n - x_0||\}$ is increasing. Since it is also bounded, the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists. It now follows from (3.4) that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.5}$$

Since $x_{n+1} = P_{C_n \bigcap E_n}(x_0) \in C_n$, from the definition of C_n we have

$$\begin{cases} ||x_{n+1} - y_n|| \le ||x_{n+1} - x_n|| \\ ||z_n - Ax_{n+1}|| = ||Ay_n - Ax_{n+1}|| \le ||A||||y_n - x_{n+1}||. \end{cases} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$||x_{n+1} - y_n|| \to 0 \quad (\text{as } n \to \infty), \tag{3.7}$$

and

$$||z_n - Ax_{n+1}|| \to 0 \quad (\text{as } n \to \infty). \tag{3.8}$$

This together with (3.7) shows that

$$\begin{aligned} ||z_n - Ay_n|| &\le ||z_n - Ax_{n+1}|| + ||Ax_{n+1} - Ay_n|| \\ &\le ||z_n - Ax_{n+1}|| + ||A||||x_{n+1} - y_n|| \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

The conclusion (3.2) is proved.

(IV) Finally, we prove $\{x_n\} \to x^* := P_\Omega(x_0)$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p \in H_1$ as $n_i \to \infty$. Since A is a bounded linear operator, we also have $Ax_{n_k} \rightharpoonup Ap$. Using (3.2), we have $y_{n_i} \rightharpoonup p$, $Ay_{n_i} \rightharpoonup Ap$ and

$$||x_{n_i} - Ux_{n_i}|| \to 0 \quad \text{and } ||Ay_{n_i} - V(Ay_{n_i})|| \to 0 \text{ (as } n_i \to \infty). \tag{3.9}$$

Since U and V are demiclosed at zero, $p \in Fix(U) = Fix(S)$ and $Ap \in Fix(V) = Fix(T)$, i.e., $p \in \Omega$.

Since $x^* = P_\Omega(x_0)$ and $p \in \Omega$, from (3.3) and Lemma 2.8 (ii) (the weakly lower semi-continuity of the norm $||\cdot||$), we have that

$$\begin{aligned} ||x_0 - x^*|| \le ||x_0 - p|| &\le \liminf_{n_i \to \infty} ||x_0 - x_{n_i}|| \\ &\le \limsup_{n_i \to \infty} ||x_0 - x_{n_i}|| \le ||x_0 - x^*|| \text{ (by 3.3)} \end{aligned}$$

Using the uniqueness of the nearest point x^*, we have $x^* = p$. Hence, we have $||x_{n_i} - x_0|| \to ||x^* - x_0||$. Again by Lemma 2.8 (i), we get that $x_{n_i} \to x^*$ as $n_i \to \infty$. Using again the uniqueness of x^*, we deduce that $\lim_{n\to\infty} x_n = x^*$.

This completes the proof of Theorem 3.2. □

Remark 3.3 Theorem 3.2 not only gives an affirmative answer to the Wang's questions in Remark 1.3, but also extends and improves the corresponding results in Yao et al. [6], Moudafi [7,8] and Wang [16,17] from the firmly nonexpansive operators and demicontractive operators to a more general quasi-pseudo-contractive operator.

The following theorem can be obtained from Theorem 3.2.

Theorem 3.4 *Let H_1 and H_2 be two real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly nonexpansive mappings and $A : H_1 \to H_2$ be a bounded linear operator, and A^* is its adjoint operator. For an arbitrary initial guess $x_0 \in H_1$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = Sx_n, \\ z_n = T(Ay_n), \\ C_n = \{z \in H_1 : ||y_n - z|| \le ||x_n - z||\} \bigcap \{z \in H_1 : ||z_n - Az|| \le ||Ay_n - Az||\}, \\ E_n = \{z \in H_1 : \langle z - x_n, \ x_0 - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap E_n}(x_0), \ n \ge 0 \end{cases} \tag{3.10}$$

If the solution set Ω of problem (1.1) is nonempty, then $\{x_n\}$ converges strongly to a solution x^* of problem (1.1).

Proof In fact, since $\Omega \neq \emptyset$, $Fix(S) \neq \emptyset$ and $Fix(T) \neq \emptyset$. This implies that

(1) S and T are quasi-nonexpansive and L−Lipschitz mappings with $L > 1$;
(2) S and T both are demiclosed at zero.

These show that S, T satisfy all the conditions of U and V in Theorem 3.2. Hence, the conclusion of Theorem 3.4 can be obtained from Theorem 3.2 immediately. □

4 Applications

In this section, we shall utilize the results presented in the paper to study the split common null point problem and the split minimum point problem in real Hilbert spaces.

4.1 Application to Split Common Null Point Problems

Let H_1 and H_2 be two real Hilbert spaces. Let $S : H_1 \to 2^{H_1}$ and $T : H_2 \to 2^{H_2}$ be two maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. The "so-called" split common null point problem for mappings S and T is to find an element $x^* \in H_1$ such that

$$0 \in S(x^*) \quad \text{and } 0 \in T(Ax^*)), \text{ i.e., } x^* \in \Omega_1 := S^{-1}(0) \bigcap A^{-1}(T^{-1}(0)) \quad (4.1)$$

It is well known that if S is a maximal monotone operator, then for each $\lambda > 0$, we can define a single-valued and firmly nonexpansive mapping $J_\lambda^S : R(I + \lambda S) \to D(S)$ by $J_\lambda^S = (I + \lambda S)^{-1}$. This mapping is called the resolvent of S. It is not difficult to see that $S^{-1}(0) = Fix(J_\lambda^S) forall \lambda > 0$.

In Theorem 3.4, if we replace S by J_λ^S and T by J_λ^T, then the following theorem can be obtained from Theorem 3.4 immediately.

Theorem 4.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $S : H_1 \to 2^{H_1}$ and $T : H_2 \to 2^{H_2}$ be two maximal monotone operators and $A : H_1 \to H_2$ be a given bounded linear operator. For an arbitrary initial guess $x_0 \in H_1$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J_\lambda^S(x_n), \\ z_n = J_\lambda^T(Ay_n), \\ C_n = \{z \in H_1 : ||y_n - z|| \le ||x_n - z||\} \bigcap \{z \in H_1 : ||z_n - Az|| \le ||Ay_n - Az||\}, \\ E_n = \{z \in H_1 : \langle z - x_n, \ x_0 - x_n \rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap E_n}(x_0), \ n \ge 0 \end{cases} \quad (4.2)$$

If the solution set Ω_1 of problem (4.1) is nonempty, then $\{x_n\}$ converges strongly to a solution x^* of problem (4.1).

4.2 Application to Split Minimum Point Problem

Let H_1 and H_2 be two real Hilbert spaces. Let $f_1 : H_1 \to (-\infty, +\infty]$ and $f_2 : H_2 \to (-\infty, +\infty]$ be two proper, lower semicontinuous and convex functions, and let $A : H_1 \to H_2$ be a bounded linear operator. The "so-called" split minimum point problem for functions f_1 and f_2 is to find a point $x^* \in H_1$ such that

$$x^* \in \Omega_2 := \arg\min_{x \in H_1} f_1(x) \bigcap A^{-1}(\arg\min_{y \in H_2} f_2(y)) \quad (4.3)$$

It is well known that since $f_i, \ i = 1, 2$ is a proper lower semicontinuous and convex function, the subdifferential ∂f_i of f_i is a maximal monotone operator and that $x^* \in argmin_{x \in H_i} f_i(x)$ if and only if $0 \in \partial f_i(x^*)$. Let us define the resolvent of ∂f_i by

$J_\lambda^{\partial f_i} = (I + \lambda \partial f_i)^{-1},\ \lambda > 0$. It is known that $Fix(J_\lambda^{\partial f_i}) = \partial f_i^{-1}(0)$. Therefore, the following theorem can be obtained from Theorem 4.1 immediately.

Theorem 4.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $f_1 : H_1 \to (-\infty, +\infty]$ and $f_2 : H_2 \to (-\infty, +\infty]$ be two proper, lower semicontinuous and convex functions, $\partial f_i,\ i = 1, 2$ be the subdifferential of f_i and $A : H_1 \to H_2$ be a given bounded linear operator. For an arbitrary initial guess $x_0 \in H_1$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J_\lambda^{\partial f_1}(x_n), \\ z_n = J_\lambda^{\partial f_2}(Ay_n), \\ C_n = \{z \in H_1 : ||y_n - z|| \le ||x_n - z||\} \bigcap \{z \in H_1 : ||z_n - Az|| \le ||Ay_n - Az||\}, \\ E_n = \{z \in H_1 : \langle z - x_n,\ x_0 - x_n \rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap E_n}(x_0),\ n \ge 0 \end{cases} \tag{4.4}$$

If the solution set Ω_2 of problem (4.3) is nonempty, then $\{x_n\}$ converges strongly to a solution x^* of problem (4.3).

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