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Some Results About the Reliability of Folded Hypercubes

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Abstract
In this paper we investigate some reliability measures, including super-connectivity, cyclic edge connectivity, etc., in the folded hypercubes.

Keywords Restricted edge-connectivity · Edge-transitive · Folded hypercubes

Mathematics Subject Classification 05C40 · 05C90

1 Introduction

Suppose that Γ is a finite, simple and undirected graph. We use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ for showing the vertex set, edge set, arc set and the automorphism group of Γ, respectively. We say that Γ is *vertex-transitive*, *edge-transitive* and *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitive on $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$, respectively.

We remind that hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $x_1x_2\cdots x_n$ on the set $\{0, 1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly one bit. It is well known that Q_n is an arc-transitive graph. As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [1], is a graph obtained from the hypercube Q_n by adding an edge, called a complementary edge, between any two vertices $x = (x_1x_2\cdots x_n)$ and $\overline{x} = (\overline{x_1}\ \overline{x_2}\cdots\overline{x_n})$, where $\overline{x_i} = 1 - x_i$. The graphs shown in Figs. 1 and 2 are FQ_3 and FQ_4, respectively. By an easy observation, FQ_n is an $(n+1)$-regular and its order is 2^n. Like Q_n, FQ_n is arc-transitive (see [15]). Also by [1], it has diameter of $\lceil n/2\rceil$ which is smaller than the diameter of Q_n. In the literature, FQ_n has received considerable attention, and a lot of its properties have been investigated. For example see ([7–9,13,22,23,25,29,34]).

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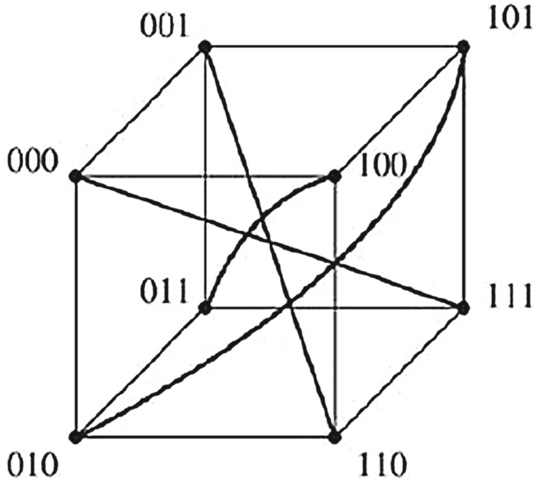

Fig. 1 (FQ_3)

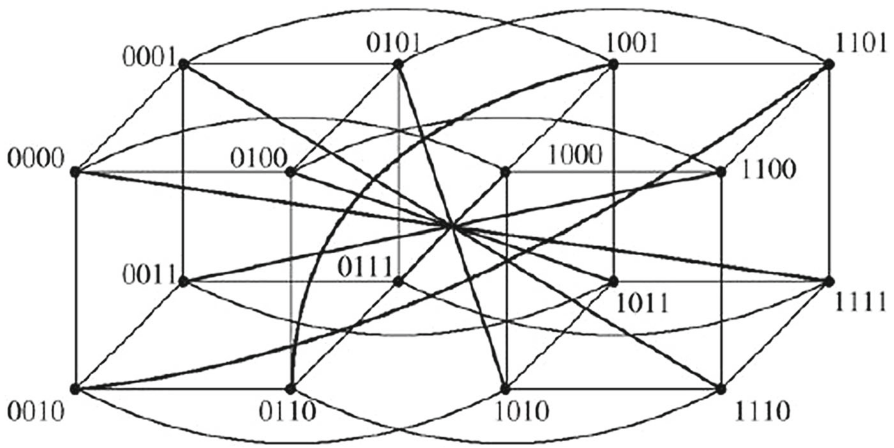

Fig. 2 (FQ_4)

Zhou et al. (see [33]) studied some reliability measures such as super-connectivity, cyclically connectivity in the balanced hypercube by using its symmetric properties. In this paper we get the similar results for n-dimensional folded hypercube FQ_n.

2 Preliminaries

In this section, we give some terminology and notation. For notation and terminology not define here we follow [3].

Let n be a positive integer. Denote by C_n the cyclic graph of order n. For a vertex v in a graph Γ, use $N_\Gamma(v)$ to denote the neighborhood of v, that is, the set of vertices adjacent to v. Also for a positive integer m, mK_1 represents the null graph with m vertices and a graph is trivial if it is a vertex.

The *vertex-connectivity* of a graph Γ denoted by $\kappa(\Gamma)$ is the minimum number of vertices whose removal results in a disconnected graph or a trivial graph. Similarly, we can define *edge-connectivity* of a graph Γ which is denoted by $\lambda(\Gamma)$. In fact $\kappa(\Gamma)$ and $\lambda(\Gamma)$ are two important factors for measuring the reliability of an interconnection network. Also a larger $\kappa(\Gamma)$ or $\lambda(\Gamma)$ means that the network Γ is more reliable. It is well known that $\kappa(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of Γ. A graph is said to be *super-*κ(resp. *super-*λ), if any minimum vertex-cut (resp. edge-cut) isolates a vertex.

In order to estimate more precisely the reliability, Esfahanian and Hakimi introduce such a kind of edge cut in [6] that separates a connected graph into a disconnected one without isolated vertices. With the properties of restricted edge connectivity, Li analyzed the reliability of circulant graphs in [11] and improved Bauer's result. For more accurate results, Ou et al. introduce the concepts of m-restricted edge cut and m-restricted edge connectivity in [6,16,17,27]. An edge set F is an *m-restricted edge cut* of a connected graph G if $G - F$ is disconnected and each component of $G - F$ contains at least m vertices (see [6]). Let $\lambda^{(m)}(G)$ be the minimum size of all m-restricted edge cuts and $\xi_m(G) = \min\{|\omega(U)| : |U| = m \text{ and} G[U] \text{is connected}\}$ where $\omega(U)$ is the set of edges with exactly one end vertex in U and $G[U]$ is the subgraph of G induced by U. A graph G is $\lambda^{(m)}$*-graph* if $\lambda^{(m)}(G) = \xi_m(G)$. Also $\lambda^{(m)}(G)$ is called *m-restricted edge connectivity* of graph G. Moreover, a graph is called *super-m-restrict edge connected*, in short, *super-*$\lambda^{(m)}$ if every minimum edge cut isolates one component $G[U]$ with $|U| = m$. In the special case, a set F of edges of a connected graph G is said to be a restricted edge-cut, if its removal disconnects G, and $G - F$ contains no isolated vertices. If G has at least one restricted edge-cut, the restricted edge-connectivity of G, denoted by $\lambda^{'}(G)$, is then defined to be the minimum cardinality over all restricted edge-cuts of G. Moreover, a graph Γ is called *super-restricted edge-connected*, in short, *super-*$\lambda^{'}$ if every minimum restricted edge cut isolates one component of size 2 . The super-restricted edge-connectivity of many interconnection networks has been studied (see [5,11,12,19,24,33]).

Similarly, if V is a vertex set then m-restricted cut, m-restricted connectivity and super-m-restricted connectivity (in the special case *super-*$\kappa^{'}$) are defined analogously.

For a graph Γ, an edge set F is a *cyclic edge-cut* if $\Gamma - F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two vertex-disjoint cycles. For a cyclically separable graph G, the *cyclic edge-connectivity* of Γ, denoted by $\lambda_c(\Gamma)$, is defined as the cardinality of a minimum cyclic edge-cut of Γ. Cyclic edge-connectivity plays an important role in many classic fields of graph theory such as measure of network reliability. A graph Γ is said to be *super-*λ_c, if the removal of any minimum cyclic edge-cut of Γ results in a component which is a shortest cycle of Γ. The cyclic edge-connectivity of many interconnection networks has been studied (see [18,21,28,30–32]).

Suppose that Γ and Δ are two graphs. The lexicographic product of Γ and Δ which is denoted by $\Gamma[\Delta]$ is a graph with vertex set $V(\Gamma) \times V(\Delta)$ and two vertices (u_1, v_1), $(u_2, v_2) \in V(\Gamma) \times V(\Delta)$ are adjacent in $\Gamma[\Delta]$ whenever either u_1 is adjacent to u_2 in Γ, or $u_1 = u_2$ and v_1 is adjacent to v_2 in Δ.

Let Γ and H be two graphs. The lexicographic product $\Gamma[H]$ is defined as the graph with vertex set $V(\Gamma) \times V(H)$, and for any two vertices (u_1, v_1), $(u_2, v_2) \in$

$V(\Gamma) \times V(H)$, they are adjacent in $\Gamma[H]$ if and only if either u_1 is adjacent to u_2 in Γ, or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

Proposition 2.1 ([14,20]) *Let Γ be a connected graph which is both vertex-transitive and edge-transitive. Then $\kappa(\Gamma) = \delta(\Gamma)$, and moreover, Γ is not super-κ if and only if $\Gamma \cong C_n[mK_1]$ $(n \geq 6)$ or $L(Q_3)[mK_1]$, where $L(Q_3)$ is the line graph of three-dimensional hypercube Q_3.*

The following results are about the connectivity of edge-transitive graphs.

Proposition 2.2 ([2,11,19,30,31]) *Let Γ be a $k(k \geq 3)$-regular edge-transitive graph. Then*

(1) Γ *is super-λ.*

(2) $\lambda^{'}(\Gamma) = 2k - 2$.

(3) Γ *is not super-$\lambda^{'}$ if and only if Γ is isomorphic to the three-dimensional hypercube Q_3 or to a four-valent edge-transitive graph of girth* 4.

(4) *If Γ is not isomorphic to K_4, K_5 or $K_{3,3}$ then $\lambda_c(\Gamma) = g(k-2)$, where g is the girth of Γ.*

By [32][Theorem 3.4] we have the following result.

Proposition 2.3 *Let Γ be a $k(k \geq 3)$-regular edge-transitive graph of girth g. Suppose that Γ is not isomorphic to K_4, K_5 or $K_{3,3}$. If Γ is not super-λ_c, then* $(g, k) = (6, 3), (4, 4), (4, 5), (4, 6)$ *or* $(3, 6)$. *Furthermore, $C_n[2K_1]$ $(n \geq 4)$ is non-super-λ_c, and if* $(g, k) = (4, 6)$ *or* $(g, k) = (4, 5)$ *then* $|\Gamma| = 16$ *or* $|\Gamma| = 12$*, respectively.*

3 Reliability Evaluation of Folded Hypercube

The reliability of an interconnection network is an important issue for multiprocessor systems. In this section we study some reliability measures, say, super-connectivity, cyclic connectivity, etc., in folded hypercube. For the folded hypercubes, in [1] it was shown that $\kappa(FQ_n) = n + 1$. However, by [15], we know that FQ_n is arc-transitive. Thus by Proposition 2.1, $\kappa(FQ_n) = \delta(FQ_n) = n + 1$. In the following we obtain the stronger result which states FQ_n is super-k for $n \geq 2$.

Theorem 3.1 *FQ_n is super-k if and only if $n \geq 2$.*

Proof If $n = 2$ then FQ_n is a complete graph K_4. Clearly, it is super-k. In what follows, assume that $n \geq 3$. Suppose to contrary that is FQ_n is not super-k. Since FQ_n is both vertex-transitive and edge-transitive, from Proposition 2.1 it follows that $FQ_n \cong C_l[mK_1]$ $(l \geq 6)$ or $L(Q_3)[mK_1]$. First suppose that $FQ_n \cong C_l[mK_1]$ $(l \geq 6)$. We know that $\kappa(FQ_n) = n + 1$. Also by [26][Theorems 1], $\kappa(C_l[mK_1]) = 2m$ and so $n + 1 = 2m$. Moreover, since $|V(C_l[mK_1])| = |V(FQ_n)|$, it follows that $2^n = ml$. Thus m and l are even. By [1] we know that $\mathrm{diam}(FQ_n) = \lceil n/2 \rceil$.

Also we know that $\operatorname{diam}(C_l[mK_1]) = l/2$ (see [10]). Now $l/2 = \lceil n/2 \rceil$ and so $\lceil n/2 \rceil = \lceil 2m - 1/2 \rceil = m = l/2$. Now $2^n = 2m^2$ and we may suppose that $m = 2^k$ for some $k \geq 0$. If $k = 1$ then $m = 2$ and $l = 4$, a contradiction. Thus $k \geq 2$. By $2^n = 2m^2 = 22^{2k} = 2^{2k+1}$ we have $n = 2k + 1$. Now by $n = 2m - 1$ we have $2k + 1 = 2^{k+1} - 1$. Thus $2^k = k + 1$, a contradiction. Now suppose that $FQ_n \cong L(Q_3)[mK_1]$. Thus, the number of vertices of these graphs is same and so $2^n = 12m$, a contradiction. □

In the following theorem we show that the delation of any minimum edge-cut of FQ_n isolates a vertex.

Theorem 3.2 *FQ_n is super-λ if and only if $n \geq 3$.*

Proof If $n = 2$ then FQ_2 is a complete graph K_4. Clearly, it is not super-λ. Suppose that $n \geq 3$. Thus the valency of FQ_n is at least 4. Since FQ_n is edge-transitive, the theorem follows from Proposition 2.2(1). □

From this theorem we immediately have the following corollary.

Corollary 3.3 *For $n \geq 2$, $\lambda(FQ_n) = n + 1$.*

In [23][Theorem 3] Xu et al. proved that the restricted edge-connectivity of FQ_n is $2n$. Here we present a simple proof for this result.

Theorem 3.4 *For $n \geq 2$, $\lambda'(FQ_n) = 2n$.*

Proof If $n = 2$ then FQ_2 is a complete graph K_4. Clearly, $\lambda'(FQ_2) = 4$. Thus we may suppose that $n \geq 3$. Therefore FQ_n has valency $n + 1 \geq 4$. Now since FQ_n is edge-transitive, the theorem follows from Proposition 2.2. □

In the following theorem we show that every minimum edge-cut of FQ_n $(n \geq 3)$ isolates an edge.

Theorem 3.5 *FQ_n is super-λ' if and only if $n \neq 3$.*

Proof If $n = 2$ then clearly FQ_2 is super-λ'. Also if $n = 3$ then again it is easy to see that FQ_3 is not super-λ' (see Fig. 1). Thus we may suppose that $n \geq 4$. Therefore FQ_n has valency $n \geq 5$. We know that FQ_n is edge-transitive. By Proposition 2.2 FQ_n is not super-λ' if and only if it is isomorphic to the three-dimensional hypercube Q_3 or a four-valent edge-transitive graph of girth 4. Since FQ_n $(n \geq 4)$ has valency at least 5 it implies that $FQ_n \cong Q_3$. Also if $FQ_n \cong Q_3$ then $n = 3$. But we know that Q_3 has valency three, but the valency of FQ_3 is four, a contradiction. Therefore FQ_n is super-λ' for $n \geq 2$. □

The following theorem shows that for $n \geq 3$, every minimum cyclic edge-cut of FQ_n isolates a shortest cycle.

Theorem 3.6 *FQ_n is super-λ_c if and only if $n \neq 2$.*

Proof If $n = 2$ then clearly FQ_2 is not super-λ_c. Also if $n = 3$ or $n = 4$ then it is easy to see that FQ_3 is super-λ_c. Thus we may suppose that $n \geq 5$. We know that FQ_n has order 2^n and valency $n + 1$. Thus FQ_n is not isomorphic to K_3, K_4 or $K_{3,3}$. Suppose to contrary that FQ_n is not super-λ_c. By Proposition 2.3, $n+1 \in \{3, 4, 5, 6\}$. Suppose that $n + 1 = 3$ or $n + 1 = 4$ then $n = 2$ or $n = 3$, a contradiction. If $n + 1 = 5$ or $n + 1 = 6$ then $n = 4$ or $n = 5$. Now by Proposition 2.3, FQ_4 or FQ_5 has 12 or 16 vertices, a contradiction.

In the following theorem we show that for $n \geq 3$, by removing $4(n-1)$ edges from FQ_n we obtain a disconnected graph which has at least two components containing cycle. □

Theorem 3.7 *For $n \geq 3$, $\lambda_c(FQ_n) = 4(n-1)$.*

Proof Since FQ_n has at least 8 vertices it is not isomorphic to K_3, K_4 and $K_{3,3}$. By [22][Theorems 3.3] we know that FQ_n has girth 4. Now by Proposition 2.2 $\lambda_c(FQ_n) = 4(n+1-2) = 4(n-1)$. □

Theorem 3.8 *FQ_n $(n \geq 8)$ is super-κ'.*

Proof By [4,34], we know that $\kappa_2 \geq \kappa_1$. Suppose that FQ_n for $(n \geq 8)$ is not super-κ'. Thus there is a restricted cut S of order κ_1, but the cut is not the neighborhood of any edge. That is, S is also a 3-restricted cut. Thus $|S| \geq \kappa_2$, a contradiction. □

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