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The Schrödinger–Bopp–Podolsky Equation Under the Effect of Nonlinearities

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Abstract
In this paper, we study the following nonlinear Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + q\phi u = f(u) \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3,$$

where $a > 0$, $q > 0$ and $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$. By means of the variational methods, we prove the existence of infinitely many nontrivial solutions, the existence of a ground state solution for $f(u) = |u|^{p-2}u + h(u)$ with $p \in [4, 6)$ and the existence of at least one positive solution for $f(u) = P(x)u^5 + \mu|u|^{p-2}u$ with $p \in (2, 6)$ under some certain assumptions.

Keywords Schrödinger–Bopp–Podolsky equation · Ground state solution · Positive solution · Multiplicity of solutions

Mathematics Subject Classification 35J60 · 35J20

1 Introduction

In the recent paper [12], the following Schrödinger–Bopp–Podolsky system has been studied for the first time

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$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a, \omega > 0$, $q \neq 0$ and $p \in (2, 6)$. The system appears when one couples the Schrödinger field $\Psi = \Psi(t, x)$ with the Bopp–Podolsky Lagrangian of the electromagnetic field and considers standing wave $\Psi(t, x) = e^{iwt} u(x)$ in the purely electrostatic case. The Bopp–Podolsky theory, which is a second-order gauge theory of the electromagnetic field, was developed by Bopp [3] and then independently by Podolsky [20]. According to Mie theory [19] and its generalizations in [4–7], the Bopp–Podolsky theory was introduced to solve the alleged infinity problem in classical Maxwell theory.

d'Avenia and Siciliano [12] used variational methods to prove the existence results of problem (1.1). Indeed, the solutions can be found as critical points of a smooth energy functional. When $p \in (2, 6)$ and $|q|$ small enough or $p \in (3, 6)$ and $q \neq 0$, they proved that the energy functional has the mountain pass geometry; hence, the above system has a nontrivial solution. In addition, they proved the above system does not admit any nontrivial solution for $p \geq 6$ by using Pohozaev-type identity, and in the radial case, as $a \to 0$, the solutions they got tend to solutions of the classical Schrödinger–Poisson system. They also showed that, if ρ is the distribution density of the given charge, then the electrostatic potential Φ satisfies the following equation

$$-\Delta \Phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\pi \delta_{x_0}$ with $x_0 \in \mathbb{R}^3$, then $\mathcal{G}(x - x_0) = \frac{1}{|x - x_0|}$ is the fundamental solution of (1.2). And the electrostatic energy is

$$\uparrow_M(\mathcal{G}) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathcal{G}|^2 = +\infty.$$

Hence, Eq. (1.2) is replaced by

$$-\Delta \phi + a^2 \Delta^2 \phi = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Podolsky theory. Moreover, we know that $\mathcal{K}(x - x_0)$ is the fundamental solution of the equation

$$-\Delta \phi + a^2 \Delta^2 \phi = 4\pi \delta_{x_0},$$

where

$$\mathcal{K}(x) := \frac{1 - e^{-|x|/a}}{|x|}, \quad \lim_{x \to x_0} \mathcal{K}(x - x_0) = \frac{1}{a}.$$

And its energy is

$$\uparrow_{\mathrm{BP}}(\mathcal{K}) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathcal{K}|^2 + \frac{a^2}{2} \int_{\mathbb{R}^3} |\Delta \mathcal{K}|^2 < +\infty,$$

more details can be found in [12].

Remark 1.1 The operator $-\Delta + \Delta^2$ appears also in other different physical and mathematical problems (see [2,14]).

After that, Gaetano and Kaye [15] supplemented and improved some results in [12]. It has been showed that (1.1) has no solution for large values of q's and has two radial solutions for small q's.

Chen and Tang [10] extended the subcritical case to more general cases and dealt with the following system

$$\begin{cases} -\Delta u + V(x)u + \phi u = \mu f(u) + u^5 \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3,$$

where $a > 0$, $\mu > 0$, $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$ with $V_\infty = \lim_{|y|\to\infty} V(y) \geq \sup_{x\in\mathbb{R}^3} V(x) > 0$, and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies that there exists a constant $p \in (2, 6)$ such that $\int_0^t f(s)\mathrm{d}s \geq t^p$ for all $t \geq 0$. They obtained the existence of ground state solutions for $p \in (4, 6)$ and $\mu > 0$ or $p \in (2, 4]$ and $\mu > \mu_*$, where μ_* is a positive constant.

Motivated by the cited papers [10,12], our aim here is to study the existence of ground state solutions, positive solutions and infinitely many nontrivial solutions for the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + q\phi u = f(u) \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $a > 0$, $q > 0$ and V satisfies the following conditions:

(V_1) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$;
(V_2) there exists a constant $d_0 > 0$ such that

$$\lim_{|y|\to+\infty} \text{meas}\left\{x \in \mathbb{R}^3 : |x - y| \leq d_0, V(x) \leq M\right\} = 0, \quad \forall M > 0.$$

We work in the Hilbert space

$$E := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2\mathrm{d}x < +\infty\right\},$$

endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Theorem 1.2 [23] *Assume that* (V_1)–(V_2) *are satisfied. Then,* E *is continuously embedded into* $L^s(\mathbb{R}^3)$ *for* $s \in [2, 6]$*. Moreover, the embedding* $E \hookrightarrow L^s(\mathbb{R}^3)$ *is compact for* $s \in [2, 6)$.

We first consider the following system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Assume that f is a continuous function and satisfies the following conditions:

(f_1) $f(t) = -f(-t)$;
(f_2) there exist $1 < \iota < 5$ such that $\lim_{|t|\to 0} \frac{f(t)}{|t|} = \lim_{|t|\to+\infty} \frac{f(t)}{|t|^\iota} = 0$;
(f_3) $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^4} = \infty$, and there exists $\mu \geq 4$, $\kappa > 0$ such that

$$\mu F(t) \leq tf(t) + \kappa t^2,$$

where $F(t) = \int_0^t f(r)\,\mathrm{d}r$.

Our first result is as follows.

Theorem 1.3 *Assume that* (V_1)–(V_2) *and* (f_1)–(f_3) *are satisfied. Then, the problem* (1.4) *possesses infinitely many nontrivial solutions.*

Remark 1.4 To our knowledge, there are few results about existence of multiply solutions to Schrödinger–Bopp–Podolsky equation.

Remark 1.5 In order to assure the boundedness of the Palais–Smale sequences of the energy functional, the following condition is usually supposed.

(AR) There exists $\mu > 4$ such that

$$0 < \mu H(s) \leq sh(s), \quad \text{for all} \quad s \neq 0.$$

Obviously, (f_3) is weaker than Ambrosetti–Rabinowitz condition.

We next consider the case of $f(u) = |u|^{p-2}u + h(u)$, namely the following system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u + h(u) \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi u^2 \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where $h \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(h_1) for some $\varrho \in (1, 5)$,

$$\lim_{|u|\to+\infty} \frac{h(u)}{|u|^\varrho} = \lim_{|u|\to 0} \frac{h(u)}{|u|} = 0;$$

(h_2) for any $p \in [4, 6)$, $H(u) + \frac{1}{p}|u|^p \geq 0$, where $H(u) = \int_0^u h(r)\,\mathrm{d}r$,
(h_3) there exists $\theta_0 \in (0, 1)$ such that for any $t > 0$ and $u \neq 0$,

$$\left[\frac{h(u)}{u^3} - \frac{h(tu)}{(tu)^3}\right]\mathrm{sign}(1-t) + \theta_0 \frac{V(x)|1-t^2|}{(tu)^2} \geq 0.$$

The following is the second result of our paper.

Theorem 1.6 *Assume that* (V_1)–(V_2) *and* (h_1)–(h_3) *are satisfied. If* $p \in [4, 6)$*, then the problem* (1.5) *has a ground state solution.*

Remark 1.7 The condition (h_3) comes from [11], and it is weaker than Ambrosetti–Rabinowitz condition and also weaker than the following condition:

$$\frac{h(u)}{|u|^3} \text{ is nondecreasing on } (-\infty, 0) \cup (0, +\infty).$$

Finally, we consider the case of $f(u) = P(x)u^5 + \mu|u|^{p-2}u$, namely the following system:

$$\begin{cases} -\Delta u + V(x)u + q\phi u = P(x)u^5 + \mu|u|^{p-2}u \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \end{cases} \text{in } \mathbb{R}^3. \tag{1.6}$$

We assume that $P(x)$ is a continuous function and satisfies the following conditions:

(P_1) $P(x) > 0$ and $P(x) \in L^\infty(\mathbb{R}^3)$;
(P_2) $|P(x) - P(x_0)| = O(|x - x_0|^2)$ as $x \to x_0$, where $P(x_0) = \sup_{x\in\mathbb{R}^3} P(x)$.

The difficulty in obtaining solutions of the problem (1.6) lies in two aspects. On the one hand, since the problem involves critical exponent, the difficulty lies in the lack of compactness. On the other hand, when $p \in (2, 4)$, the boundedness of the Palais–Smale sequences of the energy functional is hard to get. Inspired by [1,9,12,13,16,21], we construct a truncated function and use the mountain pass theorem with mountain pass lever value under $\frac{1}{3}S^{\frac{3}{2}}\|P(x)\|_\infty^{\frac{1}{2}}$.

Now we are in a position to state the following results of our paper on the existence of positive solutions.

Theorem 1.8 *Assume that* (V_1)–(V_2) *and* (P_1)–(P_2) *are satisfied. If* $p \in (4, 6)$ *with* $\mu > 0$ *or* $p = 4$ *with* μ *sufficiently large, then the problem* (1.6) *has at least one positive solution for any* $q > 0$.

Theorem 1.9 *Assume that* (V_1)–(V_2) *and* (P_1)–(P_2) *are satisfied. If* $p \in (2, 4)$ *with* μ *sufficiently large, then there exists* $q_* > 0$ *such that, for all* $q \in (0, q_*)$*, the problem* (1.6) *has at least one positive solution.*

Notations Throughout the paper, we denote by meas$(\cdot)$ the Lebesgue measure in $\mathbb{R}^3$. E^* is the dual space of E. $\|\cdot\|_p$ denotes the usual norm of Lebesgue space $L^p(\mathbb{R}^3)$. S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. The symbol C denotes a positive constant and may vary from line to line.

2 Preliminary

In this section, we want to introduce the variational setting, the functional setting and some main results.

2.1 The Variational Setting

We first consider the nonlinear Schrödinger Lagrangian density

$$\mathcal{L}_{\mathrm{Sc}} = i\hbar\bar{\psi}\partial_t\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + 2F(\psi), \tag{2.1}$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ and $\hbar, m > 0$. If the gauge potential of the electromagnetic field $(\mathbf{E}, \mathbf{H})$ is $(\phi, \mathbf{A})$, where $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, then the following equations hold

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{H} = \nabla\times\mathbf{A}.$$

To study the interaction between ψ and its electromagnetic field $(\mathbf{E}, \mathbf{H})$, we can replace the derivatives ∂_t and ∇ in (2.1) with the covariant ones

$$D_t = \partial_t + \frac{i\sqrt{q}}{\hbar}\phi, \quad \mathbf{D} = \nabla - \frac{i\sqrt{q}}{\hbar c}\mathbf{A}.$$

Thus, we have

$$\begin{aligned}\mathcal{L}_{\mathrm{CSc}} &= i\hbar\bar{\psi}D_t\psi - \frac{\hbar^2}{2m}|\mathbf{D}\psi|^2 + 2F(\psi)\\ &= i\hbar\bar{\psi}\left(\partial_t + \frac{i\sqrt{q}}{\hbar}\phi\right)\psi - \frac{\hbar^2}{2m}\left|\left(\nabla - \frac{i\sqrt{q}}{\hbar c}\mathbf{A}\right)\psi\right|^2 + 2F(\psi).\end{aligned}$$

According to [3], the Bopp–Podolsky Lagrangian density is

$$\begin{aligned}\mathcal{L}_{\mathrm{BP}} &= \frac{1}{8\pi}\left\{|\mathbf{E}|^2 - |\mathbf{H}|^2 + a^2\left[(\mathrm{div}\mathbf{E})^2 - \left|\nabla\times\mathbf{H} - \frac{1}{c}\partial_t\mathbf{E}\right|^2\right]\right\}\\ &= \frac{1}{8\pi}\left\{|\nabla\phi + \frac{1}{c}\partial_t\mathbf{A}|^2 - |\nabla\times\mathbf{A}|^2\right.\\ &\quad \left. + a^2\left[\left(\Delta\phi + \frac{1}{c}\mathrm{div}\partial_t\mathbf{A}\right)^2 - \left|\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\partial_t(\nabla\phi + \frac{1}{c}\partial_t\mathbf{A})\right|^2\right]\right\}.\end{aligned}$$

Now, we add the Lagrangian density of the electromagnetic field to $\mathcal{L}_{\mathrm{CSc}}$ so that we can get the total Lagrangian density

$$\mathcal{L} := \mathcal{L}_{\mathrm{CSc}} + \mathcal{L}_{\mathrm{BP}}.$$

Therefore, the total action is

$$\mathcal{S}(\psi, \phi, \mathbf{A}) = \int_{\mathbb{R}^3}\mathcal{L}\mathrm{d}x\mathrm{d}t.$$

More details can be found in [12].

2.2 The Operator $-\Delta + a^2\Delta^2$

Let $\mathcal{D}$ be the completion of $C_c^\infty(\mathbb{R}^3)$ with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\varphi\nabla\psi + a^2 \int_{\mathbb{R}^3} \Delta\varphi\Delta\psi.$$

Obviously, $\mathcal{D}$ is continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and therefore in $L^6(\mathbb{R}^3)$.
The following lemmas were obtained in [12].

Lemma 2.1 [12] *The Hilbert space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$.*

Lemma 2.2 [12] *The space $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{A}$, where*

$$\mathcal{A} := \left\{ \phi \in D^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3) \right\}$$

normed by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, consequently $\mathcal{D} = \mathcal{A}$.

Lemma 2.3 [12] *For all $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y) = \frac{1-e^{-|\cdot-y|/a}}{|\cdot-y|}$ is the fundamental solution of*

$$-\Delta\phi + a^2\Delta^2\phi^2 = 4\pi\delta_y.$$

Furthermore,

(i) *if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and, for almost everywhere $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \mapsto \frac{f(y)}{|x-y|}$ is summable, then $\mathcal{K} * f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$;*
(ii) *if $f \in L^s(\mathbb{R}^3)$ with $1 \le s < \frac{3}{2}$, then $\mathcal{K} * f \in L^r(\mathbb{R}^3)$ for $r \in \left(\frac{3s}{3-2s}, +\infty\right]$.*

In both cases, $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi^2 = 4\pi f.$$

Moreover, almost everywhere in $\mathbb{R}^3$, we have

$$\nabla(\mathcal{K} * f) = (\nabla\mathcal{K}) * f \quad \text{and} \quad \Delta(\mathcal{K} * f) = (\Delta\mathcal{K}) * f.$$

Then, for any fixed $u \in H^1(\mathbb{R}^3)$, $\phi_u := \mathcal{K} * u^2$ is the unique solution in $\mathcal{D}$ of the second equation in (1.3), and the following lemma holds.

Lemma 2.4 [12] *For every $u \in H^1(\mathbb{R}^3)$, we have:*

(i) *for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;*
(ii) *$\phi_u \ge 0$;*
(iii) *for every $s \in (3, +\infty]$, $\phi_u \in L^s(\mathbb{R}^3) \bigcap C_0(\mathbb{R}^3)$;*

(iv) *for every* $s \in \left(\frac{3}{2}, +\infty\right]$, $\nabla \phi_u = \nabla \mathcal{K} * u^2 \in L^s(\mathbb{R}^3) \bigcap C_0(\mathbb{R}^3)$;
(v) $\phi_u \in \mathcal{D}$;
(vi) $\|\phi_u\|_6 \leq C\|u\|^2$;
(vii) ϕ_u *is the unique minimizer of the functional*

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{a^2}{2}\|\Delta\phi\|_2^2 - \int_{\mathbb{R}^3} \phi u^2, \quad \phi \in \mathcal{D};$$

(viii) *if* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{v_n} \rightharpoonup \phi_v$ *in* $\mathcal{D}$.

2.3 The Functional Setting

It is easy to see that the weak solutions of (1.3) are the critical points of the C^1 energy functional

$$R(u,\phi) = \frac{1}{2}\|u\|^2 + \frac{q}{2}\int_{\mathbb{R}^3} \phi u^2 \mathrm{d}x - \frac{q}{16\pi}\|\phi\|_{\mathcal{D}}^2 - \int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

on $E \times \mathcal{D}$. If $(u,\phi) \in E \times \mathcal{D}$ is a critical point of R, then we have

$$\begin{aligned} 0 = \partial_u R(u,\phi)[v] &= \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)uv + q\phi uv]\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} f(u)v\mathrm{d}x, \quad \text{for all } \ v \in E, \end{aligned}$$

and

$$0 = \partial_\phi R(u,\phi)[\xi] = \frac{q}{2}\int_{\mathbb{R}^3} \xi u^2 \mathrm{d}x - \frac{q}{8\pi}\int_{\mathbb{R}^3}\left[\nabla\phi\nabla\xi + a^2\Delta\phi\Delta\xi\right]\mathrm{d}x, \quad \text{for all } \ \xi \in \mathcal{D}.$$

Distinctly, the functional R is strongly indefiniteness, and hence, we adopt a reduction procedure which used in [12]. Noting that $\partial_\phi R \in \mathcal{C}^1(\mathbb{R}^3)$, by implicit function theorem, we have

$$G_\Phi = \left\{(u,\phi) \in E \times \mathcal{D} : \partial_\phi R(u,\phi) = 0\right\} \quad \text{and} \quad \Phi \in \mathcal{C}^1(E,\mathcal{D}),$$

where G_Φ is the graph of the map $\Phi : u \in E \mapsto \phi_u \in \mathcal{D}$. Hence, we have the reduced form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + V(x)u^2\right]\mathrm{d}x + \frac{q}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x,$$

and it is of class $\mathcal{C}^1$ on E. For all $u, v \in E$,

$$\begin{aligned} I'(u)[v] &= \partial_u R(u,\Phi(u))[v] + \partial_\phi R(u,\Phi(u)) \circ \Phi'(u)[v] \\ &= \partial_u R(u,\Phi(u))[v] \end{aligned}$$

$$= \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)uv + q\phi_u uv]\,dx - \int_{\mathbb{R}^3} f(u)v dx.$$

In fact, to find solutions of Eq. (1.3), we only need to find critical points of I.

3 Existence of Multiple Solutions

In this section, assume that (V_1)–(V_2) and (f_1)–(f_3) are satisfied. We prove that the problem (1.4) possesses infinitely many nontrivial solutions. And we give the energy functional I by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

Lemma 3.1 [22] *Let X be an infinite-dimensional Banach space, and there exists a finite-dimensional space W such that $X = W \oplus V$. $I \in \mathcal{C}^1(X, \mathbb{R})$ satisfies the (PS) condition, and*

(i) *$I(u) = I(-u)$ for all $u \in X$, $I(0) = 0$;*
(ii) *there exist $\rho > 0$, $\alpha > 0$ such that $I|_{\partial B_\rho \cap V} \geq \alpha$;*
(iii) *for any finite-dimensional subspace $Y \subset X$, there exists $R = R(Y) > 0$ such that $I(u) \leq 0$ on $Y \backslash B_R$.*

Then, I possesses an unbounded sequence of critical values.

Lemma 3.2 *Under the assumptions of Theorem 1.3, if $\{u_n\} \subset E$ satisfies the following conditions:*

(i) *$\{I(u_n)\}$ is bounded above by a positive constant;*
(ii) *$I'(u_n)[u_n] \to 0$, then $\{u_n\}$ is bounded in E.*

Proof If $\{u_n\}$ is unbounded in E, we can find a subsequence still denoted by $\{u_n\}$ such that $\|u_n\| \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, we have $\|v_n\| = 1$. Thus, we may assume that $v_n \rightharpoonup v$ in E. It follows from (f_3) and (i) that there exists a constant $c > 0$ such that

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{\mu} I'(u_n)[u_n] \\ &\geq \frac{\mu - 2}{2\mu}\|u_n\|^2 - \frac{\kappa}{\mu}\|u_n\|_2^2 \\ &= \frac{\mu - 2}{2\mu}\|u_n\|^2\|v_n\|^2 - \frac{\kappa}{\mu}\|u_n\|^2\|v_n\|_2^2, \end{aligned}$$

as $n \to \infty$, which implies $1 \leq \frac{2\kappa}{\mu - 2}\limsup_{n\to\infty}\|v_n\|_2^2$. Therefore, $v \not\equiv 0$.

By (f_3) and Fatou's lemma, one has

$$0 = \lim_{n\to\infty}\frac{c}{\|u_n\|^4} = \lim_{n\to\infty}\frac{I(u_n)}{\|u_n\|^4}$$

$$\leq \lim_{n\to\infty}\left(\frac{1}{2\|u_n\|^2}+\frac{\|u_n\|_2^4}{4a\|u_n\|^4}-\int_{\mathbb{R}^N}\frac{F(u)}{u_n^4}v_n^4\mathrm{d}x\right)=-\infty,$$

as $n \to +\infty$. This is a contradiction, and hence, $\{u_n\}$ is bounded in E. □

Lemma 3.3 *Under the assumptions of Theorem 1.3, then the functional I satisfies the (PS) condition.*

Proof To prove that I satisfies the (PS) condition, we only need to prove $\{u_n\} \subset E$ which obtained by Lemma 3.2, has a convergent subsequence. As $\{u_n\}$ is bounded in E, there exists a subsequence still denoted by $\{u_n\}$ and $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ in $L^q(\mathbb{R}^3)$ for $2 < q < 6$. It follows from $u_n \rightharpoonup u_0$ in E and $I'(u_0) \in E^*$ that $\langle I'(u_0), u_n - u_0\rangle \to 0$. And as $I'(u_n) \to 0$ in E^*, it is easy to obtain

$$\langle I'(u_n), u_n - u_0\rangle \leq \|I'(u_n)\|_{E^*}\|u_n - u_0\| \to 0.$$

Therefore,

$$\langle I'(u_n) - I'(u_0), u_n - u_0\rangle = \langle I'(u_0), u_n - u_0\rangle - \langle I'(u_n), u_n - u_0\rangle \to 0.$$

By (f_2), for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(u) \leq \epsilon|u| + C_\epsilon|u|^\iota.$$

Hence, as $n \to +\infty$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}((f(u_n)-f(u_0))(u_n-u_0))\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^3}|f(u_n)||u_n-u_0|\,\mathrm{d}x+\int_{\mathbb{R}^3}|f(u_0)||u_n-u_0|\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^3}\left(\epsilon|u_n|+C_\varepsilon|u_n|^\iota\right)|u_n-u_0|\,\mathrm{d}x+\int_{\mathbb{R}^3}\left(\epsilon|u_0|+C_\varepsilon|u_0|^\iota\right)|u_n-u_0|\,\mathrm{d}x\\
&\leq \epsilon\left(\|u_n\|_2+\|u_0\|_2\right)\|u_n-u_0\|_2\\
&\quad +C_\epsilon\left(\|u_n\|_{\iota+1}^\iota+\|u_0\|_{\iota+1}^\iota\right)\|u_n-u_0\|_{\iota+1}\to 0,
\end{aligned}$$

as $\epsilon \to 0$ and $\iota \in (1, 5)$.

And then,

$$\begin{aligned}
\|u_n-u_0\|^2 &= \langle I'(u_n)-I'(u_0), u_n-u\rangle\\
&\quad + \int_{\mathbb{R}^3}((f(u_n)-f(u_0))(u_n-u_0))\,\mathrm{d}x \to 0.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3 We have verified that I satisfies the (PS) condition. It follows from (f_1) that I is an even functional. As E is a separable space, E has orthonormal basis $\{e_i\}$. For all $k, j \in \mathbb{Z}$, we define $E_j := \mathbb{R}e_j$, $W_k := \oplus_{j=1}^{k} E_j$, $V_k := \overline{\oplus_{j=k+1}^{\infty} E_j}$. Let $W = W_k$, $V = V_k$, clearly $E = W \oplus V$ and $\dim W < \infty$.

Next, we verify that I satisfies (ii) in Lemma 3.1. By (f_2), for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \epsilon \int_{\mathbb{R}^3} u^2 \mathrm{d}x - C_\epsilon \int_{\mathbb{R}^3} u^{\iota+1}\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - C\left(\epsilon\|u\|^2 + C_\epsilon \|u\|^{\iota+1}\right). \end{aligned}$$

Therefore, there exists $\rho > 0$ small enough, $\alpha > 0$ such that $I(u) \geq \alpha > 0$ as $\|u\| = \rho$.

Now, we verify that I satisfies (iii) in Lemma 3.1. If $I(u) > 0$ for any finite-dimensional subspace $Y \subset E$, then we can find a sequence $\{u_n\}$ such that $\|u_n\| \to +\infty$ and $I(u_n) > 0$. Let $v_n = \frac{u_n}{\|u_n\|}$, from the proof in Lemma 3.2, we have

$$0 = \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|^4} \leq \lim_{n\to\infty} \left(\frac{1}{2\|u_n\|^2} + \frac{\|u_n\|_2^4}{4a\|u_n\|^4} - \int_{\mathbb{R}^N} \frac{F(u)}{u_n^4} v_n^4 \mathrm{d}x \right) = -\infty,$$

as $n \to +\infty$, which is a contradiction. This shows that there exists $R = R(Y) > 0$ such that $I(u) \leq 0$ on $Y \backslash B_R$. The proof is completed. □

4 Existence of Ground State Solutions for $p \in [4, 6)$

In this section, to solve the problem (1.5), we apply Jeanjean's trick [17] and give a family of energy functionals

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} \left[\frac{1}{p}|u|^p + H(u)\right] \mathrm{d}x.$$

Lemma 4.1 [24] *Assume that $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^3)$. If*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^2 \,\mathrm{d}x = 0 \quad \text{for some} \quad r > 0,$$

then $u_n \to 0$ in $L^q\left(\mathbb{R}^3\right)$ for all $q \in (2, 6)$.

Lemma 4.2 [17] *Let $(X, \|\cdot\|)$ be a Banach space and $T \subset \mathbb{R}^+$ be an interval. Consider a family of $\mathcal{C}^1$ functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u) \quad \forall \lambda \in T,$$

with $B(u) \geq 0$ *and either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to +\infty$. *If there are two points* $v_1, v_2 \in X$ *such that*

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \forall \lambda \in T,$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in T$, *there exists a bounded* $(PS)_{c_\lambda}$ *sequence in* X, *and the mapping* $\lambda \to c_\lambda$ *is non-increasing and continuous from the left.*

Lemma 4.3 *Suppose that* V, h *satisfy* (V_1)–(V_2) *and* (h_1)–(h_2). *Then, for almost every* $\lambda \in [\frac{1}{2}, 1]$, *there is a bounded sequence* $\{v_m\}$, *such that* $I_\lambda(v_m) \to c_\lambda$ *in* E *and* $I'_\lambda(v_m) \to 0$ *in the dual* E^* *of* E.

Proof Obviously that

$$\int_{\mathbb{R}^3} \left[\frac{1}{p}|u|^p + H(u)\right] \mathrm{d}x \geq 0,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \to +\infty \quad \text{as} \quad \|u\| \to +\infty.$$

Let $u_\tau = \tau^2 u(\tau\cdot)$, then

$$\begin{aligned} I_\lambda(u_\tau) = &\frac{\tau^3}{2}\|\nabla u\|_2^2 + \frac{\tau}{2}\int_{\mathbb{R}^3} V(x)u^2 \,\mathrm{d}x + \frac{\tau^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-\frac{|x-y|}{\tau a}}}{|x-y|} u^2(x)u^2(y) \,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{\lambda\tau^{2p-3}}{p}\|u\|_p^p - \lambda\tau^3\int_{\mathbb{R}^3} H(u)\,\mathrm{d}x \\ \leq &\frac{\tau^3}{2}\|\nabla u\|_2^2 + \frac{\tau}{2}\int_{\mathbb{R}^3} V(x)u^2 \,\mathrm{d}x + \frac{\tau^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{\lambda\tau^{2p-3}}{p}\|u\|_p^p. \end{aligned}$$

Since $p \in [4, 6)$, there exists $\tau_0 > 0$ large enough such that $I_\lambda(\tau_0 u) < 0$. By taking $v = u_{\tau_o}$, we have $I_\lambda(v) < 0$.

Moreover, it follows from (h_1) that

$$|h(u)| \leq \epsilon|u| + C_\epsilon|u|^\varrho, \quad \text{for any } \epsilon > 0.$$

By simple calculations, we derive

$$\int_{\mathbb{R}^3} H(u)\,\mathrm{d}x \leq \frac{\epsilon}{2}\|u\|_2^2 + \frac{C_\epsilon}{\varrho+1}\|u\|_{\varrho+1}^{\varrho+1}.$$

Hence,

$$
\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \lambda \frac{1}{p}\|u\|_p^p - \lambda \int_{\mathbb{R}^3} H(u)\,\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_2^2 - \frac{1}{p}\|u\|_p^p - \frac{\epsilon}{2}\|u\|_2^2 - \frac{C_\epsilon}{\varrho+1}\|u\|_{\varrho+1}^{\varrho+1} \\
&\geq \frac{1-\epsilon C}{2}\|u\|^2 - \frac{C}{p}\|u\|^p - \frac{CC_\epsilon}{\varrho+1}\|u\|^{\varrho+1}.
\end{aligned}
$$

Then, there exists $\rho > 0$ small enough such that $I_\lambda(u) > 0$ as $\|u\| = \rho$. Now, Lemma 4.2 leads to the conclusion. □

Lemma 4.4 *If $\{v_m\}$ is a bounded sequence in E obtained by Lemma 4.3, then*

$$
\lim_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_m|^2 \,\mathrm{d}x > 0.
$$

Proof Assume that $\lim_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_m|^2 \,\mathrm{d}x = 0$. From Lemma 4.1, we have $v_m \to 0$ in $L^q\left(\mathbb{R}^3\right)$ for all $q \in (2, 6)$.

Hence,

$$
\int_{\mathbb{R}^3} h(v_m)v_m \,\mathrm{d}x \leq \epsilon\|v_m\|_2^2 + C_\epsilon\|v_m\|_{\varrho+1}^{\varrho+1} = o(1).
$$

Moreover, by the Hardy–Littlewood–Sobolev inequality (see [18]), we obtain

$$
\int_{\mathbb{R}^3} \phi_{v_m} v_m^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v_m^2(x)v_m^2(y)}{|x-y|} \,\mathrm{d}x\,\mathrm{d}y \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|v_m\|_{\frac{12}{5}}^4 = o(1).
$$

Thus, we have

$$
\begin{aligned}
o(1) = I_\lambda'(v_m)[v_m] &= \|v_m\|^2 + \int_{\mathbb{R}^3} \phi_{v_m} v_m^2 \,\mathrm{d}x - \lambda\|v_m\|_p^p - \lambda \int_{\mathbb{R}^3} h(v_m)v_m \,\mathrm{d}x \\
&= \|v_m\|^2 + o(1),
\end{aligned}
$$

and then

$$
\begin{aligned}
c_\lambda + o(1) = I_\lambda(v_m) = \frac{1}{2}\|v_m\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{v_m} v_m^2 \,\mathrm{d}x - \frac{\lambda}{p}\|v_m\|_p^p \\
- \lambda \int_{\mathbb{R}^3} H(v_m)\,\mathrm{d}x = o(1),
\end{aligned}
$$

which is a contradiction. □

Lemma 4.5 *If $\{v_m\} \subset E$ is the sequence obtained by Lemma 4.3, then for a.e. $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a sequence of points $\{y_m\} \subset \mathbb{R}^3$, set $u_m(x) := v_m(x - y_m)$, we have*

(i) $u_m \rightharpoonup u_\lambda \not\equiv 0$ *in E;*
(ii) $I'_\lambda(u_\lambda) = 0$ *in E^*;*
(iii) $I_\lambda(u_\lambda) \le c_\lambda$ *in E;*
(iv) *there exists $M > 0$ such that $I_\lambda(u_\lambda) \ge M$.*

Proof Putting together Lemmas 4.3 and 4.4, we know that for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a bounded sequence $\{v_m\}$ which satisfies $I_\lambda(v_m) \to c_\lambda$ in E and $I'_\lambda(v_m) \to 0$ in E^* as $m \to +\infty$. Furthermore, by Lemma 4.4, there exists a sequence of points $\{y_m\} \subset \mathbb{R}^3$ and $\alpha > 0$, such that

$$\lim_{m\to+\infty} \int_{B_1(y_m)} v_m^2 \,\mathrm{d}x \ge \alpha > 0. \tag{4.1}$$

Let $u_m(x) := v_m(x - y_m)$. By the invariance translations of I_λ, as $m \to +\infty$, we have that $I_\lambda(u_m) \to c_\lambda$ in E and $I'_\lambda(u_m) \to 0$ in E^*. Since $\{u_m\}$ is bounded, there exists $u_\lambda \in E$ such that

$$\begin{cases} u_m \rightharpoonup u_\lambda & \text{in } E; \\ u_m \to u_\lambda & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3), \ \forall s \in [1, 6]; \\ u_m \to u_\lambda & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{4.2}$$

In the following, we complete the proof of this lemma.

(i) It follows from (4.1) that

$$C\|u_\lambda\|^2 \ge \|u_\lambda\|_2^2 \ge \int_{B_1(0)} u_\lambda^2 \,\mathrm{d}x = \lim_{m\to+\infty} \int_{B_1(0)} u_m^2 \,\mathrm{d}x \ge \alpha > 0,$$

and thus $u_\lambda \not\equiv 0$ in E.

(ii) We only need to show that $\langle I'_\lambda(u_\lambda), \varphi\rangle = 0$ for any $\varphi \in E$. Observe that

$$\begin{aligned} &\langle I'_\lambda(u_m), \varphi\rangle - \langle I'_\lambda(u_\lambda), \varphi\rangle \\ &= \int_{\mathbb{R}^3} \left((\nabla u_m - \nabla u_\lambda)\varphi + V(x)(u_m - u_\lambda)\varphi\right) \mathrm{d}x + \int_{\mathbb{R}^3} \left(\phi_{u_m} u_m - \phi_{u_\lambda} u_\lambda\right)\varphi \,\mathrm{d}x \\ &\quad - \lambda \int_{\mathbb{R}^3} (f(u_m) - f(u_\lambda))\varphi \,\mathrm{d}x - \lambda \int_{\mathbb{R}^3} \left(|u_m|^{p-2}u_m - |u_\lambda|^{p-2}u_\lambda\right)\varphi \,\mathrm{d}x. \end{aligned}$$

Moreover,

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \left(\phi_{u_m} u_m - \phi_{u_\lambda} u_\lambda\right)\varphi \,\mathrm{d}x\right| &\le \int_{\mathbb{R}^3} \phi_{u_m}|u_m - u_\lambda||\varphi| \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} |\phi_{u_m} - \phi_{u_\lambda}||u_\lambda\varphi| \,\mathrm{d}x := J_1 + J_2. \end{aligned}$$

By Lemma 2.4, $\{\phi_{u_m}\}$ is bounded in $L^6(\mathbb{R}^3)$ and $\phi_{u_m} \rightharpoonup \phi_{u_\lambda}$ in $L^6(\mathbb{R}^3)$. Using the Hölder inequality and the strong convergence of u_m to u_λ in $L^3(\mathbb{R}^3)$, we get

$$J_1 \leq \|\phi_{u_m}\|_6 \|u_m - u_\lambda\|_3 \|\varphi\|_2 \to 0.$$

Since $\varphi \in E$, for any $\epsilon > 0$ there exists $R > 0$ such that

$$\left(\int_{B_R^c(0)} |\varphi|^s \, \mathrm{d}x \right)^{\frac{1}{s}} < \epsilon, \quad s \in [2, 6]. \tag{4.3}$$

Hence,

$$\begin{aligned} J_2 &= \int_{B_R(0)} |\phi_{u_m} - \phi_{u_\lambda}||u_\lambda \varphi| \, \mathrm{d}x + \int_{B_R^c(0)} |\phi_{u_m} - \phi_{u_\lambda}||u_\lambda \varphi| \, \mathrm{d}x \\ &\leq \|\phi_{u_m} - \phi_{u_\lambda}\|_{L^6(B_R(0))} \|u_\lambda\|_2 \|\varphi\|_3 + \|\phi_{u_m} - \phi_{u_\lambda}\|_6 \|u_\lambda\|_2 \|\varphi\|_{L^3(B_R^c(0))} \\ &\to 0, \end{aligned}$$

as $m \to +\infty$ and $R \to +\infty$.
Furthermore, notice that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(|u_m|^{p-2} u_m - |u_\lambda|^{p-2} u_\lambda \right) \varphi \, \mathrm{d}x \right| \\ &\leq \int_{B_R(0)} \left| |u_m|^{p-2} u_m - |u_\lambda|^{p-2} u_\lambda \right| |\varphi| \, \mathrm{d}x \\ &\quad + \int_{B_R^c(0)} \left| |u_m|^{p-2} u_m - |u_\lambda|^{p-2} u_\lambda \right| |\varphi| \, \mathrm{d}x \\ &:= J_3 + J_4. \end{aligned}$$

From (4.2), we infer that there exists $Q(x) \in L^p(B_R(0))$ such that $|u_m| \leq Q(x)$. It follows from dominated convergence theorem that $J_3 \to 0$ as $m \to 0$.
On the other hand, as $R \to \infty$, we obtain

$$\begin{aligned} J_4 &\leq \int_{B_R^c(0)} |u_m|^{p-1} |\varphi| + \int_{B_R^c(0)} |u_\lambda|^{p-1} |\varphi| \, \mathrm{d}x \\ &\leq \|u_m\|_p^{p-1} \|\varphi\|_{L^p(B_R^c(0))} + \|u_\lambda\|_p^{p-1} \|\varphi\|_{L^p(B_R^c(0))} \\ &\leq \epsilon C \to 0. \end{aligned}$$

Moreover, as $u_m \to u_\lambda$ a.e. in $\mathbb{R}^3$, by continuity,

$$h(u_m) \to h(u_\lambda) \quad \text{a.e. in} \quad \mathbb{R}^3.$$

It follows from (h_1) that $\{h(u_m)\}$ is bounded in $L^{\frac{6}{5}}(\mathbb{R}^3)$, and so $h(u_m) \rightharpoonup h(u_\lambda)$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$. Hence, as $R \to \infty$, we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3} (h(u_m) - h(u_\lambda))\, \varphi \,\mathrm{d}x \right| \\
&\leq \int_{B_R(0)} |h(u_m) - h(u_\lambda)|\, |\varphi| \,\mathrm{d}x + \int_{B_R^c(0)} |h(u_m) - h(u_\lambda)|\, |\varphi| \,\mathrm{d}x \\
&\leq \int_{B_R(0)} |h(u_m) - h(u_\lambda)|\, |\varphi| \,\mathrm{d}x + \int_{B_R^c(0)} |h(u_m)|\, |\varphi| \,\mathrm{d}x \\
&\quad + \int_{B_R^c(0)} |h(u_\lambda)|\, |\varphi| \,\mathrm{d}x \\
&\leq \|h(u_m) - h(u_\lambda)\|_{L^{\frac{6}{5}}(B_R(0))} \|\varphi\|_6 + \|u_m\|_{\frac{6}{5}} \|\varphi\|_{L^6(B_R^c(0))} \\
&\quad + \|u_\lambda\|_{\frac{6}{5}} \|\varphi\|_{L^6(B_R^c(0))} \\
&\leq o(1) + \epsilon C \to 0.
\end{aligned}
$$

Now, we get $I'_\lambda(u_\lambda) = 0$.

(iii) It follows from (h_3) that

$$
\begin{aligned}
&\frac{1}{4} h(u)u - H(u) + \frac{\theta_0 V(x)}{4} u^2 \\
&\quad = \int_0^1 \left[\frac{h(u)}{u^3} - \frac{h(su)}{su^3} + \frac{\theta_0 V(x)(1-s^2)}{(su)^2} \right] s^3 u^4 \mathrm{d}s \geq 0.
\end{aligned}
$$

Hence, by Fatou's lemma we get

$$
\begin{aligned}
c_\lambda &= \lim_{m\to+\infty} \left[I_\lambda(u_m) - \frac{1}{4} \left\langle I'_\lambda(u_m), u_m \right\rangle \right] \\
&\geq \left(\frac{\lambda}{4} - \frac{\lambda}{p} \right) \liminf_{m\to+\infty} \|u_m\|_p^p + \frac{1}{4} \liminf_{m\to+\infty} \left(\|u_m\|^2 - \lambda\theta_0 \int_{\mathbb{R}^3} V(x)|u_m|^2 \mathrm{d}x \right) \\
&\quad + \lambda \liminf_{m\to+\infty} \int_{\mathbb{R}^3} \left(\frac{1}{4} h(u_m)u_m - H(u_m) + \frac{\theta_0 V(x)|u_m|^2}{4} \right) \mathrm{d}x \\
&\geq \left(\frac{\lambda}{4} - \frac{\lambda}{p} \right) \|u_\lambda\|_p^p + \frac{1}{4} \left(\|u_\lambda\|^2 - \lambda\theta_0 \int_{\mathbb{R}^3} V(x)|u_\lambda|^2 \mathrm{d}x \right) \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{4} h(u_\lambda)u_\lambda - H(u_\lambda) + \frac{\theta_0 V(x)|u_\lambda|^2}{4} \right) \mathrm{d}x \\
&= I_\lambda(u_\lambda) - \frac{1}{4} \left\langle I'_\lambda(u_\lambda), u_\lambda \right\rangle = I_\lambda(u_\lambda) > 0.
\end{aligned}
$$

(iv) Recall that $C\|u_\lambda\|^2 \geq \alpha$, we infer

$$
I_\lambda(u_\lambda) \geq \frac{1-\theta_0}{4} \|u_\lambda\|^2 \geq \frac{\alpha}{C} := M.
$$

The proof is completed. □

Now, according to Lemmas 4.3 and 4.5, there exists a sequence $\{(\lambda_n, u_{\lambda_n})\} \subset [\frac{1}{2}, 1] \times E$, such that

(i) $\lambda_n \to 1$ as $n \to +\infty$;
(ii) $u_{\lambda_n} \not\equiv 0$, $M \le I_{\lambda_n}(u_{\lambda_n}) \le c_{\lambda_n}$;
(iii) $I'_{\lambda_n}(u_{\lambda_n}) = 0$.

Proof of Theorem 1.6 By Lemma 4.5, we have

$$
\begin{aligned}
c_1 \ge I_{\lambda_n}(u_{\lambda_n}) &= I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{4} I'_{\lambda_n}(u_{\lambda_n})[u_{\lambda_n}] \\
&= \frac{1}{4}\|u_{\lambda_n}\|^2 + (\frac{\lambda}{4} - \frac{\lambda}{p})\|u_{\lambda_n}\|_p^p + \lambda_n \int_{\mathbb{R}^3} \left(\frac{1}{4} h(u_{\lambda_n})u_{\lambda_n} - H(u_{\lambda_n})\right) \mathrm{d}x \\
&\ge \frac{1}{4}\|u_{\lambda_n}\|^2 - \frac{\lambda_n \theta_0}{4} \int_{\mathbb{R}^3} V(x)|u_{\lambda_n}|^2 \mathrm{d}x \\
&\ge \frac{1-\theta_0}{4}\|u_{\lambda_n}\|^2,
\end{aligned}
$$

which implies that there exists a constant $K > 0$ such that$\|u_{\lambda_n}\| \le K$.

Using the facts that for any $\varphi \in E$,

$$
\langle I'(u_{\lambda_n}), \varphi\rangle = \langle I'_{\lambda_n}(u_{\lambda_n}), \varphi\rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} h(u_{\lambda_n}) \varphi \, \mathrm{d}x,
$$

$$
I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} H(u_{\lambda_n}) \, \mathrm{d}x,
$$

and $\{u_{\lambda_n}\}$ is bounded in E, we obtain that $M \le \lim_{n\to+\infty} I(u_{\lambda_n}) \le c_1$ and $\lim_{n\to+\infty} I'(u_{\lambda_n}) = 0$. Up to a subsequence, there exists a subsequence still denoted by $\{u_{\lambda_n}\}$ and $u_0 \in E$ such that $u_{\lambda_n} \rightharpoonup u_0$ in E. By using the preceding method in Lemma 4.5, we can obtain the existence of a nontrivial solution u_0 for I such that $I'(u_0) = 0$ and $M \le I(u_0) \le c_1$. Thus, u_0 is a nontrivial solution of (1.5). Define $m := \inf\{I(u) : u \not\equiv 0, I'(u) = 0\}$. Let $\{u_n\}$ be a sequence such that $I'(u_n) = 0$ and $I(u_n) \to m$. Similar to the argument in Lemma 4.5, we can prove that there exists $\bar{u} \in E$ such that $I'(\bar{u}) = 0$ and $I(\bar{u}) \le m$. By the definition of m, we have $m \le I(\bar{u})$. Hence, $I(\bar{u}) = m$, which shows that $\bar{u}$ is a ground state solution of (1.5). □

5 Existence of Positive Solutions for $p \in (2, 6)$

In this section, assume that (V_1)–(V_2) and (P_1)–(P_2) are satisfied and $p \in (2, 6)$. We prove the existence of positive solutions for problem (1.6).

By simple calculation, we have

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \frac{q}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x$$
$$- \int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|u|^6 + \frac{\mu}{p}|u|^p\right)\mathrm{d}x,$$
$$I'(u)[v] = \int_{\mathbb{R}^3}(\nabla u \nabla v + V(x)uv + q\phi_u uv)\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^3}\left(P(x)u^5 v + \mu|u|^{p-2}uv\right)\mathrm{d}x.$$

Lemma 5.1 [12] *If $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 \mathrm{d}x - \int_{\mathbb{R}^3}\phi_v v^2 \mathrm{d}x - \int_{\mathbb{R}^3}\phi_{v_n - v}(v_n - v)^2 \mathrm{d}x \to 0.$$

Lemma 5.2 *Let $\{v_n\}$ be a sequence such that $v_n \rightharpoonup v$ in E, then*

$$\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 \mathrm{d}x \to \int_{\mathbb{R}^3}\phi_v v^2 \mathrm{d}x \quad \text{as} \quad n \to +\infty.$$

Proof To prove $\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 \to \int_{\mathbb{R}^3}\phi_v v^2$, we only need to prove that

$$\left|\int_{\mathbb{R}^3}\left(\phi_{v_n}v_n^2 - \phi_v v^2\right)\mathrm{d}x\right| \le \left|\int_{\mathbb{R}^3}\left(\phi_{v_n}v_n - \phi_v v\right)v_n\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}\left(\phi_{v_n}v_n - \phi_v v\right)v\,\mathrm{d}x\right|$$
$$\le \int_{\mathbb{R}^3}\phi_{v_n}|v_n - v||v_n|\,\mathrm{d}x + \int_{\mathbb{R}^3}|\phi_{v_n} - \phi_v||v_n v|\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3}\phi_{v_n}|v_n - v||v|\,\mathrm{d}x + \int_{\mathbb{R}^3}|\phi_{v_n} - \phi_v||v|^2\,\mathrm{d}x$$
$$:= J_1 + J_2 + J_3 + J_4 \to 0.$$

We just give the proof of $J_1 \to 0$, and the proof of $J_3 \to 0$ can be proved in the same way. Lemma 2.4 and Theorem 1.2 imply that

$$J_1 \le \|\phi_{v_n}\|_6\|v_n - v\|_3\|v_n\|_2 \to 0 \quad \text{as} n \to +\infty.$$

Moreover, since $v \in E$, for any $\epsilon > 0$ there exists $R > 0$ such that

$$\left(\int_{B_R^c(0)}|v|^s\,\mathrm{d}x\right)^{\frac{1}{s}} < \epsilon, \quad s \in [2, 6].$$

Obviously,

$$J_2 = \int_{B_R(0)}|\phi_{v_n} - \phi_v||v_n v|\,\mathrm{d}x + \int_{B_R^c(0)}|\phi_{v_n} - \phi_v||v_n v|\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \|\phi_{v_n} - \phi_v\|_{L^6(B_R(0))}\|v_n\|_2\|v\|_3 + \|\phi_{v_n} - \phi_v\|_6\|v_n\|_2\|v\|_{L^3(B_R^c(0))} \\
&\to 0,
\end{aligned}
$$

as $m \to +\infty$ and $R \to +\infty$, and $J_4 \to 0$ can be proved in the same way. □

5.1 The Case $p \in [4, 6)$

Lemma 5.3 *Assume that* (V_1)–(V_2) *and* (P_1)–(P_2) *are satisfied. Then, the functional* I *verifies the mountain pass geometry, that is,*

(i) *there exist* $\alpha, \rho > 0$ *such that* $I(v) \geq \alpha$ *for all* $\|v\| = \rho$;
(ii) *there exists* $e \in H^1((R)^3)\backslash\{0\}$ *such that* $I(e) < 0$ *with* $\|e\| > \rho$.

Proof Since $P(x) \leq P(x_0)$, we have

$$
\begin{aligned}
I(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|u|^6 + \frac{\mu}{p}|u|^p\right)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{6}P(x_0)\|u\|_6^6 - \frac{\mu}{p}\|u\|_p^p \\
&\geq \frac{1}{2}\|u\|^2 - C\left(\frac{1}{6}P(x_0)\|u\|^6 + \frac{\mu}{p}\|u\|^p\right).
\end{aligned}
$$

Then, there exist $\alpha > 0$ and $\rho > 0$ small enough such that $I(u) \geq \alpha$ for all $\|u\| = \rho$.
Moreover, let $u_\tau = \tau u$, then

$$
I(u_\tau) = \frac{\tau^2}{2}\|u\|^2 + \frac{\tau^4 q}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \frac{\mu\tau^p}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x - \frac{\tau^6}{6}\int_{\mathbb{R}^3}P(x)|u|^6\,\mathrm{d}x.
$$

Hence, there exists $\tau_0 > 0$ large enough such that $I(\tau_0 u) < 0$. By taking $e = u_{\tau_o}$, we have $I(e) < 0$. □

Combine Lemma 5.3 and Mountain–Pass lemma, we infer that there exists a sequence $\{v_n\}$ in E at the level c, such that

$$
I(v_n) \to c \quad \text{and} \quad I'(v_n) \to 0, \quad \text{as} \quad n \to +\infty, \tag{5.1}
$$

where

$$
\begin{aligned}
&c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0, \\
&\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1((R)^3)) : \gamma(0) = 0, \gamma(1) = e\}.
\end{aligned}
$$

Lemma 5.4 *The sequence* $\{v_n\}$ *in* (5.1) *is bounded in* E *for* $p \in [4, 6)$.

Proof As $n \to +\infty$, we have

$$
c + o(1) = I(v_n) - \frac{1}{4}I'(v_n)[v_n]
$$

$$= \frac{1}{4}\|v_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} \mu|v_n|^p \mathrm{d}x + \left(\frac{1}{4} - \frac{1}{6}\right)\int_{\mathbb{R}^3} P(x)|v_n|^6 \mathrm{d}x$$
$$\geq \frac{1}{4}\|v_n\|^2.$$

This prove Lemma 5.4. □

Lemma 5.5 *The sequence* $\{v_n\}$ *in* (5.1) *is compact in* E *if*

$$c < \frac{1}{3} S^{\frac{3}{2}} \|P(x)\|_\infty^{-\frac{1}{2}}.$$

Proof As the sequence $\{v_n\}$ is given by (5.1), it satisfies that

$$I(v_n) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + V(x)v_n^2\right)\mathrm{d}x + \frac{q}{4}\int_{\mathbb{R}^3}\phi_{v_n} v_n^2 \mathrm{d}x$$
$$- \int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|v_n|^6 + \frac{\mu}{p}|v_n|^p\right)\mathrm{d}x = c + o(1), \tag{5.2}$$
$$I'(v_n)[v_n] = \int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + V(x)v_n^2\right)\mathrm{d}x + q\int_{\mathbb{R}^3}\phi_{v_n} v_n^2 \mathrm{d}x$$
$$- \int_{\mathbb{R}^3}\left(P(x)v_n^6 + \mu|v_n|^p\right)\mathrm{d}x = o(1). \tag{5.3}$$

Since the sequence $\{v_n\}$ is bounded, there exists $v \in E\backslash\{0\}$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } E; \\ v_n \to v & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3), \ \ \forall s \in [1,6]; \\ v_n \to v & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{5.4}$$

From $I'(v_n) \to 0$, we can obtain

$$I'(v)[v] = \int_{\mathbb{R}^3}\left(|\nabla v|^2 + V(x)v^2\right)\mathrm{d}x + q\int_{\mathbb{R}^3}\phi_v v^2 \mathrm{d}x$$
$$- \int_{\mathbb{R}^3}\left(P(x)|v|^6 + \mu|v|^p\right)\mathrm{d}x = 0. \tag{5.5}$$

Set $v_n' = v_n - v$. By Brezis–Lieb's lemma in [8],

$$I(v_n) - I(v) - I(v_n') \to 0 \quad \text{as} \quad n \to +\infty, \tag{5.6}$$
$$I'(v_n)[v_n] - I'(v)[v] - I'(v_n')[v_n'] \to 0 \quad \text{as} \quad n \to +\infty. \tag{5.7}$$

Theorem 1.2 and Lemmas 5.1, 5.2 and (5.4) imply that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|v_n'|^p \mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{v_n'}|v_n'|^2 \mathrm{d}x = 0 \quad \text{as} \quad n \to +\infty. \tag{5.8}$$

Putting (5.8) in (5.6) and (5.7), as $n \to +\infty$, we have

$$\begin{aligned} I(v) &= c + o(1) - I(v_n') \\ &= c + o(1) - \frac{1}{2}\|v_n'\|^2 - \frac{q}{4}\int_{\mathbb{R}^3} \phi_{v_n'}|v_n'|^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{1}{6}P(x)|v_n'|^6 + \frac{\mu}{p}|v_n'|^p\right) \mathrm{d}x \\ &= c - \frac{1}{2}\|v_n'\|^2 + \int_{\mathbb{R}^3} \frac{1}{6}P(x)|v_n'|^6 \mathrm{d}x + o(1), \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} o(1) &= I'(v)[v] + I'(v_n')[v_n'] \\ &= \int_{\mathbb{R}^3} \left(|\nabla v_n'|^2 + V(x)|v_n'|^2\right) \mathrm{d}x + q\int_{\mathbb{R}^3} \phi_{v_n'}|v_n'|^2 \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \left(P(x)|v_n'|^6 + \mu|v_n'|^p\right) \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left(|\nabla v_n'|^2 + V(x)|v_n'|^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} P(x)|v_n'|^6 \mathrm{d}x + o(1). \end{aligned}$$

Assume that $\int_{\mathbb{R}^3} P(x)|v_n'|^6 \mathrm{d}x \to l$ as $n \to +\infty$, then $\int_{\mathbb{R}^3} |\nabla v_n'|^2 + V(x)|v_n'|^2 \mathrm{d}x \to l$.

It follows from Sobolev inequality that

$$\begin{aligned} \int_{\mathbb{R}^3} \left(|\nabla v_n'|^2 + V(x)|v_n'|^2\right) \mathrm{d}x\mathrm{d}x &\geq S\left(\int_{\mathbb{R}^3} |v_n'|^6 \mathrm{d}x\right)^{\frac{1}{3}} \\ &\geq S\|P(x)\|_\infty^{-\frac{1}{3}}\left(\int_{\mathbb{R}^3} P(x)|v_n'|^6 \mathrm{d}x\right)^{\frac{1}{3}}. \end{aligned}$$

If $l > 0$, we have

$$l \geq S^{\frac{3}{2}}\|P(x)\|_\infty^{-\frac{1}{2}}. \tag{5.10}$$

(5.9), (5.10) and Lemma 5.4 imply that

$$I(v) \to c - \frac{1}{3}l \leq c - \frac{1}{3}S^{\frac{3}{2}}\|P(x)\|_\infty^{-\frac{1}{2}} < 0. \tag{5.11}$$

However, (5.5) shows that

$$\begin{aligned} I(v) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v_n'|^2 + V(x)|v_n'|^2\right) \mathrm{d}x + \frac{q}{4}\int_{\mathbb{R}^3} \phi_v v^2 \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \left(\frac{1}{6}P(x)|v|^6 + \frac{\mu}{p}|v|^p\right) \mathrm{d}x, \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^3}\left(P(x)|v|^6+\mu|v|^p\right)\mathrm{d}x-\int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|v|^6+\frac{\mu}{p}|v|^p\right)\mathrm{d}x\\
&\geq 0,
\end{aligned}
$$

which is a contradiction to (5.11), thus $l=0$. This concludes that $\{v_n\}$ is compact. □

Proof of Theorem 1.8 Assume that (V_1)–(V_2) and (P_1)–(P_2) are satisfied. For each $\epsilon>0$, x_0 is chosen such that $P(x_0)=\sup_{x\in\mathbb{R}^3}P(x)$, we consider the following function

$$
\omega_{\epsilon,x_0}(x)=\frac{(3\epsilon)^{\frac{1}{4}}}{\left(\epsilon+|x-x_0|^2\right)^{\frac{1}{2}}},
$$

which is a solution of the minimization problem $S=\inf\{\|\nabla v\|_2^2 : v\in\mathcal{D}^{1,2}(\mathbb{R}^3),\ \|v\|_6=1\}$, and ω_{ϵ,x_0} satisfies

$$
\frac{\|\nabla\omega_{\epsilon,x_0}\|_2^2}{\|\omega_{\epsilon,x_0}\|_6^2}=S.
$$

Let $\eta\in C_0^\infty(\mathbb{R}^3,[0,1])$ be a cutoff function such that $\eta(x)=1$ for $|x-x_0|\leq r$, $0<\eta(x)<1$ for $r<|x-x_0|\leq 2r$ and $\eta(x)=0$ for $|x-x_0|\geq 2r$. Set $v_\epsilon(x)=\eta(x)\omega_{\epsilon,x_0}(x)$, then as $\epsilon\to 0$, the following estimations hold (see [9,25])

$$
\begin{aligned}
\|\nabla v_\epsilon\|_2^2 &= S^{\frac{3}{2}}+O(\epsilon^{\frac{1}{2}}),\\
\|v_\epsilon\|_6^6 &= S^{\frac{3}{2}}+O(\epsilon^{\frac{3}{2}}),\\
\|v_\epsilon\|_s^s &= \begin{cases} O(\epsilon^{\frac{s}{4}}), & s\in[2,3),\\ O(\epsilon^{\frac{s}{4}}|\ln\epsilon|), & s=3,\\ O(\epsilon^{\frac{6-s}{4}}), & s\in(3,6). \end{cases}
\end{aligned}
$$

As $I(0)=0$, $\lim_{t\to+\infty}I(tv_\epsilon)=-\infty$, there exist $t_\epsilon>0$ and $M>0$ such that

$$
I(t_\epsilon v_\epsilon)=\sup_{t\geq 0}I(tv_\epsilon)\geq M \quad\text{and}\quad \frac{\mathrm{d}I(tv_\epsilon)}{\mathrm{d}t}|_{t=t_\epsilon}=0.
$$

Hence, we get

$$
\begin{aligned}
t_\epsilon\|\nabla v_\epsilon\|_2^2 &= t_\epsilon\left(S^{\frac{3}{2}}+O(\epsilon^{\frac{1}{2}})\right)\\
&\leq t_\epsilon\|\nabla v_\epsilon\|_2^2+t_\epsilon\int_{\mathbb{R}^3}V(x)v_\epsilon^2\mathrm{d}x+qt_\epsilon^3\int_{\mathbb{R}^3}\phi_{v_\epsilon}v_\epsilon^2\mathrm{d}x\\
&= t_\epsilon^5\int_{\mathbb{R}^3}P(x)v_\epsilon^6\mathrm{d}x+\frac{\mu}{t_\epsilon}\int_{\mathbb{R}^3}|t_\epsilon v_\epsilon|^p\mathrm{d}x\\
&\leq t_\epsilon^5P(x_0)\left(S^{\frac{3}{2}}+O(\epsilon^{\frac{3}{2}})\right)+\mu\left(t_\epsilon O(\epsilon^{\frac{1}{2}})+t_\epsilon^5S^{\frac{3}{2}}+t_\epsilon^5O(\epsilon^{\frac{3}{2}})\right),
\end{aligned}
$$

which shows that there exists a constant $A_1 > 0$ such that $t_\epsilon \geq A_1$.

Moreover,

$$\begin{aligned} M &\leq I(t_\epsilon v_\epsilon) \leq \frac{t_\epsilon^2}{2}\|\nabla v_\epsilon\|_2^2 + \frac{t_\epsilon^2}{2}\left(\max_{x\in B_{2r}(x_0)} V(x)\right)\|v_\epsilon\|_2^2 + \frac{qt_\epsilon^4}{4a}\|v_\epsilon\|_2^4 \\ &\quad - \frac{t_\epsilon^6}{6}\left(\min_{x\in B_{2r}(x_0)} P(x)\right)\|v_\epsilon\|_6^6 \\ &= \frac{t_\epsilon^2}{2}\left(S^{\frac{3}{2}} + O(\epsilon^{\frac{1}{2}}) + \left(\max_{x\in B_{2r}(x_0)} V(x)\right) O(\epsilon^{\frac{1}{2}})\right) + \frac{qt_\epsilon^4}{4a}\left(O(\epsilon^{\frac{1}{2}})\right)^2 \\ &\quad - \frac{t_\epsilon^6}{6}\left(\min_{x\in B_{2r}(x_0)} P(x)\right)\left(S^{\frac{3}{2}} + O(\epsilon^{\frac{3}{2}})\right), \end{aligned}$$

which gives that there exists a constant $A_2 > 0$ such that $t_\epsilon \leq A_2$.

Next, we claim that

$$\sup_{t>0}\left(\frac{t_\epsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2 \mathrm{d}x - \frac{t_\epsilon^6}{6}\int_{\mathbb{R}^3}P(x)|v_\epsilon|^6 \mathrm{d}x\right) \leq \frac{1}{3}S^{\frac{3}{2}}\|P\|_\infty^{-\frac{1}{2}} + O(\epsilon^{\frac{1}{2}}).$$

Choose $\delta > 0$ small enough such that $|x - x_0| < \delta\epsilon^{\frac{1}{2}} < r$. Using the condition (P_2), there exists $\gamma > 0$ such that

$$|P(x) - P(x_0)| \leq \gamma|x - x_0|^2, \quad \text{for} \quad |x - x_0| < \delta\epsilon^{\frac{1}{2}},$$

hence

$$\begin{aligned} \int_{\mathbb{R}^3}|P(x) - P(x_0)||v_\epsilon|^6 \mathrm{d}x &\leq \int_{|x-x_0|<\delta\epsilon^{\frac{1}{2}}} \gamma|x - x_0|^2|v_\epsilon|^6 \mathrm{d}x \\ &\quad + C\int_{|x-x_0|\geq\delta\epsilon^{\frac{1}{2}}} |v_\epsilon|^6 \mathrm{d}x \\ &\leq \gamma\delta\epsilon\|v_\epsilon\|_6^6 + C\epsilon^{\frac{3}{2}}\int_{|x-x_0|=\delta\epsilon^{\frac{1}{2}}}^{|x-x_0|=2r} \frac{1}{\left(\epsilon + |x - x_0|^2\right)^3}\mathrm{d}x \\ &\leq C\epsilon + O(\epsilon^{\frac{5}{2}}) + O(\epsilon^2). \end{aligned}$$

Then, we have that

$$\begin{aligned} &\frac{t_\epsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2 \mathrm{d}x - \frac{t_\epsilon^6}{6}\int_{\mathbb{R}^3}P(x)|v_\epsilon|^6 \mathrm{d}x \\ &= \frac{t_\epsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2 \mathrm{d}x - \frac{t_\epsilon^6}{6}\int_{\mathbb{R}^3}P(x_0)|v_\epsilon|^6 \mathrm{d}x + \int_{\mathbb{R}^3}|P(x_0) - P(x)||v_\epsilon|^6 \mathrm{d}x \\ &\leq \frac{1}{3}S^{\frac{3}{2}}\|P\|_\infty^{-\frac{1}{2}} + O(\epsilon^{\frac{1}{2}}) + O(\epsilon^{\frac{5}{2}}) + O(\epsilon^{\frac{3}{2}}) + C\epsilon + O(\epsilon^2), \end{aligned}$$

which implies the claim holds.

Thus,

$$
\begin{aligned}
I(t_\epsilon v_\epsilon) &= \frac{t_\epsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2\mathrm{d}x + \frac{t_\epsilon^2}{2}\int_{\mathbb{R}^3}V(x)v_\epsilon^2\mathrm{d}x + \frac{qt_\epsilon^4}{4}\int_{\mathbb{R}^3}\phi_{v_\epsilon}v_\epsilon^2\mathrm{d}x\\
&\quad -\frac{\mu t_\epsilon^p}{p}\int_{\mathbb{R}^3}|v_\epsilon|^p\mathrm{d}x - \frac{t_\epsilon^6}{6}\int_{\mathbb{R}^3}P(x)|v_\epsilon|^6\mathrm{d}x\\
&\le \frac{1}{3}S^{\frac{3}{2}}\|P\|_\infty^{-\frac{1}{2}} + \frac{qA_2^4}{4a}\|v_\epsilon\|_2^4 - \mu\frac{A_1^p}{p}\|v_\epsilon\|_p^p\\
&\quad + O(\epsilon^{\frac{1}{2}}) + C\epsilon + O(\epsilon^{\frac{3}{2}}) + O(\epsilon^{\frac{5}{2}}) + O(\epsilon^2)\\
&\le \frac{1}{3}S^{\frac{3}{2}}\|P\|_\infty^{-\frac{1}{2}} + O(\epsilon^{\frac{1}{2}}) + O(\epsilon) + O(\epsilon^{\frac{3}{2}}) + O(\epsilon^{\frac{5}{2}}) + O(\epsilon^2) - C\mu\epsilon^{\frac{6-p}{4}}\\
&:= \frac{1}{3}S^{\frac{3}{2}}\|P\|_\infty^{-\frac{1}{2}} + A.
\end{aligned}
$$

If $4 < p < 6$ with $\mu > 0$ or $p = 4$ with μ sufficient large, we have $A < 0$.

Combine Lemmas 5.2 and 5.4, we can easily verity that if there exists $v_0 \in E$ and $v_0 \not\equiv 0$ such that

$$
\sup_{t\ge 0} I(tv_0) < \frac{1}{3}S^{\frac{3}{2}}\|P(x)\|_\infty^{-\frac{1}{2}}. \tag{5.12}
$$

Then, the problem (1.6) has at least one positive solution. Taking $v_0 = v_\epsilon$ for ϵ small enough, then (5.12) holds, this completes our proof. □

5.2 The Case of $p \in (2, 4)$

Inspired by [12], we consider the truncated functional

$$
\begin{aligned}
I_T(u) &:= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \frac{q}{4}\chi\left(\frac{\|u\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|u|^6 + \frac{\mu}{p}|u|^p\right)\mathrm{d}x,
\end{aligned}
$$

where $\chi(s) = 1$ for $0 \le s \le 1$, $\chi(s) \in (0, 1)$ for $1 \le s \le 2$, $\chi(s) = 0$ for $2 \le s$ and $\|\chi'\|_\infty \le 2$.

Lemma 5.6 *The functional I_T satisfies the mountain pass geometry:*

(i) *there exist $\alpha, \rho > 0$ such that $I(u) \ge \alpha$ for all $\|u\| = \rho$;*
(ii) *there exists $e \in H^1((R)^3)\backslash\{0\}$ such that $I(e) < 0$ with $\|e\| > \rho$.*

Proof As $q > 0$, we have

$$
I_T(u) \ge \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^3}\left(\frac{1}{6}P(x)|u|^6 + \frac{\mu}{p}|u|^p\right)\mathrm{d}x
$$

$$\geq \frac{1}{2}\|u\|^2 - C\left(\frac{P(x_0)}{6}\|u\|^6 + \frac{\mu}{p}\|u\|^p\right).$$

Then, there exists $\rho > 0$ small enough such that $I_T(u) > 0$ as $\|u\| = \rho$.

In addition, consider $u_\tau = \tau u$, we get

$$I_T(\tau u) = \frac{\tau^2}{2}\|u\|^2 + \frac{q\tau^4}{4}\chi\left(\tau^2\frac{\|u\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x \\ - \int_{\mathbb{R}^3}\left(\frac{\tau^6}{6}P(x)|u|^6 + \frac{\mu\tau^p}{p}|u|^p\right)\mathrm{d}x,$$

and there exists $\tau_0 > 0$ large enough such that $I_\lambda(\tau_0 u) < 0$. By taking $v = u_{\tau_o}$, we have $I_T(v) < 0$. □

By mountain pass theorem, there exists a sequence $\{u_n\}$ in E at the level c_T, such that

$$I_T(u_n) \to c_T \quad \text{and} \quad I_T'(u_n) \to 0, \quad \text{as} \quad n \to +\infty, \tag{5.13}$$

where

$$c_T := \inf_{\gamma\in\Gamma_T}\max_{t\in[0,1]} I_T(\gamma(t)) > 0,$$
$$\Gamma_T = \{\gamma \in \mathcal{C}([0,1], H^1((R)^3)) : \gamma(0) = 0, I_T(\gamma(1)) < 0\}.$$

Remark 5.7 It follows from $I_T \leq I$ that $c_T \leq c$, where the constant c is given in (5.1).

Lemma 5.8 *There exist $T_0 > 0$ independent of q and $q_* > 0$ dependent on T_0 such that if $q < q_*$, then the sequence $\{u_n\}$ in (5.13) satisfies*

$$\limsup_n \|u_n\| < T_0.$$

Proof Similar to the proof in [12]. Arguing by contradiction, suppose for any $T > 0$, there exists $q > 0$ such that

$$\limsup_n \|u_n\| \geq T. \tag{5.14}$$

Notice that

$$pI_T(u_n) - I_T'(u_n)[u_n] = \left(\frac{p}{2} - 1\right)\|u_n\|^2 + \left(\frac{pq}{4} - q\right)\chi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \\ - \frac{q\|u_n\|^2}{2T^2}\chi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \\ - \left(\frac{p}{6} - 1\right)\int_{\mathbb{R}^3}P(x)|u_n|^6\mathrm{d}x. \tag{5.15}$$

Let e be as in Lemma 5.6. As $n \to +\infty$, we have

$$\begin{aligned} I_T(u_n) &\le 2c_T \le 2 \max_{t\in[0,1]} I_T(te) \\ &\le \|e\|^2 \max_{t\in[0,1]} t^2 + \frac{q}{2} \max_{t\in[0,1]} \left[t^4 \chi\left(\frac{t^2\|e\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_e e^2 \mathrm{d}x \right] \\ &:= C_1 + \frac{q}{2} J_1. \end{aligned}$$

If $t^2\|e\|^2 > 2T^2$ then $J_1 = 0$ and if $t^2\|e\|^2 \le 2T^2$, then

$$J_1 \le \frac{4T^4}{\|e\|^4} \int_{\mathbb{R}^3} \phi_e e^2 \mathrm{d}x \le \frac{4T^4}{a\|e\|^4} \|e\|_2^4.$$

By Theorem 1.2, we have

$$I_T(u_n) \le C_1 + C_2 q T^4. \tag{5.16}$$

Through similar discussions, we obtain

$$\chi\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \le C_3 T^4, \tag{5.17}$$

and

$$\|u_n\|^2 \chi'\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \le \|u_n\|^2 \left|\chi'\left(\frac{\|u_n\|^2}{T^2}\right)\right| \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \le C_4 T^6. \tag{5.18}$$

Combining (5.14)–(5.18), we get

$$\begin{aligned} &C_5 T^2 - T \\ &\le \left(\frac{p}{2} - 1\right) \|u_n\|^2 - I_T'(u_n)[u_n] \\ &\le p I_T(u_n) + \left(q^2 - \frac{pq^2}{4}\right) \chi\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \\ &\quad + \frac{q\|u_n\|^2}{2T^2} \chi'\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \\ &\quad - \left(1 - \frac{p}{6}\right) \int_{\mathbb{R}^3} P(x)|u_n|^6 \mathrm{d}x \\ &\le p I_T(u_n) + \left(q - \frac{pq}{4}\right) \chi\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \\ &\quad + \frac{q\|u_n\|^2}{2T^2} \chi'\left(\frac{\|u_n\|^2}{T^2}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \\ &\le C_6 + q C_7 T^4. \end{aligned} \tag{5.19}$$

Obviously, the inequality (5.19) does not hold if q small enough and for large T. □

Remark 5.9 Since $\limsup_{n} \|u_n\| < T_0$, we have $I_{T_0}(u_n) = I(u_n)$.

Proof of Theorem 1.9 Assume that (V_1)–(V_2) and (P_1)–(P_2) are satisfied. From Lemmas 5.6, 5.8 and Remark 5.9, there exists $q_* > 0$ such that if $q \in (0, q_*)$ then I possesses a bounded (PS) sequence $\{u_n\}$ at level c_{T_0}. Moreover, we already know that if $c_{T_0} < \frac{1}{3} S^{\frac{3}{2}} \|P(x)\|_\infty^{-\frac{1}{2}}$, then $\{u_n\}$ is compact in E. It remains to show that there exists $v_0 \in E$ and $v_0 \not\equiv 0$ such that

$$\sup_{t \geq 0} I(tv_0) < \frac{1}{3} S^{\frac{3}{2}} \|P(x)\|_\infty^{-\frac{1}{2}},$$

for $p \in (2, 4)$ with μ sufficiently large. As its proof is similar with the Proof of Theorem 1.8, we omit it here. □

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