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Some Results on the 3-Vertex-Rainbow Index of a Graph

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Abstract

Let G be a nontrivial connected graph with a vertex-coloring c: $V(G) \rightarrow \{1, 2, \ldots, q\}, q \in N$. For a set $S \subseteq V(G)$ and $|S| \geq 2$, a subtree T of G satisfying $S \subseteq V(T)$ is said to be an S-Steiner tree or simply S-tree. The S-tree T is called a vertex-rainbow S-tree if the vertices of $V(T) \setminus S$ have distinct colors. Let k be a fixed integer with $2 \leq k \leq |V(G)|$, if every k-subset S of $V(G)$ has a vertex-rainbow S-tree, then G is said to be vertex-rainbow k-tree connected. The k-vertex-rainbow index of G, denoted by $rvx_k(G)$, is the minimum number of colors that are needed in order to make G vertex-rainbow k-tree connected. In this paper, we study the 3-vertex-rainbow index of unicyclic graphs and complementary graphs, respectively.

Keywords Vertex-coloring · S-tree · Vertex-rainbow S-tree · k-vertex-rainbow index

Mathematics Subject Classification 05C15 · 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of [1] for those not described here. In recent years, colored notions of connectivity in graphs become a new and active subject in graph theory. Stating from rainbow connection, rainbow-vertex connection and total rainbow con-

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nection appeared later. Many researchers are working in this field, and a lot of papers have been published in journals, see [2,3,5–12,15–19,21] for details. The reader also can see [14] for dynamic survey and [13] for a new monograph on this topic.

The concept of rainbow path was generalized to rainbow tree by Chartrand et al. [4]. For every set S of $V(G)$ and $|S| \geq 2$, a subtree T of G such that $S \subseteq V(T)$ is called an S-Steiner tree or simply S-tree. For every set S of $V(G)$, an S-tree is called a rainbow S-tree if any two edges of the tree are assigned distinct colors. Let k be a fixed integer with $2 \leq k \leq |V(G)|$, the edge-coloring c of G is said to be a k-rainbow coloring if for every set S of k vertices of G, there exists a rainbow S-tree. The k-rainbow index of a connected graph, denoted by $rx_k(G)$, is the minimum number of colors that are required in a k-rainbow coloring of G.

As a natural counterpart of the k-rainbow index, Mao introduced the concept of k-vertex-rainbow index $rvx_k(G)$ and investigated the Nordhaus–Gaddum problem with $k = 3$ in [20]. An S-tree T is called a vertex-rainbow S-tree or a vertex-rainbow tree connecting S if no two vertices of $V(T)\backslash S$ share same color. Let k be a fixed integer with $2 \leq k \leq |V(G)|$, the vertex-coloring c of G is said to be a k-vertex-rainbow coloring if for every set S of k vertices of G, there exists a vertex-rainbow S-tree. If such vertex-coloring c exists, then G is called vertex-rainbow k-tree connected. The k-vertex-rainbow index of a connected graph, denoted by $rvx_k(G)$, is the minimum number of colors that are needed in a k-vertex-rainbow coloring of G.

In 2010, Chartrand et al. [4] considered the 3-rainbow index of unicyclic graphs. In Sect. 2, we investigated the 3-vertex-rainbow index of unicyclic graphs. Moreover, we studied the 3-vertex-rainbow index of complementary graphs in Sect. 3.

2 The 3-Vertex-Rainbow Index of Unicyclic Graphs

For a subset X of $V(G)$, we use $G[X]$ to denote the induced subgraph by X. The distance between two vertices u and v in a connected graph G, denoted by $d(u, v)$, is the length of a shortest u-v path in G. The eccentricity of a vertex v is defined as $\mathrm{ecc}_G(v) := \max_{x \in V(G)} d(v, x)$. The Steiner distance $d(S)$ of a set S of $V(G)$ is the minimum number of edge of a tree in G containing S. The Steiner k-diameter $\mathrm{sdiam}_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. In [20], Mao obtained the following results.

Proposition 2.1 [20] *Let G be a nontrivial connected graph of order n. Then, $rvx_k(G) = 0$ if and only if $sdiam_k(G) = k - 1$.*

Proposition 2.2 [20] *Let G be a nontrivial connected graph of order n ($n \geq 5$), and let k be an integer with $2 \leq k \leq n$. Then, $0 \leq rvx_k(G) \leq n - 2$.*

Let $G_1 = C_1 \cup C_2 \cup \{uv\}$, where $C_1 = uu_1u_2 \ldots u_iu$ and $C_2 = vv_1v_2 \ldots v_jv$. For every k-subset S of $\{u_1, u_2, \ldots, u_i, v_1, v_2, \ldots, v_j\}$ satisfying $S \cap \{u_1, u_2, \ldots, u_i\} \neq \emptyset$ and $S \cap \{v_1, v_2, \ldots, v_j\} \neq \emptyset$, we know that every S-tree contains the two cut vertices u and v. The following observation holds.

Observation 2.3 *Let G be a connected graph of order n containing two cut vertices u and v. For each integer k with $2 \leq k \leq n$, every k-vertex-rainbow coloring of G must assign distinct colors to u and v.*

In [4], Chartrand et al. proved that the 3-rainbow index of cycles is $n-1$ for $n=3$, or $n-2$ for $n \geq 4$. Now we consider the 3-vertex-rainbow index of cycles.

Theorem 2.4 *For integer n,*

$$rvx_3(C_n) = \begin{cases} 0 & \text{if } \ 3 \leq \text{n} \leq 4, \\ n-4 & \text{if } \ \text{n} \geq 5. \end{cases}$$

Proof For $3 \leq n \leq 4$, by Proposition 2.1, we have $rvx_3(C_3) = 0$ and $rvx_3(C_4) = 0$.

For $n = 5$, we assign one color to all vertices of C_5. Thus, $rvx_3(C_5) \leq 1$. Since $\text{sdiam}_3(C_5) = 3$, we have $rvx_3(C_5) \geq 1$, and so $rvx_3(C_5) = 1 = n-4$.

For $n = 6$, we assign a vertex-coloring c to C_6 as follows: $c(v_1) = c(v_2) = c(v_3) = c(v_6) = 1, c(v_4) = c(v_5) = 2$. It is easy to verify that C_6 is vertex-rainbow 3-tree connected with the above vertex-coloring, and so $rvx_3(C_6) \leq 2$. Since every tree connecting $\{v_1, v_3, v_5\}$ has size at least 4, every tree has at least two vertices which are different from $\{v_1, v_3, v_5\}$. Note that $rvx_3(C_6) \geq 2$. Thus $rvx_3(C_6) = 2 = n-4$.

For $n \geq 7$, let $C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Assign a vertex-coloring c to C_n as follows: $c(v_1) = c(v_4) = c(v_7) = 1, c(v_2) = c(v_5) = 2$ and $c(v_3) = c(v_6) = 3$ if $n = 7$; $c(v_i) = i$ for $1 \leq i \leq n-4$ and $c(v_i) = i - n + 4$ for $n-3 \leq i \leq n$ if $n \geq 8$. For $n = 7$, we can check that the coloring c is a 3-vertex-rainbow coloring, and thus, $rvx_3(C_7) \leq 3 = n-4$. For $n \geq 8$, we need to show that there exists a vertex-rainbow tree connecting any three different vertices v_i, v_j and v_z of C_n, where $1 \leq i < j < z \leq n$. Let $S = \{v_i, v_j, v_z\}$. Without loss of generality, we verify $rvx_3(C_n) \leq n-4$ by considering the following four cases.

Case 1 v_i, v_j and v_z are three distinct vertices of the path $P = v_1v_2 \ldots v_{n-4}$ ($P = v_5v_6 \ldots v_n$), where $1 \leq i < j < z \leq n-4$ ($5 \leq i < j < z \leq n$). Obviously, there is a vertex-rainbow S-tree $T = v_i \ldots v_j \ldots v_z$, and so $rvx_3(C_n) \leq n-4$.

Case 2 Two vertices of S which lie on the path $P = v_1v_2 \ldots v_{n-4}$, and the remaining vertex of S lies on the path $P' = v_{n-3}v_{n-2}v_{n-1}v_n$. Let $1 \leq i < j \leq 4$ and $n-3 \leq z \leq n$. If $i = 1$ and $j = 2$, then there exists a vertex-rainbow S-tree T, where $T = v_z \ldots v_iv_j$. If $i = 1$, $j = 3$ and $n-2 \leq z \leq n$, then there exists a vertex-rainbow S-tree T, where $T = v_z \ldots v_iv_2v_j$. If $i = 1$, $j = 3$ and $z = n-3$, then there exists a vertex-rainbow S-tree T, where $T = v_iv_2v_jv_4 \ldots v_z$. If $i = 1$ and $j = 4$, then there exists a vertex-rainbow S-tree T, where $T = v_jv_5 \ldots v_z \ldots v_i$. If $i = 2$ and $j = 3$, then there exists a vertex-rainbow S-tree T, where $T = v_z \ldots v_1v_iv_j$. If $i = 2$, $j = 4$ and $n-3 \leq z \leq n-1$, then there exists a vertex-rainbow S-tree T, where $T = v_iv_3v_jv_5 \ldots v_z$. If $i = 2$, $j = 4$ and $z = n$, then there exists a vertex-rainbow S-tree T, where $T = v_zv_1v_iv_3v_j$. If $i = 3$ and $j = 4$, then there exists a vertex-rainbow S-tree T, where $T = v_iv_jv_5 \ldots v_z$. Let $1 \leq i \leq 4, 5 \leq j \leq n-4$ and $n-3 \leq z \leq n$. If $n-3 \leq z \leq n-2$, then there exists a vertex-rainbow S-tree T, where $T = v_i \ldots v_j \ldots v_z$. If $i = 1$ and $z = n-1$, then there exists a vertex-rainbow

S-tree T, where $T = v_j \dots v_z \dots v_i$. If $2 \leq i \leq 4$ and $z = n-1$, then there exists a vertex-rainbow S-tree T, where $T = v_i \dots v_j \dots v_z$. If $z = n$, then there exists a vertex-rainbow S-tree T, where $T = v_z \dots v_i \dots v_j$. Let $5 \leq i < j \leq n-4$ and $n-3 \leq z \leq n$. Then, there exists a vertex-rainbow S-tree T, where $T = v_i \dots v_j \dots v_z$. Hence, $rvx_3(C_n) \leq n-4$.

Case 3 One vertex of $\{v_i, v_j, v_z\}$ which lies on the path $P = v_1 v_2 \dots v_{n-4}$ and the remaining vertices of $\{v_i, v_j, v_z\}$ lie on the path $P^{'} = v_{n-3} v_{n-2} v_{n-1} v_n$. Let $1 \leq i \leq 4$ and $n-3 \leq j < z \leq n$. Then, an argument similar to the one used in the proof of $1 \leq i < j \leq 4$ and $n-3 \leq z \leq n$. Let $5 \leq i \leq n-4$ and $n-3 \leq j < z \leq n$. Then, there exists a vertex-rainbow S-tree T, where $T = v_i \dots v_j \dots v_z$. Thus, $rvx_3(C_n) \leq n-4$.

Case 4 v_i, v_j and v_z are three distinct vertices of the path $P = v_{n-3} v_{n-2} v_{n-1} v_n$, where $n-3 \leq i < j < z \leq n$. Thus, there exists a vertex-rainbow S-tree T, where $T = v_i \dots v_j \dots v_z$. Therefore, $rvx_3(C_n) \leq n-4$.

Next, we verify that $rvx_3(C_n) \geq n-4$ by proving the following three claims.

Claim 1 No five vertices of C_n can be colored the same.

Proof Suppose that $c(v_i) = c(v_j) = c(v_m) = c(v_p) = c(v_q)$, where $1 \leq i < j < m < p < q \leq n$. According to the adjacency of vertices of $\{v_i, v_j, v_m, v_p, v_q\}$, we only need to consider the seven types of $\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$ shown in Fig. 1. (For all the graphs of this paper, if two vertices of a graph are joined by a solid line, then the two vertices are said to be adjacent.) Let $S = \{v_g, v_h, v_m\}$. Then, any tree connecting S is not a vertex-rainbow S-tree, a contradiction. This completes the proof of the Claim 1. □

Claim 2 At most one pair of four vertices of C_n can be colored the same. Moreover, the four vertices share same color if and only if P_4 contains them, and other vertices have distinct colors.

Proof Suppose that there are two pairs $\pi_1 = \{v_i, v_j, v_p, v_q\}$ and $\pi_2 = \{v_a, v_b, v_c, v_d\}$ of vertices, where the colors of four of the eight vertices are the same if and only if the four vertices belong to the same pair with $i < j < p < q$ and $a < b < c < d$. If v_i, v_j, v_p and v_q are four internally distinct vertices of the path $P = v_a \dots v_b$, then there is no vertex-rainbow S-tree, where $S = \{v_i, v_b, v_q\}$. If v_i, v_j and v_p are three internally distinct vertices of the path $P = v_a \dots v_b$, v_q is an internal vertex of the path $P^{'} = v_b \dots v_c \dots v_d \dots v_a$, then there is no vertex-rainbow tree connecting $\{v_i, v_b, v_q\}$. If v_i and v_j are two internally distinct vertices of the path $P = v_a \dots v_b$, v_p and v_q are two internally distinct vertices of the path $P^{'} = v_b \dots v_c \dots v_d \dots v_a$, then there is no vertex-rainbow tree connecting $\{v_i, v_c, v_q\}$. If v_i, v_j, v_p and v_q are four internally distinct vertices of the paths $P = v_a \dots v_b$, $P^{'} = v_b \dots v_c$, $P^{''} = v_c \dots v_d$ and $P^{'''} = v_d \dots v_a$, respectively, then there is no vertex-rainbow S-tree, where $S = \{v_i, v_c, v_q\}$.

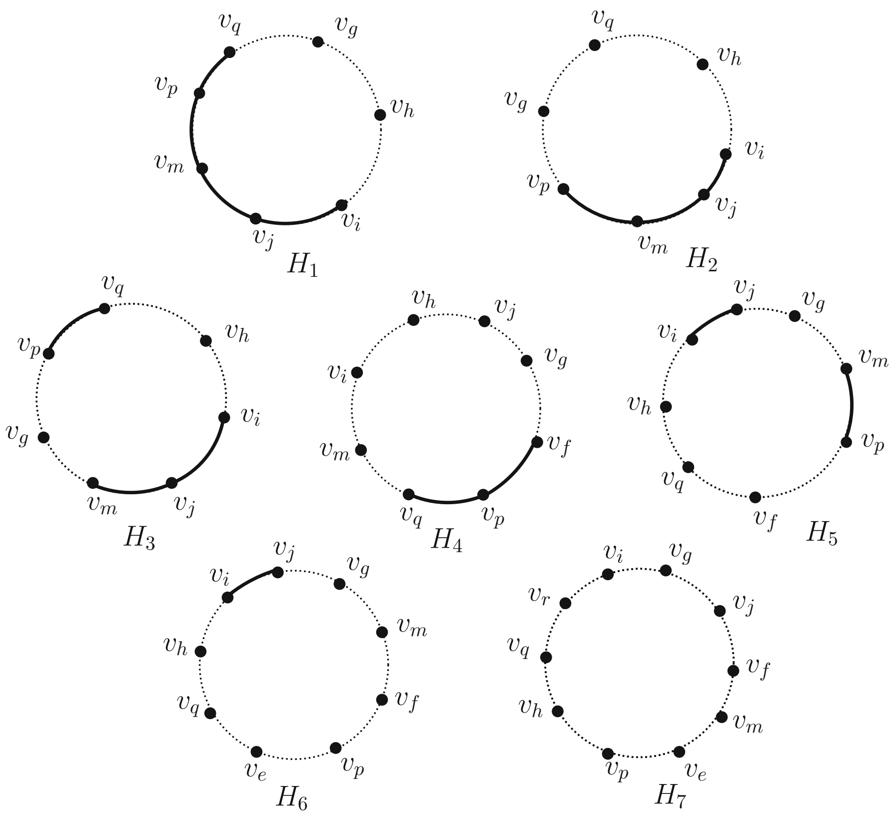

Fig. 1 Seven types of claim 2

Without loss of generality, assume that $\pi_1 = \{v_i, v_j, v_p, v_q\}$ satisfies $c(v_i) = c(v_j) = c(v_p) = c(v_q)$. Next, we verify that the four vertices of π_1 can be colored the same if and only if P_4 contains π_1, and the others have distinct colors. Consider the four graphs G_1, G_2, G_3 and G_4 shown in Fig. 2. For the type shown in G_1, there is no vertex-rainbow tree connecting $\{v_g, v_j, v_h\}$; In G_2, G_3 and G_4, there is no vertex-rainbow tree connecting $\{v_g, v_h, v_f\}$. If P_4 contains π_1, then we can check that there exists a vertex-rainbow tree connecting any three distinct vertices of C_n. Suppose that there exist two distinct vertices v_a and v_b satisfying $c(v_a) = c(v_b)$. By Claim 1, we have $c(v_a) \neq c(v_i)$. According to the adjacency of vertices of $\{v_i, v_j, v_p, v_q, v_a, v_b\}$, we consider the graphs G_5, G_6, G_7, G_8 and G_9 in Fig. 2. For the type shown in G_5, there is no vertex-rainbow tree connecting $\{v_i, v_j, v_g\}$; In G_6, there is no vertex-rainbow tree connecting $\{v_i, v_q, v_g\}$; In G_7, G_8 and G_9, there is no vertex-rainbow tree connecting $\{v_j, v_g, v_f\}$, a contradiction. This completes the proof of the Claim 2. □

Claim 3 If there are not four vertices of C_n with the same color, then one of the following cases holds.

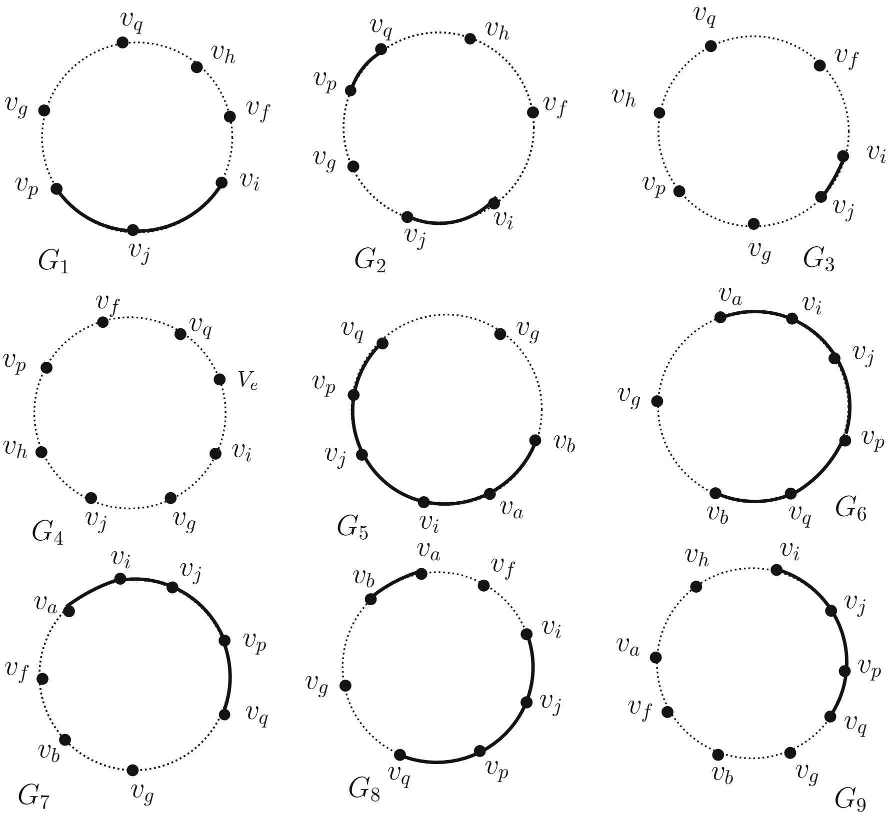

Fig. 2 Nine types of Claim 2

(i) At most two pairs of three vertices of C_n can be colored the same and other vertices must be colored the different.
(ii) If C_n contains only one pair of three vertices with the same color, then other vertices contain at most two pairs of two vertices where the vertices in each pair can be colored the same.
(iii) At most four pairs of two vertices of C_n satisfy that any two vertices of C_n have the same color if and only if the two vertices in the same pair.

Proof (i) Let $\pi_1 = \{v_a, v_a^{'}, v_a^{''}\}$ and $\pi_2 = \{v_b, v_b^{'}, v_b^{''}\}$ which satisfy $c(v_a) = c(v_a^{'}) = c(v_a^{''})$ and $c(v_b) = c(v_b^{'}) = c(v_b^{''})$, where the vertices of π_1 (π_2) are encountered in the clockwise order $v_a, v_a^{'}, v_a^{''}$ ($v_b, v_b^{'}, v_b^{''}$) in C_n. Since $n \geq 7$, we know that there exists a vertex $v_g \notin \pi_1 \cup \pi_2$. There is a vertex-rainbow tree connecting any three distinct vertices if and only if the following two cases hold: (1) v_b and $v_b^{'}$ are two internally distinct vertices of the path $P = v_a \ldots v_a^{'}, v_b^{''}$ and v_g are the internal vertices of the paths $P^{'} = v_a^{'} \ldots v_a^{''}$ and $P^{''} = v_a^{'} \ldots v_b^{''}$, respectively; (2) v_b and $v_b^{'}$ are two internally distinct vertices of the path $P = v_a \ldots v_a^{'}, v_b^{''}$ and v_g are the internal vertices of the paths $P^{'} = v^{''}_a \dots v_a$ and $P^{''} = v^{''}_b \dots v_a$, respectively. (Isomorphism is no longer discussed.) If v_b and $v^{'}_b$ are the internal vertices of the path $P = v_a \dots v^{'}_a$, $v^{''}_b$ is the internal vertex of $P^{'} = v^{'}_a \dots v^{''}_a \dots v_a$, and the location of v_g that is different from (1) and (2), then we can check that there exists a set $S^{'}$ of three vertices, any tree connecting $S^{'}$ is not vertex-rainbow. If v_b, $v^{'}_b$ and $v^{''}_b$ are three internally distinct vertices of the path $P = v_a \dots v^{'}_a$, then any tree connecting $\{v_b, v^{'}_a, v_g\}$ is not vertex-rainbow. If v_b, $v^{'}_b$ and $v^{''}_b$ are the internal vertices of the paths $P = v_a \dots v_b \dots v^{'}_a$, $P^{'} = v^{'}_a \dots v^{'}_b \dots v^{''}_a$ and $P^{''} = v^{''}_a \dots v^{''}_b \dots v_a$, respectively, then any tree connecting $\{v_a, v^{'}_a, v^{''}_a\}$ is not vertex-rainbow.

Let v_b and $v^{'}_b$ be two internally distinct vertices of the path $P = v_a \dots v^{'}_a$, $v^{''}_b$ be an internal vertex of the path $P^{'} = v^{'}_a \dots v^{''}_a \dots v_a$. Suppose $\pi_3 = \{v_c, v^{'}_c, v^{''}_c\}$, where $c(v_c) = c(v^{'}_c) = c(v^{''}_c) \neq c(v_a) \neq c(v_b)$, and the vertices in π_3 are encountered in the clockwise order $v_c, v^{'}_c, v^{''}_c$ in C_n. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are three internal vertices of the path $P = v_a \dots v^{'}_b \dots v^{'}_a$. Then, any tree connecting $\{v_a, v^{''}_c, v^{''}_b\}$ is not vertex-rainbow. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are three internal vertices of the path $P = v^{'}_a \dots v^{''}_a$. Then, any tree connecting $\{v_a, v_c, v^{''}_b\}$ is not vertex-rainbow. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are three internal vertices of the path $P = v^{''}_a \dots v_a$. Then, any tree connecting $\{v_a, v_c, v^{''}_b\}$ is not vertex-rainbow. Assume that v_c and $v^{'}_c$ are two internal vertices of the path $P = v_a \dots v^{'}_b \dots v^{'}_a$, $v^{''}_c$ is an internal vertex of the path $P^{'} = v^{'}_a \dots v^{''}_a \dots v_a$. If v_c and $v^{'}_c$ are two internal vertices of the path $P = v_a \dots v_b \dots v^{'}_b$, then any tree connecting $\{v_a, v^{'}_b, v^{''}_c\}$ is not vertex-rainbow. If v_c is the internal vertex of the path $P = v_a \dots v^{'}_b \dots v^{'}_a$ and $v^{'}_c$ is the internal vertex of the path $P^{'} = v^{'}_b \dots v^{'}_a$, then any tree connecting $\{v_a, v^{'}_c, v^{''}_c\}$ is not vertex-rainbow. Assume that v_c and $v^{'}_c$ are two internal vertices of the path $P = v^{'}_a \dots v^{''}_a$, $v^{''}_c$ is an internal vertex of the path $P^{'} = v^{''}_a \dots v_a \dots v^{'}_a$. Then, any tree connecting $\{v_a, v^{'}_a, v^{''}_b\}$ is not vertex-rainbow. Assume that v_c and $v^{'}_c$ are two internal vertices of the path $P = v^{''}_a \dots v_a$, $v^{''}_c$ is an internal vertex of the path $P^{'} = v_a \dots v^{'}_a \dots v^{''}_a$. Then, there is no vertex-rainbow tree connecting $\{v_a, v^{'}_a, v^{''}_b\}$. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are the internal vertices of the paths $P = v_a \dots v^{'}_a$, $P^{'} = v^{'}_a \dots v^{''}_a$ and $P^{''} = v^{''}_a \dots v_a$, respectively. Then, any tree connecting $\{v_a, v^{'}_a, v_c\}$ is not vertex-rainbow. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are the internal vertices of the paths $P = v^{'}_a \dots v^{''}_a$, $P^{'} = v^{''}_a \dots v_a$ and $P^{''} = v_a \dots v^{'}_a$, respectively. Then, any tree connecting $\{v_a, v^{'}_a, v^{''}_c\}$ is not vertex-rainbow. Assume that v_c, $v^{'}_c$ and $v^{''}_c$ are the internal vertices of the paths $P = v^{''}_a \dots v_a$, $P^{'} = v_a \dots v^{'}_a$ and $P^{''} = v^{'}_a \dots v^{''}_a$, respectively. Then, any tree connecting $\{v_a, v^{'}_a, v^{'}_c\}$ is not vertex-rainbow, a contradiction.

Next, we verify that other vertices must be colored different. Note that v_b and $v^{'}_b$ are two internal vertices of the path $P = v_a \dots v^{'}_a$, and $v^{''}_b$ is the internal vertex of the path $P^{'} = v^{'}_a \dots v^{''}_a \dots v_a$. Suppose that $\pi_4 = \{v_d, v^{'}_d\}$ satisfies $c(v_d) = c(v^{'}_d)$, and the vertices of π_4 are encountered in the clockwise order $v_d, v^{'}_d$ in C_n. Let v_d and $v^{'}_d$ be two internal vertices of the path $P = v_a \dots v^{'}_b \dots v^{'}_a$. If at least one vertex of $\{v_d, v_d^{'}\}$ is the internal vertex of the path $P = v_a \dots v_b$, then there is no vertex-rainbow tree connecting $\{v_d, v_d^{'}, v_a^{'}\}$. If v_d and $v_d^{'}$ are the internal vertices of the path $P = v_b \dots v_b^{'}$, then there is no vertex-rainbow tree connecting $\{v_d, v_a, v_a^{'}\}$. If v_d and $v_d^{'}$ are the internal vertices of the path $P = v_b^{'} \dots v_a^{'}$, then there is no vertex-rainbow tree connecting $\{v_a, v_a^{'}, v_b^{'}\}$. If v_d and $v_d^{'}$ are the internal vertices of the paths $P = v_b \dots v_b^{'}$ and $P^{'} = v_b^{'} \dots v_a^{'}$, respectively, then there is no vertex-rainbow tree connecting $\{v_d, v_a, v_a^{'}\}$. Let v_d and $v_d^{'}$ be two internal vertices of the path $P = v_a^{'} \dots v_a^{''}$. Then, there is no vertex-rainbow tree connecting $\{v_a, v_a^{'}, v_a^{''}\}$. Let v_d and $v_d^{'}$ be two internal vertices of the path $P = v_a^{''} \dots v_a$. Then, any tree connecting $\{v_a, v_a^{'}, v_a^{''}\}$ is not vertex-rainbow. Let v_d and $v_d^{'}$ be the internal vertices of the paths $P = v_a \dots v_a^{'}$ and $P^{'} = v_a^{'} \dots v_a^{''} \dots v_a$, respectively. If v_d is the internal vertex of the path $P = v_a \dots v_b$, then there is no vertex-rainbow tree connecting $\{v_d, v_d^{'}, v_a^{'}\}$. If v_d is the internal vertex of the path $P = v_b \dots v_b^{'}$, then there is no vertex-rainbow tree connecting $\{v_a, v_a^{'}, v_d\}$. If v_d is the internal vertex of the path $P = v_b^{'} \dots v_a^{'}$, then there is no vertex-rainbow tree connecting $\{v_a, v_b^{''}, v_d\}$. Let v_d and $v_d^{'}$ be the internal vertices of the paths $P = v_a^{'} \dots v_a^{''}$ and $P^{'} = v_a^{''} \dots v_a$, respectively. Then, there is no vertex-rainbow tree connecting $\{v_b, v_d, v_d^{'}\}$. Let v_d and $v_d^{'}$ be the internal vertices of the paths $P = v_a^{'} \dots v_a^{''}$ and $P^{'} = v_a \dots v_b^{'} \dots v_a^{'}$, respectively, or v_d and $v_d^{'}$ be the internal vertices of the paths $P = v_a^{''} \dots v_a$ and $P^{'} = v_a \dots v_a^{'} \dots v_a^{''}$, respectively. By the above similar argument, we can find a set of three vertices, and there is no vertex-rainbow tree connecting it, a contradiction.

(ii) Let $\pi_1 = \{v_a, v_a^{'}, v_a^{''}\}$ such that $c(v_a) = c(v_a^{'}) = c(v_a^{''})$. To the contrary, assume that there exist three pairs $\pi_2 = \{v_b, v_b^{'}\}, \pi_3 = \{v_c, v_c^{'}\}$ and $\pi_4 = \{v_d, v_d^{'}\}$ of vertices where the colors of two of the six vertices are the same if and only if the two vertices belong to the same pair. Let $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}\} = \{v_b, v_b^{'}, v_c, v_c^{'}, v_d, v_d^{'}\}$, where $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$. If $v_a, v_a^{'}$ and $v_a^{''}$ are the internal vertices of the path $P = v_{i_1} \dots v_{i_2}$, then any tree connecting $\{v_a, v_{i_1}, v_{i_2}\}$ is not vertex-rainbow. If v_a and $v_a^{'}$ are the internal vertices of the path $P = v_{i_1} \dots v_{i_2}$, $v_a^{''}$ is the internal vertex of the path $P^{'} = v_{i_2} \dots v_{i_3} \dots v_{i_6} \dots v_{i_1}$, then any tree connecting $\{v_a^{''}, v_{i_1}, v_{i_2}\}$ is not vertex-rainbow. If v_a is the internal vertex of the path $P = v_{i_1} \dots v_{i_2}$, $v_a^{'}$ and $v_a^{''}$ are the internal vertices of the paths $P^{'} = v_{i_2} \dots v_{i_3}$ and $P^{''} = v_{i_3} \dots v_{i_6} \dots v_{i_1}$, respectively, then any tree connecting $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ is not vertex-rainbow. If v_a is the internal vertex of the path $P = v_{i_1} \dots v_{i_2}$, $v_a^{'}$ and $v_a^{''}$ are the internal vertices of the paths $P^{'} = v_{i_3} \dots v_{i_4}$ and $P^{''} = v_{i_4} \dots v_{i_6} \dots v_{i_1}$, respectively, then any tree connecting $\{v_{i_1}, v_{i_2}, v_{i_4}\}$ is not vertex-rainbow, a contradiction.

(iii) Suppose that there are five pairs $\pi_1 = \{v_a, v_a^{'}\}, \pi_2 = \{v_b, v_b^{'}\}, \pi_3 = \{v_c, v_c^{'}\}, \pi_4 = \{v_d, v_d^{'}\}$ and $\pi_5 = \{v_e, v_e^{'}\}$ of vertices where the colors of two of the ten vertices are the same if and only if the two vertices belong to the same pair. Let $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}, v_{i_7}, v_{i_8}, v_{i_9}, v_{i_{10}}\} = \{v_a, v_a^{'}, v_b, v_b^{'}, v_c, v_c^{'}, v_d, v_d^{'}, v_e, v_e^{'}\}$, where $i_1 < i_2 < \ldots < i_{10}$. Without loss of generality, we consider the two trees $T_1 =$

$v_{i_1} \dots v_{i_2} \dots v_{i_3} \dots v_{i_4} \dots v_{i_5} \dots v_{i_6} \dots v_{i_7}$ and $T_2 = v_{i_4} \dots v_{i_5} \dots v_{i_6} \dots v_{i_7} \dots v_{i_8} \dots v_{i_9} \dots v_{i_{10}} \dots v_{i_1}$. Let $S = \{v_{i_1}, v_{i_4}, v_{i_7}\}$. We prove (iii) by consider the following two cases.

Case 1 Two vertices of the set S can be colored the same. Without loss of generality, we assume that $c(v_{i_4}) = c(v_{i_7}) \neq c(v_{i_1})$. Denote $S_1 = \{v_{i_5}, v_{i_6}, v_{i_8}, v_{i_9}, v_{i_{10}}\}$. Since $c(v_{i_4}) = c(v_{i_7})$, we know that at most four colors are assigned to S_1; obviously, T_2 is not a vertex-rainbow S-tree. Suppose $c(v_{i_1}) = c(v_{i_j})$ with $8 \leq j \leq 10$. If $c(v_{i_5}) \neq c(v_{i_6})$, then we have $c(v_{i_2}) = c(v_{i_3})$, or only one color of $\{v_{i_2}, v_{i_3}\}$ is among the colors that are assigned to $\{v_{i_5}, v_{i_6}\}$ and the another color of $\{v_{i_2}, v_{i_3}\}$ is among the colors that are assigned to $\{v_{i_8}, v_{i_9}, v_{i_{10}}\} \setminus \{v_{i_j}\}$, or the two colors of $\{v_{i_2}, v_{i_3}\}$ are among the colors that are assigned to $\{v_{i_5}, v_{i_6}\}$, and so there is no vertex-rainbow S-tree. If $c(v_{i_5}) = c(v_{i_6})$, then there are two colors are assigned to $\{v_{i_2}, v_{i_3}, v_{i_8}, v_{i_9}, v_{i_{10}}\} \setminus \{v_{i_j}\}$, and so there is no vertex-rainbow S-tree. Suppose $c(v_{i_1}) = c(v_{i_5})$. If $c(v_{i_6}) = c(v_{i_j})$ and $c(v_{i_2}) = c(v_{i_3})$, where $8 \leq j \leq 10$, then there is no vertex-rainbow S-tree. If $c(v_{i_6}) = c(v_{i_j})$, where $8 \leq j \leq 10$, then the colors that are assigned to $\{v_{i_2}, v_{i_3}\}$ are among the colors that are assigned to $\{v_{i_8}, v_{i_9}, v_{i_{10}}\} \setminus \{v_{i_j}\}$, and so there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_6}, v_{i_j}\}$. If $c(v_{i_6}) \neq c(v_{i_j})$, where $8 \leq j \leq 10$, then there are two vertices of $\{v_{i_8}, v_{i_9}, v_{i_{10}}\}$ having the same color and the color of v_{i_6} is among the colors that are assigned to $\{v_{i_2}, v_{i_3}\}$, and so there is no vertex-rainbow S-tree. Suppose $c(v_{i_1}) = c(v_{i_6})$. If $c(v_{i_5}) = c(v_{i_8})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_5}, v_{i_8}\}$. If $c(v_{i_5}) = c(v_{i_9})$, then there is no vertex-rainbow tree connecting $\{v_{i_4}, v_{i_7}, v_{i_{10}}\}$. If $c(v_{i_5}) = c(v_{i_{10}})$ and $c(v_{i_2}) = c(v_{i_8})$, then there is no vertex-rainbow tree connecting $\{v_{i_2}, v_{i_5}, v_{i_8}\}$. If $c(v_{i_5}) = c(v_{i_{10}})$ and $c(v_{i_2}) = c(v_{i_9})$, then there is no vertex-rainbow tree connecting $\{v_{i_1}, v_{i_3}, v_{i_8}\}$. If $c(v_{i_5}) = c(v_{i_{10}})$ and $c(v_{i_2}) = c(v_{i_3})$, then there is no vertex-rainbow S-tree. If $c(v_{i_5}) \neq c(v_{i_j})$, where $8 \leq j \leq 10$, then there are two vertices of $\{v_{i_8}, v_{i_9}, v_{i_{10}}\}$ having the same color, and the color of v_{i_5} is among the colors that are assigned to $\{v_{i_2}, v_{i_3}\}$, and so there is no vertex-rainbow S-tree. Suppose $c(v_{i_1}) = c(v_{i_2})$ ($c(v_{i_1}) = c(v_{i_3})$). We can check that there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_6}, v_{i_{10}}\}$ ($\{v_{i_2}, v_{i_4}, v_{i_{10}}\}$), a contradiction.

Case 2 No vertex of S can be colored the same. Moreover, we have the following four subcases.

Subcase 2.1 T_2 is vertex-rainbow, and T_1 is not vertex-rainbow. Note that at least one color of $\{v_{i_2}, v_{i_3}\}$ is among the colors that are assigned to $\{v_{i_5}, v_{i_6}\}$. Suppose $c(v_{i_2}) = c(v_{i_5})$. Then, the colors that are assigned to S are among the colors that are assigned to $\{v_{i_6}, v_{i_8}, v_{i_9}, v_{i_{10}}\}$. If $c(v_{i_2}) = c(v_{i_5})$ and $c(v_{i_6}) = c(v_{i_7})$, then any tree connecting $\{v_{i_2}, v_{i_5}, v_{i_8}\}$ is not vertex-rainbow. If $c(v_{i_2}) = c(v_{i_5})$ and $c(v_{i_6}) = c(v_{i_j})$, where $8 \leq j \leq 10$, then any tree connecting $\{v_{i_1}, v_{i_3}, v_{i_6}\}$ is not vertex-rainbow. Suppose $c(v_{i_2}) = c(v_{i_6})$. Then, the colors are assigned to $\{v_{i_1}, v_{i_3}, v_{i_4}, v_{i_7}\}$ are among the colors that are assigned to $\{v_{i_5}, v_{i_8}, v_{i_9}, v_{i_{10}}\}$, we know that one of vertices of $\{v_{i_1}, v_{i_3}, v_{i_4}, v_{i_7}\}$ has the same color as the vertex v_{i_5}. If $c(v_{i_2}) = c(v_{i_6})$, $c(v_{i_1}) = c(v_{i_5})$ and $c(v_{i_7}) = c(v_{i_{10}})$, then any tree connecting $\{v_{i_2}, v_{i_6}, v_{i_9}\}$ is not vertex-rainbow. If $c(v_{i_2}) = c(v_{i_6})$, $c(v_{i_1}) = c(v_{i_5})$ and $c(v_{i_7}) = c(v_{i_l})$, where $8 \leq l \leq 9$, then any tree connecting $\{v_{i_2}, v_{i_6}, v_{i_{10}}\}$ is not vertex-rainbow. If $c(v_{i_2}) = c(v_{i_6})$ and $c(v_{i_3}) = c(v_{i_5})$, then any tree connecting $\{v_{i_2}, v_{i_6}, v_{i_9}\}$ is not vertex-rainbow. If $c(v_{i_2}) = c(v_{i_6})$

and $c(v_{i_4}) = c(v_{i_5})$, then any tree connecting $\{v_{i_2}, v_{i_6}, v_{i_9}\}$ is not vertex-rainbow. If $c(v_{i_2}) = c(v_{i_6})$ and $c(v_{i_5}) = c(v_{i_7})$, then any tree connecting $\{v_{i_2}, v_{i_4}, v_{i_8}\}$ is not vertex-rainbow, a contradiction.

Subcase 2.2 Both T_1 and T_2 are vertex-rainbow. Obviously, the colors that are assigned to $\{v_{i_2}, v_{i_3}\}$ are among the colors that are assigned to $\{v_{i_8}, v_{i_9}, v_{i_{10}}\}$, and the colors that are assigned to S are among the colors that are assigned to $\{v_{i_5}, v_{i_6}, v_{i_8}, v_{i_9}, v_{i_{10}}\}$. For $c(v_{i_1}) = c(v_{i_{10}})$, we know that the colors of $\{v_{i_4}, v_{i_7}\}$ are same with the colors that are assigned to $\{v_{i_5}, v_{i_6}\}$. Then, there is no vertex-rainbow tree connecting $\{v_{i_2}, v_{i_5}, v_{i_9}\}$. For $c(v_{i_1}) = c(v_{i_9})$ ($c(v_{i_1}) = c(v_{i_8})$), we have a similar argument with $c(v_{i_1}) = c(v_{i_{10}})$. Then, there is no vertex-rainbow tree connecting $\{v_{i_2}, v_{i_8}, v_{i_{10}}\}$ ($\{v_{i_1}, v_{i_4}, v_{i_8}\}$). For $c(v_{i_1}) = c(v_{i_6})$, we can find that one of vertices of $\{v_{i_4}, v_{i_7}\}$ has the same color with the vertex v_{i_5}. If $c(v_{i_1}) = c(v_{i_6}), c(v_{i_2}) = c(v_{i_{10}})$ ($c(v_{i_2}) = c(v_{i_9})$) and $c(v_{i_4}) = c(v_{i_5})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_6}, v_{i_8}\}$. If $c(v_{i_1}) = c(v_{i_6}), c(v_{i_2}) = c(v_{i_{10}})$ ($c(v_{i_2}) = c(v_{i_9})$) and $c(v_{i_7}) = c(v_{i_5})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_6}, v_{i_8}\}$. If $c(v_{i_1}) = c(v_{i_6}), c(v_{i_2}) = c(v_{i_8})$ and $c(v_{i_4}) = c(v_{i_5})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_7}, v_{i_{10}}\}$. If $c(v_{i_1}) = c(v_{i_6}), c(v_{i_2}) = c(v_{i_8})$ and $c(v_{i_7}) = c(v_{i_5})$, then there is no vertex-rainbow tree connecting $\{v_{i_1}, v_{i_4}, v_{i_8}\}$. For $c(v_{i_1}) = c(v_{i_5})$, we can find one of vertices of $\{v_{i_4}, v_{i_7}\}$ has the same color with the vertex v_{i_6}. If $c(v_{i_1}) = c(v_{i_5}), c(v_{i_2}) = c(v_{i_{10}})$ ($c(v_{i_2}) = c(v_{i_9})$) and $c(v_{i_4}) = c(v_{i_6})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_5}, v_{i_8}\}$. If $c(v_{i_1}) = c(v_{i_5}), c(v_{i_2}) = c(v_{i_{10}})$ ($c(v_{i_2}) = c(v_{i_9})$) and $c(v_{i_7}) = c(v_{i_6})$, then any tree connecting $\{v_{i_3}, v_{i_5}, v_{i_8}\}$ is not vertex-rainbow. If $c(v_{i_1}) = c(v_{i_5}), c(v_{i_2}) = c(v_{i_8})$ and $c(v_{i_4}) = c(v_{i_6})$, then there is no vertex-rainbow tree connecting $\{v_{i_3}, v_{i_7}, v_{i_{10}}\}$. If $c(v_{i_1}) = c(v_{i_5}), c(v_{i_2}) = c(v_{i_8})$ and $c(v_{i_7}) = c(v_{i_6})$, then there is no vertex-rainbow tree connecting $\{v_{i_1}, v_{i_4}, v_{i_8}\}$, a contradiction.

Subcase 2.3 T_2 is not vertex-rainbow, and T_1 is vertex-rainbow. We have $c(v_{i_2}) \neq c(v_{i_3}) \neq c(v_{i_5}) \neq c(v_{i_6})$, and there are two vertices of $\{v_{i_8}, v_{i_9}, v_{i_{10}}\}$ having the same color. Therefore. the colors that are assigned to S are among the colors that are assigned to $\{v_{i_2}, v_{i_3}, v_{i_5}, v_{i_6}\}$. If $c(v_{i_8}) = c(v_{i_9})$, then there is no vertex-rainbow tree connecting $\{v_{i_4}, v_{i_7}, v_{i_{10}}\}$. If $c(v_{i_9}) = c(v_{i_{10}})$, then there is no vertex-rainbow tree connecting $\{v_{i_1}, v_{i_4}, v_{i_8}\}$. If $c(v_{i_8}) = c(v_{i_{10}})$, then there is no vertex-rainbow tree connecting $\{v_{i_1}, v_{i_7}, v_{i_9}\}$, a contradiction.

Subcase 2.4 Neither T_1 nor T_2 is vertex-rainbow. Consider the tree $T_3 = v_{i_7} \ldots v_{i_8} \ldots v_{i_9} \ldots v_{i_{10}} \ldots v_{i_1} \ldots v_{i_2} \ldots v_{i_3} \ldots v_{i_4}$. Assume that T_3 is vertex-rainbow. Note that T_1 is not vertex-rainbow, and so an argument similar to the one used in the proof of Subcase 2.1 shows a contradiction.

Combining the above three claims, we have $rvx_3(C_n) \geq n - 4$, and hence $rvx_3(C_n) = n - 4$. □

The girth of a graph G with some cycles, denoted by $g(G)$ or simply g, is the length of the smallest cycle in G. Let $N_k = \{1, 2, \ldots, k\}$ for each positive integer k, and $d(v, H) = min\{d_G(v, x) : x \in V(H)\}$, where H is a subgraph of a connected graph G and $v \in V(G) \backslash V(H)$. Now we determine the 3-vertex-rainbow index of all unicyclic graphs that are not cycles.

Theorem 2.5 *If G is a unicyclic graph of order $n \geq 4$ and $3 \leq g(G) \leq 4$ that is not a cycle, then $rvx_3(G) \leq n-3$, and the bound is tight.*

Proof Let G' be a subgraph of G , where $V(G') = \{v_1, v_2, v_3, v_4, \ldots, v_t\}$, $E(G') = \{v_1v_2, v_2v_3, v_1v_3, v_iv_{i+1} | 3 \leq i \leq t-1\}$ for $g(G) = 3$, or $E(G') = \{v_1v_2, v_2v_3, v_3v_4, v_1v_4, v_iv_{i+1} | 4 \leq i \leq t-1\}$ for $g(G) = 4$, and $d_G(v_t) = 1$. Let $V(G) \setminus V(G') = \{v_{t+1}, v_{t+2}, \ldots, v_n\}$. Suppose that there are $h_1 + 1$ leaves in G, denote $h = h_1 + 1$ (May be $h_1 = 0$). Assign a vertex-coloring c to G' as follows: $c(v_i) = i - 2$ for $4 \leq i \leq t-1$ and other vertices with color 1. We can find that G' is vertex-rainbow 3-tree connected with $3 \leq g(G) \leq 4$, thus $rvx_3(G') \leq t-3$. For $g(G) = 3$, we know that G' contains $t-3$ cut vertices. Then, by Observation 2.3, $rvx_3(G') \geq t-3$. Therefore, $rvx_3(G') = t-3$. For $g(G) = 4$, assume that $rvx_3(G') \leq t-4$, by Observation 2.3, we assign $t-4$ colors to the cut vertices of G'. Then, the colors that are assigned to $\{v_1, v_2, v_3, v_t\}$ are among the colors that are assigned to $\{v_4, v_5, \ldots, v_{t-1}\}$. If $c(v_1) = c(v_3) = c(v_4)$, then there is no vertex-rainbow tree connecting $\{v_2, v_{t-1}, v_t\}$. If $c(v_3) = c(v_4) \neq c(v_1)$ and $c(v_1) \neq c(v_{t-1})$, then there is no vertex-rainbow tree connecting $\{v_2, v_{t-1}, v_t\}$. If $c(v_3) = c(v_4) \neq c(v_1)$ and $c(v_1) = c(v_{t-1})$, then there is no vertex-rainbow tree connecting $\{v_2, v_{t-2}, v_t\}$. If $c(v_1) = c(v_4) \neq c(v_3)$, then we draw a same conclusion with $c(v_3) = c(v_4) \neq c(v_1)$. If $c(v_1) \neq c(v_4)$ and $c(v_3) \neq c(v_4)$, then there is no vertex-rainbow tree connecting $\{v_2, v_4, v_t\}$. Therefore, $rvx_3(G') \geq t-3$. Hence, $rvx_3(G') = t-3$. Next, we assign color 1 to the leaves of G, and colors $t-2, t-1, t, t+1, \ldots, n-3-h_1$ to the cut vertices in G but not in G'.

Suppose $g(G) = 3$. Let $d(v_1) = d(v_2) = 2$ and $d(v_3) \geq 3$. Then, by checking the given 3-vertex-rainbow coloring, we have $rvx_3(G) = n-3-h_1 = n-2-h$, where $h \geq 1$. Let $d(v_1) = 2$ and $d(v_2) \geq 3$. Thus, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_2)$, we assign a new color $n-2-h_1$ to the cut vertex v_2 instead of the color 1; otherwise, any tree connecting $\{v_4, v_5, v_m\}$ is not vertex-rainbow, and so $rvx_3(G) = n-2-h_1 = n-1-h$, where $h \geq 2$. Let $d(v_1) \geq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_1)$, and we assign a new color $n-2-h_1$ to the cut vertex v_1 instead of the color 1; otherwise, any tree connecting $\{v_4, v_5, v_m\}$ is not vertex-rainbow. Assume that there exists another leaf v_{m_1} satisfying $d(v_{m_1}, C) = d(v_{m_1}, v_j)$, where $2 \leq j \leq 3$. Then, we must assign a new color $n-1-h_2$ to the cut vertex v_j; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_4\}$. Therefore, $rvx_3(G) = n-1-h_2 = n-h$ with $h \geq 3$. If $h = 1$, then $G = G'$, and so $rvx_3(G) = n-3$. Finally, we have $rvx_3(G) \leq n-3$, and $G = G'$ is a tight example.

Suppose $g(G) = 4$. Let $d(v_1) = d(v_2) = d(v_3) = 2$ and $d(v_4) \geq 3$. Then, by checking the given 3-vertex-rainbow coloring, we have $rvx_3(G) = n-3-h_1 = n-2-h$, where $h \geq 1$. Let $d(v_1) = d(v_2) = 2$ and $d(v_3) \geq 3$. Then, $rvx_3(G) = n-3-h_1 = n-2-h$ by checking the given 3-vertex-rainbow coloring, where $h \geq 1$. Let $d(v_1) = 2$ and $d(v_2) \geq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_2)$, and so we must assign a new color $n-2-h_1$ to the cut vertex v_2 instead of the color 1, otherwise, any tree connecting $\{v_4, v_5, v_m\}$ is not vertex-rainbow. Thus, $rvx_3(G) = n-2-h_1 = n-1-h$, where $h \geq 2$. Let $d(v_1) \geq 3$. If $d(v_2) = d(v_3) = 1$, then $rvx_3(G) = n-3-h_1 = n-2-h$ with $h \geq 2$. If

$d(v_j) \geq 3$, where $2 \leq j \leq 3$, then there exist two leaves v_m and v_{m_1} satisfying $d(v_m, C) = d(v_m, v_1)$ and $d(v_{m_1}, C) = d(v_{m_1}, v_j)$. Note that we must color v_1 with a new color $n - 2 - h_1$ instead of color 1; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_t\}$, and so $rvx_3(G) = n - 2 - h_1 = n - 1 - h$ with $h \geq 3$. If $d(v_2) = d(v_3) \geq 3$, then there exist three leaves v_m, v_{m_1} and v_{m_2} satisfying $d(v_m, C) = d(v_m, v_1)$, $d(v_{m_1}, C) = d(v_{m_1}, v_2)$ and $d(v_{m_2}, C) = d(v_{m_2}, v_3)$. In order to ensure there exists a vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_{m_2}\}$, we assign the new colors $n - 2 - h_1$ and $n - 1 - h_1$ to v_1 and v_2, respectively, and so $rvx_3(G) = n - 1 - h_1 = n - h$, where $h \geq 4$. For $h = 1$, we have $rvx_3(G) = n - 3$. Finally, $rvx_3(G) \leq n - 3$, and $G = G^{'}$ is a tight example. □

Theorem 2.6 *If G is a unicyclic graph of order $n \geq 6$ and girth 5 that is not a cycle, then $rvx_3(G) \leq n - 4$, and the bound is tight.*

Proof Let $G^{'}$ be a subgraph of G , where $V(G^{'}) = \{v_1, v_2, \ldots, v_t\}$, $E(G^{'}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_5, v_iv_{i+1} | 5 \leq i \leq t - 1\}$ and $d_G(v_t) = 1$. Let $V(G) \setminus V(G^{'}) = \{v_{t+1}, v_{t+2}, \ldots, v_n\}$. Suppose that there are $h_1 + 1$ leaves in G, and denote $h = h_1 + 1$. Assign a vertex-coloring c to $G^{'}$ as follows: $c(v_i) = i - 3$ for $5 \leq i \leq t - 1$, and other vertices with color 1. By checking the given vertex-coloring, we have $rvx_3(G^{'}) \leq t - 4$. To the contrary, assume that $rvx_3(G^{'}) \leq t - 5$. Let $X_1 = \{v_1, v_2, v_3, v_4, v_t\}$ and $X_2 = \{v_5, v_6, \ldots, v_{t-1}\}$, by Observation 2.3, we have the colors that are assigned to X_1 are among the colors that are assigned to X_2. Then, there is no vertex-rainbow tree connecting $\{v_1, v_3, v_t\}$, and so $rvx_3(G^{'}) \geq t - 4$. Hence, $rvx_3(G^{'}) = t - 4$. Moreover, an argument similar to the one used in the proof of Theorem 2.5 shows that $rvx_3(G) \leq n - 4$. Finally, $rvx_3(G) \leq n - 4$, and $G = G^{'}$ is a tight example. □

Theorem 2.7 *If G is a unicyclic graph of order $n \geq 7$ and girth at least 6 that is not a cycle, then $rvx_3(G) \leq n - 5$, and the bound is tight.*

Proof Let $G^{'}$ be a subgraph of G with $V(G^{'}) = \{v_1, v_2, \ldots, v_g, v_{g+1}, \ldots, v_t\}$ and $E(G^{'}) = \{v_1v_g, v_iv_{i+1} | 1 \leq i \leq t - 1\}$, where $g \geq 6$ and $d_G(v_t) = 1$. Let $V(G) \setminus V(G^{'}) = \{v_{t+1}, v_{t+2}, \ldots, v_n\}$. Suppose that there are $h_1 + 1$ leaves in G and $h = h_1 + 1$. Assign a vertex-coloring c to $G^{'}$ as follows: $c(v_i) = 1$ for $i = 1, 4, t$, $c(v_i) = 2$ for $2 \leq i \leq 3$ and $c(v_i) = i - 4$ for $5 \leq i \leq t - 1$. We can check that $G^{'}$ is vertex-rainbow 3-tree connected with the vertex-coloring c, and thus $rvx_3(G^{'}) \leq t - 5$. To the contrary, assume that $rvx_3(G^{'}) \leq t - 6$. Then, there exists a 3-vertex-rainbow coloring c of $G^{'}$ using colors in N_{t-6}. Let A_1 be the set of colors assigned to the g vertices of the set $S_1 = \{v_1, v_2, \ldots, v_{g-1}, v_t\}$ and A_2 be the set of colors assigned to the $t - g$ cut vertices of the set $S_2 = \{v_g, v_{g+1}, \ldots, v_{t-1}\}$. Then, $A_1 \cup A_2 = N_{t-6}$. Furthermore, $|A_2| = t - g$ by Observation 2.3. We may therefore assume that $A_2 = N_{t-g}$.

If $g = 6$, then $A_2 = N_{t-6} = A_1 \cup A_2$, and so $A_1 \subseteq A_2$. Obviously, any tree connecting $\{v_2, v_4, v_t\}$ is not vertex-rainbow. If $g \geq 7$, then there are $g - 6$ colors in A_1 different from A_2. Let $S_3 = \{v_1, v_2, \ldots, v_{g-1}\}$ and A_3 be the set of colors assigned to S_3. We consider the following two cases.

Case 1 $c(v_i) \neq c(v_t)$, where $1 \leq i \leq g - 1$. Then, there are $g - 7$ colors in A_3 different from A_2, and $A_3 = A_1 \setminus c(v_t)$. Let $S_3^{'} \subseteq S_3$ and $|S_3^{'}| = g - 7$. Then, we assign $g - 7$

colors in A_3 that are different from A_2 to the vertices set S_3'. Suppose $A_3 \cap A_2 = \emptyset$. Then $A_3 = g-7$, and so $g-6$ colors are assigned to C_g, which contradicts $rvx_3(C_g) = g-4$; Suppose $A_3 \cap A_2 \neq \emptyset$. According to $rvx_3(C_g) = g-4$, we have that at least two colors of A_3 are among the colors that are assigned to $\{v_{g+1}, v_{g+2}, \ldots, v_{t-1}\}$. Then, $c(v_p) = c(v_j)$, where $1 \leq p \leq g-1$ and $g+1 \leq j \leq t-1$. Furthermore, there exist four vertices of $S_3 \setminus S_3' \cup \{v_p\}$ satisfying the colors that are assigned to them are among the colors that are assigned to the other vertices of G'. If $p = 1$, then there is no vertex-rainbow tree connecting $\{v_2, v_3, v_t\}$; If $2 \leq p \leq g-1$, then any tree connecting $\{v_{p-1}, v_{p+1}, v_t\}$ is not vertex-rainbow.

Case 2 $c(v_i) = c(v_t)$, where $1 \leq i \leq g-1$. Then, there are $g-6$ colors of A_3 that are not contained in A_2 and $A_3 = A_1$. Let $S_3' \subseteq S_3$ and $|S_3'| = g-6$. Then, we assign $g-6$ colors in A_3 that are different from A_2 to the vertices set S_3'. Suppose $A_3 \cap A_2 = \emptyset$. Then, $|A_3| = g-6$, and so $g-5$ colors are assigned to C_g, which contradicts $rvx_3(C_g) = g-4$. Suppose $A_3 \cap A_2 \neq \emptyset$. Since $rvx_3(C_g) = g-4$, we have at least one color of A_3 is among the colors that are assigned to $\{v_{g+1}, v_{g+2}, \ldots, v_{t-1}\}$. Then, $c(v_p) = c(v_j)$, where $1 \leq p \leq g-1, g+1 \leq j \leq t-1$. Moreover, an argument similar to the one used in the proof of Case 1 shows that if $p = 1$, then any tree connecting $\{v_2, v_3, v_t\}$ is not vertex-rainbow, and if $2 \leq p \leq g-1$, then there is no vertex-rainbow tree connecting $\{v_{p-1}, v_{p+1}, v_t\}$. Finally, $rvx_3(G') \geq t-5$. Therefore, $rvx_3(G') = t-5$.

Next, we assign the color 1 to the leaves of G, the colors $t-4, t-3, \ldots, n-5-h_1$ to the cut vertices in G but not in G'. Let $d(v_i) = 2$, where $1 \leq i \leq 6$. Then, $rvx_3(G) = n-5-h_1 = n-4-h$ by checking the given 3-vertex-rainbow coloring, where $h \geq 1$. Let $d(v_i) = 2$ and $d(v_6) \geq 3$, where $1 \leq i \leq 5$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_6)$. Note that we must assign a new color $n-4-h_1$ to v_3 or v_4; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_2, v_4\}$ or $\{v_m, v_3, v_7\}$, and so $rvx_3(G) = n-4-h_1 = n-3-h$, where $h \geq 2$. Let $d(v_i) = 2$ and $d(v_5) \geq 3$, where $1 \leq i \leq 4$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_5)$. Note that we must assign a new color $n-4-h_1$ to v_1 or v_4; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_2, v_7\}$ or $\{v_m, v_3, v_7\}$, and so $rvx_3(G) = n-4-h_1 = n-3-h$, where $h \geq 2$. Let $d(v_i) = 2$ and $d(v_4) \geq 3$, where $1 \leq i \leq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_4)$. Note that we must assign a new color $n-4-h_1$ to v_4; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_2, v_7\}$. Suppose that there exists another leaf v_{m_1} of G satisfying $d(v_{m_1}, C) = d(v_{m_1}, v_6)$. We must assign a new color $n-3-h_1$ to v_6; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_2\}$, and so $rvx_3(G) = n-3-h_1 = n-2-h$, where $h \geq 3$. Let $d(v_1) = d(v_2) = 2$ and $d(v_3) \geq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_3)$. Note that we must assign the new color $n-4-h_1$ to v_3; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_1, v_5\}$. Suppose that there exists another leaf v_{m_1} of G satisfying $d(v_{m_1}, C) = d(v_{m_1}, v_5)$. Then, assign a new color $n-3-h_1$ to v_5; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_1\}$, and so $rvx_3(G) = n-3-h_1 = n-2-h$, where $h \geq 3$. Let $d(v_1) = 2$ and $d(v_2) \geq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_2)$.

If $d(v_4) = d(v_5) = d(v_6) = 2$, then $c(v_2) = n - 4 - h_1$; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_1, v_4\}$, and so $rvx_3(G) = n - 4 - h_1 = n - 3 - h$, where $h \geq 2$. If $d(v_6) = 2$ and $d(v_i) \geq 3$, where $4 \leq i \leq 5$, then $c(v_2) = n - 4 - h_1$ and $c(v_i) = n - 3 - h_1$; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_7\}$, and so $rvx_3(G) = n - 3 - h_1 = n - 2 - h$, where $d(v_{m_1}, C) = d(v_{m_1}, v_i)$ and $h \geq 3$. If $d(v_6) \geq 3$ and $d(v_3) = d(v_4) = d(v_5) = 2$, then $c(v_2) = n-4-h_1$ and $c(v_3) = n-3-h_1$; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_4\}$, and so $rvx_3(G) = n - 3 - h_1 = n - 2 - h$, where $d(v_{m_1}, C) = d(v_{m_1}, v_6)$ and $h \geq 3$. If $d(v_6) \geq 3$ and $d(v_4) \geq 3$, then $c(v_2) = n-4-h_1$, $c(v_3) = n-3-h_1$ and $c(v_4) = n-2-h_1$; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_{m_1}, v_{m_2}\}$, and so $rvx_3(G) = n-2-h_1 = n-1-h$, where $d(v_{m_1}, C) = d(v_{m_1}, v_4)$, $d(v_{m_2}, C) = d(v_{m_2}, v_6)$ and $h \geq 4$. Let $d(v_1) \geq 3$. Then, there exists a leaf v_m of G satisfying $d(v_m, C) = d(v_m, v_1)$. Note that we must assign the color $n-4-h_1$ to v_4 or v_5; otherwise, there is no vertex-rainbow tree connecting $\{v_m, v_3, v_5\}$ or $\{v_m, v_4, v_6\}$, and so $rvx_3(G) = n-4-h_1 = n-3-h$, where $h \geq 2$. Suppose that there exist other leaves $v_{m_1}, v_{m_2}, v_{m_3}, v_{m_4}$ and v_{m_5} satisfying $d(v_{m_i}, C) = d(v_{m_i}, v_{i+1})$, $1 \leq i \leq 5$. In order to ensure that there exists a vertex-rainbow tree connecting any three vertices of $\{v_m, v_{m_1}, v_{m_2}, v_{m_3}, v_{m_4}, v_{m_5}\}$, we assign some new colors $n-4-h_1, n-3-h_1, n-2-h_1$ and $n-1-h_1$ to v_3, v_4, v_5 and v_6, respectively. Therefore, $rvx_3(G) = n-1-h_1 = n-h$, where $h \geq 7$. For $h = 1$, we have $rvx_3(G) = rvx_3(G^{'}) = n - 5$. Finally, $rvx_3(G) \leq n - 5$, and $G = G^{'}$ is a tight example. □

3 The 3-Vertex-Rainbow Index of Complementary Graphs

Let G be a simple graph with order n. The complement graph $\overline{G}$ of G is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. If $\overline{G}$ is an disconnected graph with $t \geq 2$ connected components, then G contains a complete t-partite spanning subgraph. We immediately draw the following conclusion.

Proposition 3.1 *Let $\overline{G}$ be a graph with $t \geq 2$ connected components $\overline{G_i}$ and $n_i' = n(\overline{G_i})$ $(1 \leq i \leq t)$. Then, $rvx_3(G) \leq rvx_3(K_{n_1', \ldots, n_t'})$.*

Now we investigate the 3-vertex-rainbow index of a connected graph G by use of its complement graph $\overline{G}$ with diameter at least 3.

Theorem 3.2 *Let G be a connected graph of order n.*

(i) *If $\overline{G}$ is connected and $diam(\overline{G}) \geq 4$, then $rvx_3(G) = 1$;*
(ii) *If $\overline{G}$ is connected and $diam(\overline{G}) = 3$, then $1 \leq rvx_3(G) \leq 2$, and the bounds are tight;*
(iii) *If $\overline{G}$ is disconnected, then $0 \leq rvx_3(G) \leq 1$, and the bounds are tight.*

Proof Choose a vertex x in $\overline{G}$, where x satisfies $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G}) = d$. Denote $N^i_{\overline{G}}(x) = \{v : d(x, v) = i\}$ for $0 \leq i \leq d$, especially, $N^0_{\overline{G}}(x) = \{x\}$ and $N^1_{\overline{G}}(x) =$

$N_{\overline{G}}(x)$. Then, $\cup_{0\le i\le d} N^i_{\overline{G}}(x)$ is a vertex partition of $V(\overline{G})$. Let $X = \cup_{i\equiv 0 (\mathrm{mod}\ 2)} N^i_{\overline{G}}(x)$ and $Y = \cup_{i\equiv 1(\mathrm{mod}\ 2)} N^i_{\overline{G}}(x)$.

(i) $\overline{G}$ is connected and $d \ge 4$. Then, $G[X]$ ($G[Y]$) contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph) by the definition of complement graphs, where $k_1 = \lceil \frac{d+1}{2} \rceil$ ($k_2 = \lceil \frac{d}{2} \rceil$). Now, we show a vertex-coloring c of G as follows: $c(z) = 1$ for $z \in G$. Let $S = \{u, v, w\}$, where u, v and w are any three different vertices of G.

Suppose $u, v, w \in N^i_{\overline{G}}(x)$, where i is even. Then, T is a vertex-rainbow S-tree with $E(T) = \{ux, vx, wx\}$. Suppose $u, v \in N^i_{\overline{G}}(x)$ and $w \in N^j_{\overline{G}}(x)$, where i, j are even and $i \ne j$. Then, T is a vertex-rainbow S-tree with $E(T) = \{uw, vw\}$. Suppose $u \in N^i_{\overline{G}}(x)$, $v \in N^j_{\overline{G}}(x)$ and $w \in N^r_{\overline{G}}(x)$, where i, j and r are even and $i \ne j \ne r$. Then, T is a vertex-rainbow S-tree with $E(T) = \{uv, uw\}$. Suppose $u, v \in N^i_{\overline{G}}(x)$ and $w \in Y$, where i is even. Then, one of the following two cases holds. (1) $uw, vw \in E(G)$; (2) $uy, vy, wy \in E(G)$, where $y \in N^j_{\overline{G}}(x)$, j is even and $i \ne j$. For (1), there is a vertex-rainbow S-tree T with $E(T) = \{uw, vw\}$. For (2), there is a vertex-rainbow S-tree T with $E(T) = \{uy, vy, wy\}$. Suppose $u \in N^i_{\overline{G}}(x)$, $v \in N^j_{\overline{G}}(x)$ and $w \in N^r_{\overline{G}}(x)$, where i, j are even and r is odd and $i \ne j$. Then, either $uv, vw \in E(G)$ or $uy, vy, wy \in E(G)$, where $y \in N^q_{\overline{G}}(x)$, q is even and $i \ne j \ne q$. If $uv, vw \in E(G)$, then T is a vertex-rainbow S-tree with $E(T) = \{uv, vw\}$. If $uy, vy, wy \in E(G)$, then T is a vertex-rainbow S-tree with $E(T) = \{uy, vy, wy\}$. Suppose $u \in X$ and $v, w \in Y$ or $u, v, w \in Y$. Then, a similar argument as above shows that we can find a vertex-rainbow S-tree. Therefore, $rvx_3(G) \le 1$.

Next, we prove $rvx_3(G) \ge 1$. Let $u \in N^{d-2}_{\overline{G}}(x)$, $v \in N^{d-1}_{\overline{G}}(x)$ and $w \in N^d_{\overline{G}}(x)$. Then, $uv, vw \in E(\overline{G})$ and $uw \notin E(\overline{G})$. It follows that $uv, vw \notin E(G)$ and $uw \in E(G)$, and so $rvx_3(G) \ge 1$. Hence, $rvx_3(G) = 1$.

(ii) $\overline{G}$ is connected and $d = 3$. According to the definition of complement graphs, we know that $G[X]$ ($G[Y]$) contains a spanning complete 2-partite subgraph. Now, we give the graph G a vertex-coloring c as follows: $c(z_1) = 1$ for $z_1 \in X$ and $c(z_2) = 2$ for $z_2 \in Y$. Denote $S = \{u, v, w\}$, where u, v and w are any three different vertices of G.

Assume that $u, v, w \in N^2_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{ux, vx, wx\}$. Assume that $u, v \in N^2_{\overline{G}}(x)$ and $w = x \in N^0_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{uw, vw\}$. Assume that $u, v \in N^2_{\overline{G}}(x)$ and $w \in N_{\overline{G}}(x)$. Then, there are two vertices $x \in N^0_{\overline{G}}(x)$ and $y \in N^3_{\overline{G}}(x)$ such that $xu, xv, xy, yw \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{xu, xv, xy, yw\}$. Assume that $u, v \in N^2_{\overline{G}}(x)$ and $w \in N^3_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{ux, vx, wx\}$. Assume that $u \in N^2_{\overline{G}}(x)$, $v \in N^0_G(x)$ and $w \in N_{\overline{G}}(x)$. Then, there is a vertex $y \in N^3_{\overline{G}}(x)$ such that $vu, yv, yw \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{vu, yv, yw\}$. Assume that $u \in N^2_{\overline{G}}(x)$, $v \in N^0_G(x)$ and $w \in N^3_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{vu, vw\}$. Assume that $u, v \in Y$ and $w = x \in N^0_{\overline{G}}(x)$. If $u, v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{wu, wv\}$; If $u, v \in N_{\overline{G}}(x)$, then there is a vertex $y \in N^3_{\overline{G}}(x)$ such that $yu, yv, wy \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{uy, vy, wy\}$; If $u \in N^3_{\overline{G}}(x)$ and $v \in N_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{uw, uv\}$. Assume that $u, v \in Y$ and $w \in N^2_{\overline{G}}(x)$. If $u, v \in N^1_{\overline{G}}(x)$, then there are two vertices $y \in N^3_{\overline{G}}(x)$ and $x \in N^0_{\overline{G}}(x)$ satisfying $xw, xy, yu, yv \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{xw, xy, yu, yv\}$; If $u, v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xv, xu, xw\}$; If $u \in N^1_{\overline{G}}(x)$ and $v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xv, xw, uv\}$. Assume that $u, v, w \in Y$. If $u, v, w \in N^1_{\overline{G}}(x)$ ($u, v, w \in N^3_{\overline{G}}(x)$), then there is a vertex $y \in N^3_{\overline{G}}(x)$ ($y \in N^1_{\overline{G}}(x)$) satisfying $uy, vy, wy \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{uy, vy, wy\}$; If $u, v \in N^1_{\overline{G}}(x)$ ($u, v \in N^3_{\overline{G}}(x)$) and $w \in N^3_{\overline{G}}(x)$ ($w \in N^1_{\overline{G}}(x)$), then the tree T connecting S is vertex-rainbow, where $V(T) = \{u, v, w\}$ and $E(T) = \{uw, vw\}$. From what has been discussed above, we get that $rvx_3(G) \leq 2$.

Next, we show $rvx_3(G) \geq 1$. Let $u \in N_{\overline{G}}(x)$, $v \in N^2_{\overline{G}}(x)$ and $w \in N^3_{\overline{G}}(x)$. Then, $uv, vw \in E(\overline{G})$ and $uw \notin E(\overline{G})$. It follows that $uv, vw \notin E(G)$ and $uw \in E(\overline{G})$, and so $rvx_3(G) \geq 1$. Hence $1 \leq rvx_3(G) \leq 2$.

Tight Example 1. Let $\overline{G}$ be a connected graph of order n with $\mathrm{diam}(\overline{G}) = 3$. We can prove that $rvx_3(G) = 1$.

Pick a vertex x of $\overline{G}$ that satisfies $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G}) = 3$. Denote $N^0_{\overline{G}}(x) = \{x\}$, $N_{\overline{G}}(x) = \{y_1, y_2\}$, $N^2_{\overline{G}}(x) = \{u_1, u_2\}$, $N^3_{\overline{G}}(x) = \{v_1, v_2, \ldots, v_{n-5}\}$ and $E(\overline{G}) = \{xy_1, xy_2, u_1y_1, u_2y_2, u_1u_2, u_1v_i, u_2v_i | 1 \leq i \leq n-5\}$. Then, $V(G) = \{x, y_1, y_2, u_1, u_2, v_1, v_2, \ldots, v_{n-5}\}$ and $E(G) = \{y_1y_2, u_1y_2, u_2y_1, xu_1, xu_2, xv_i, y_1v_i, y_2v_i, v_iv_j | 1 \leq i, j \leq n-5\}$. Since the tree connecting $\{u_1, u_2, y_1\}$ has at least three edges, we have $rvx_3(G) \geq 1$. Now we only need to prove $rvx_3(G) \leq 1$.

Assign color 1 to all vertices of G. Suppose $S = \{x, u_1, u_2\}$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xu_2\}$. Suppose $S = \{u_1, u_2, v_i\}$, where $1 \leq i \leq n-5$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xu_2, xv_i\}$. Suppose $S = \{u_1, u_2, y_j\}$, where $1 \leq j \leq 2$. If $j = 1$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xu_2, u_2y_j\}$; If $j = 2$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xu_2, u_1y_j\}$. Suppose $S = \{x, u_j, v_i\}$, where $1 \leq i \leq n-5$, $1 \leq j \leq 2$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_j, xv_i\}$. Suppose $S = \{x, u_1, y_j\}$, where $1 \leq j \leq 2$. If $j = 1$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xu_2, u_2y_j\}$; If $j = 2$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, u_1y_j\}$. Suppose $S = \{x, u_2, y_j\}$, where $1 \leq j \leq 2$. Then, we have a similar argument with $S = \{x, u_1, y_j\}$. Suppose $S = \{x, u, v\}$, where $u, v \in N_{\overline{G}}(x) \cup N^3_{\overline{G}}(x)$. If $u, v \in N_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xv_1, uv_1, vv_1\}$. If $u, v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu, xv\}$. If $u \in N_{\overline{G}}(x)$ and $v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xv, uv\}$. Suppose $S = \{u_1, u, v\}$, where $u, v \in N_{\overline{G}}(x) \cup N^3_{\overline{G}}(x)$. If $u, v \in N_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{u_1u, uv\}$. If $u, v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{u_1y_2, uv, uy_2\}$. If $u \in N_{\overline{G}}(x)$ and $v \in N^3_{\overline{G}}(x)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{xu_1, xv, uv\}$. Suppose $S = \{u_2, u, v\}$, where $u, v \in N_{\overline{G}}(x) \cup N^3_{\overline{G}}(x)$. Then, we have a similar argument with $S = \{u_1, u, v\}$. Suppose $S = \{u, v, w\}$, where $u, v, w \in N_{\overline{G}}(x) \cup N^3_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{uv, wu\}$. Therefore, $rvx_3(G) \leq 1$.

Tight Example 2 Let $\overline{G}$ be a connected graph of order n with $\mathrm{diam}(\overline{G}) = 3$. We can prove that $rvx_3(G) = 2$.

Pick a vertex x of $\overline{G}$ such that $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G}) = 3$. Suppose that $N^0_{\overline{G}}(x) = \{x\}, N^1_{\overline{G}}(x) = N_{\overline{G}}(x) = \{y_1, y_2, \ldots, y_{n-3}\}, N^2_{\overline{G}}(x) = \{u_1\}, N^3_{\overline{G}}(x) = \{v_1\}$ and $E(\overline{G}) = \{xy_i, u_1y_i, u_1v_1 | 1 \leq i \leq n-3\}$. Then, $V(G) = \{x, u_1, v_1, y_i | 1 \leq i \leq n-3\}$ and $E(G) = \{xu_1, xv_1, v_1y_i, y_iy_j | 1 \leq i, j \leq n-3\}$. Since the tree T connecting $S = \{u_1, y_1, y_2\}$ has at least four edges and $V(T) \backslash S = \{x, v_1\}$, we have $rvx_3(G) \geq 2$. Now we only need to prove $rvx_3(G) \leq 2$.

Define a vertex-coloring c of G as follows: $c(x) = c(u_1) = 1, c(v_1) = c(y_i) = 2$, where $1 \leq i \leq n-3$. Let $S = \{x, u_1, v_1\}$. Then, T satisfying $E(T) = \{xu_1, xv_1\}$ is a vertex-rainbow S-tree. Let $S = \{x, u_1, y_i\}$. Then, T satisfying $E(T) = \{xu_1, xv_1, v_1y_i\}$ is a vertex-rainbow S-tree. Let $S = \{x, v_1, y_i\}$. Then, T satisfying $E(T) = \{xu_1, xv_1, v_1y_i\}$ is a vertex-rainbow S-tree. Let $S = \{x, y_i, y_j\}$. Then, T satisfying $E(T) = \{xv_1, v_1y_i, y_iy_j\}$ is a vertex-rainbow S-tree. Let $S = \{u_1, v_1, y_i\}$. Then, T satisfying $E(T) = \{xu_1, xv_1, v_1y_i\}$ is a vertex-rainbow S-tree. Let $S = \{u_1, y_i, y_j\}$. Then, T satisfying $E(T) = \{xu_1, xv_1, v_1y_i, y_iy_j\}$ is a vertex-rainbow S-tree. Let $S = \{u, v, w\} \in N_{\overline{G}}(x) \cup N^3_{\overline{G}}(x)$. Then, T satisfying $E(T) = \{uv, uw\}$ is a vertex-rainbow S-tree. Thus, $rvx_3(G) \leq 2$.

(iii) If $\overline{G}$ is disconnected, then $\overline{G}$ has $t \geq 2$ connected components. Suppose that $\overline{G_1}, \overline{G_2}, \ldots, \overline{G_t}$ are the connected components of $\overline{G}$. It results that G contains a complete t-partite spanning subgraph. Given a vertex-coloring c of G as follows: assign the color 1 to all the vertices of G. Next, we show that there exists a vertex-rainbow tree connecting any three different vertices u, v and w of G. Let $S = \{u, v, w\}$. Let $u, v, w \in V(\overline{G_i})$, where $1 \leq i \leq t$. Then, there is a vertex $y \in V(\overline{G_j})$ such that $uy, vy, wy \in E(G)$, where $1 \leq j \leq t$ and $i \neq j$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{uy, vy, wy\}$. Let $u, v \in V(\overline{G_i})$ and $w \in V(\overline{G_j})$, where $1 \leq i \neq j \leq t$, or $u \in V(\overline{G_i})$, $v \in V(\overline{G_j})$ and $w \in V(\overline{G_r})$, where $1 \leq i \neq j \neq r \leq t$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{uw, wv\}$. Therefore, $rvx_3(G) \leq 1$. Finally, we have $0 \leq rvx_3(G) \leq 1$ by Proposition 2.2.

Tight Example 3 Let $\overline{G}$ be a graph of order n with $V(\overline{G}) = \{y_1, y_2, \ldots, y_{n-1}, y_n\}$ and $E(\overline{G}) = \{y_{n-1}y_n\}$. Then, $V(G) = V(\overline{G})$ and $E(G) = \{y_iy_j, y_iy_{n-1}, y_iy_n | 1 \leq i \neq j \leq n-2\}$. We can find that $\mathrm{sdiam}_3(G) = 2$. Therefore, $rvx_3(G) = 0$.

Tight Example 4 Let G be a connected graph of order n and $\overline{G}$ be disconnected. We can prove that $rvx_3(G) = 1$.

Suppose that $V(\overline{G}) = \{y_1, y_2, \dots, y_{n-1}, y_n\}$ and $E(\overline{G}) = \{y_{n-2}y_{n-1}, y_{n-1}y_n, y_{n-2}y_n\}$. Then, $V(G) = V(\overline{G})$ and $E(G) = \{y_{n-2}y_i, y_{n-1}y_i, y_ny_i, y_iy_j|1 \leq i, j \leq n-3\}$. Since the tree T connecting $\{y_{n-2}, y_{n-1}, y_n\}$ has at least three edges, we have $rvx_3(G) \geq 1$. Now we only need to prove $rvx_3(G) \leq 1$. Assign one color to all vertices of G. Let $X = \{y_1, y_2, \dots, y_{n-3}\}$ and $Y = \{y_{n-2}, y_{n-1}, y_n\}$. Suppose that u, v and w are any distinct vertices of G, denote $S = \{u, v, w\}$. Assume that at least one vertex of S lies on the set X. Without loss of generality, assume that $u \in X$. Then, we obtain a vertex-rainbow S-tree T, where $E(T) = \{uw, uv\}$. Assume that all vertices of S lie on the set Y. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{uy_1, vy_1, wy_1\}$, and so $rvx_3(G) \leq 1$. □

A graph $\overline{G}$ is connected with $\text{diam}(\overline{G}) = 2$. Let x be a vertex of $\overline{G}$ satisfying $\text{ecc}_{\overline{G}}(x) = \text{diam}(\overline{G}) = 2$. Suppose that $N_{\overline{G}}(x) = \{y_1, y_2, \dots, y_{n-2}\}$, $N^2_{\overline{G}}(x) = \{y_{n-1}\}$ such that y_{n-1} and y_i are adjacent in $\overline{G}$, where $1 \leq i \leq n-3$. Then, $rvx_3(G)$ can be very large if the number of cut vertices in $G[N_{\overline{G}}(x)]$ is sufficiently large. Here, we add an additional constraint to study the 3-vertex-rainbow index of G.

Theorem 3.3 *Let $\overline{G}$ be a triangle-free graph with $diam(\overline{G}) = 2$. If G is connected, then $rvx_3(G) = 1$.*

Proof Select a vertex x of $\overline{G}$ satisfying $\text{ecc}_{\overline{G}}(x) = \text{diam}(\overline{G}) = 2$. Let $|N^i_{\overline{G}}(x)| = n_i$, where $1 \leq i \leq 2$. We need to consider the following four cases: (1) $n_1 = 1, n_2 = 1$; (2) $n_1 = 1, n_2 \geq 2$; (3) $n_1 \geq 2, n_2 = 1$; (4) $n_1 \geq 2, n_2 \geq 2$. If (1), (2) or (3) occurs, then G is not connected. Thus, we only need to study (4). Let $N^0_{\overline{G}}(x) = \{x\}$, $N_{\overline{G}}(x) = \{y_1, y_2, \dots, y_k\}$ and $N^2_{\overline{G}}(x) = \{w_1, w_2, \dots, w_t\}$, where $k + t + 1 = n$. For every $w_j \in N^2_{\overline{G}}(x)$, where $1 \leq j \leq t$, define $N_{\overline{G}}(w_j) = \{y_i \in N_{\overline{G}}(x) : y_iw_j \in E(\overline{G})|1 \leq i \leq k\}$. Since $\overline{G}$ is a triangle-free graph and $\text{diam}(\overline{G}) = 2$, we have $N_{\overline{G}}(w_j) \neq \emptyset$. Without loss of generality, assume that $y_1 \notin \bigcup_{j=1}^{t} N_{\overline{G}}(w_j)$. For every $w_j \in N^2_{\overline{G}}(x)$, we have $y_1w_j \notin E(\overline{G})$. Then, $d_{\overline{G}}(y_1, w_j) \geq 3$. Hence, $\bigcup_{j=1}^{t} N_{\overline{G}}(w_j) = N_{\overline{G}}(x) = \{y_1, y_2, \dots, y_k\}$. First, we verify the following claim.

Claim 1 For every vertex y_i of $N_{\overline{G}}(x)$, where $1 \leq i \leq k$, define $N_G(y_i) = \{w_j \in N^2_{\overline{G}}(x) : y_iw_j \in E(G)|1 \leq j \leq t\}$. Then $N_G(y_i) \neq \emptyset$.

Proof Without loss of generality, we assume that $y_1 \in N_{\overline{G}}(x)$ satisfying $N_G(y_1) = \emptyset$. Then, for every vertex $w_j \in N^2_{\overline{G}}(x)$, we have $y_1w_j \notin E(G)$, and so $y_1w_j \in E(\overline{G})$. For every $y_i \in N^2_{\overline{G}}(x)\backslash\{y_1\}$. Then, we consider the following two cases: (1) There exists a vertex $w_j \in N^2_{\overline{G}}(x)$ such that $y_iw_j \notin E(\overline{G})$. (2) Every vertex $w_j \in N^2_{\overline{G}}(x)$ such that $y_iw_j \in E(\overline{G})$. If (1) occurs, then $d(y_i, w_j) \geq 3$. If (2) occurs, then G is not connected. Therefore, $N_G(y_i) \neq \emptyset$. □

Define a vertex-coloring to G as follows: assign one color to all vertices of G. We need to verify that G is vertex-rainbow 3-tree connected with the above vertex-coloring. Let u, v and w be any three distinct vertices of G and $S = \{u, v, w\}$.

Let $u, v, w \in N_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{uv, uw\}$. Let $u, v \in N_{\overline{G}}(x)$ and $w = x \in N^0_{\overline{G}}(x)$. Then, there exists a vertex $w_j \in N^2_{\overline{G}}(x)$ satisfying $ww_j, uw_j, uv \in E(G)$ by Claim 1, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{ww_j, uw_j, uv\}$. Let $u, v \in N_{\overline{G}}(x)$ and $w \in N^2_{\overline{G}}(x)$. Then, one of the following two cases holds: (1) $uw, uv \in E(G)$; (2) $uw, vw \notin E(G)$ and $uw_j \in E(G)$, where $w_j \in N^2_{\overline{G}}(x) \setminus \{w\}$. For (1), then the tree T connecting S is vertex-rainbow, where $E(T) = \{uv, uw\}$. For (2), we further consider the following subcases. If $vw_j \notin E(G)$, then $ww_j \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{uv, uw_j, ww_j\}$. If $vw_j, ww_j \in E(G)$, then the tree T connecting S is vertex-rainbow, where $E(T) = \{uv, uw_j, ww_j\}$. If $vw_j \in E(G)$ and $ww_j \notin E(G)$, then $ww_j \in E(\overline{G})$, since $\overline{G}$ is a triangle-free graph and $d = 2$, we know that there exists a vertex $y_i \in N_{\overline{G}}(x) \setminus \{u, v\}$ such that $y_i w_j \in E(\overline{G})$, $wy_i \notin E(\overline{G})$, it follows that $wy_i \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{wy_i, uy_i, uv\}$. Let $u \in N_{\overline{G}}(x)$ and $v, w \in N^2_{\overline{G}}(x)$. If $uv \in E(G)$, then T is a vertex-rainbow S-tree, where $E(T) = \{xv, xw, uv\}$. If $uv, uw \notin E(G)$, then there exists a vertex $w_j \in N^2_{\overline{G}}(x)$ such that $uw_j \in E(G)$, and so we have the following two cases: (3) $ww_j \in E(G)$; (4) $ww_j, vw_j \notin E(G)$. When (3) occurs, the tree T connecting S is vertex-rainbow, where $E(T) = \{uw_j, ww_j, wv\}$. When (4) occurs, since $\overline{G}$ is a triangle-free graph and $d = 2$, we know that there exists a vertex $y_i \in N_{\overline{G}}(x) \setminus \{u\}$ such that $y_i w_j \in E(\overline{G})$, $wy_i, vy_i \notin E(\overline{G})$, it follows that $wy_i, vy_i \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{wy_i, vy_i, uy_i\}$. Let $u \in N_{\overline{G}}(x)$, $v \in N^2_{\overline{G}}(x)$ and $w = x$. Then, there exists a vertex $w_j \in N^2_{\overline{G}}(x)$ such that $uw_j \in E(G)$, and so the tree T connecting S is vertex-rainbow, where $E(T) = \{wv, ww_j, uw_j\}$. Let u, v and $w \in N^2_{\overline{G}}(x)$. Then, the tree T connecting S is vertex-rainbow, where $E(T) = \{ux, vx, wx\}$. Let $u, v \in N^2_{\overline{G}}(x)$ and $w = x$. Note that the tree T is vertex-rainbow, where $E(T) = \{uw, vw\}$. Thus, $rvx_3(G) \leq 1$.

Obviously, we know that a tree connecting $\{x, y_i, y_j\}$ has at least three edges in G. Then, $rvx_3(G) \geq 1$, and so $rvx_3(G) = 1$. □

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References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, Berlin (2008)
2. Chandran, L.S., Das, A., Rajendraprasad, D., Varma, N.M.: Rainbow connection number and connected dominating sets. J. Graph Theory **71**, 206–218 (2012)
3. Chartrand, G., Johns, G.L., Mckeon, K.A., Zhang, P.: Rainbow connection in graphs. Math. Bohem. **133**, 85–98 (2008)
4. Chartrand, G., Okamoto, F., Zhang, P.: Rainbow trees in graphs and generalized connectivity. Networks **55**, 360–367 (2010)

5. Chen, L., Li, X., Yang, K., Zhao, Y.: The 3-rainbow index of a graph. Discuss. Math. Graph Theory **35**, 81–94 (2015)
6. Huang, X., Li, X., Shi, Y., Yue, J., Zhao, Y.: Rainbow connections for outerplanar graphs with diameter 2 and 3. Appl. Math. Comput. **242**, 277–280 (2014)
7. Krivelevich, M., Yuster, R.: The rainbow connection of a graph is (at most) reciprocal to its minimum degree. J. Graph Theory **63**, 185–191 (2010)
8. Lei, H., Li, S., Liu, H., Shi, Y.: Rainbow vertex connection of digraphs. J. Comb. Optim. **35**, 86–107 (2018)
9. Li, H., Ma, Y.: Rainbow connection number and graph operations. Discrete Appl. Math. **230**, 91–99 (2017)
10. Li, X., Liu, S., Chandran, L.S., Mathew, R., Rajendraprasad, D.: Rainbow connection number and connectivity. Electron. J. Combin. **19**, P20 (2012)
11. Li, X., Schiermeyer, I., Yang, K., Zhao, Y.: Graphs with 3-rainbow index $n-1$ and $n-2$. Discuss. Math. Graph Theory **35**, 105–120 (2015)
12. Li, X., Shi, Y.: On the rainbow vertex-connection. Discuss. Math. Graph Theory **33**, 307–313 (2013)
13. Li, X., Sun, Y.: Rainbow Connections of Graphs. Springer, New York (2012)
14. Li, X., Sun, Y.: An Updated Survey on Rainbow Connections of Graphs - A Dynamic Survey. Theory Appl, Graphs (2017)
15. Liu, H., Mestre, Â., Sousa, T.: Rainbow vertex k-connection in graphs. Discrete Appl. Math. **161**, 2549–2555 (2013)
16. Liu, H., Mestre, Â., Sousa, T.: Total rainbow k-connection in graphs. Discrete Appl. Math. **174**, 92–101 (2014)
17. Ma, Y.: Total rainbow connection number and complementary graph. Results Math. **70**, 173–182 (2016)
18. Ma, Y., Lu, Z.: Rainbow connection numbers of cayley graphs. J. Comb. Optim. **34**, 182–193 (2017)
19. Ma, Y., Lu, Z.: Rainbow connection numbers of Cayley digraphs on abelian groups. Appl. Math. Comput. **311**, 178–183 (2017)
20. Mao, Y.: The vertex-rainbow index of a graph. Discuss. Math. Graph Theory **36**, 669–681 (2016)
21. Schiermeyer, I.: On mininally rainbow k-connected graphs. Discrete Appl. Math. **161**, 702–705 (2013)