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Equivalent Norms of $cmo^p(\mathbb{R}^n)$ and Applications

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Abstract
In this paper, we prove that $cmo^p(\mathbb{R}^n)$ and $\Lambda_{n(\frac{1}{p}-1)}$, the dual spaces of local Hardy space $h^p(\mathbb{R}^n)$, are coincide with equivalent norms for $\frac{n}{n+1} < p \leq 1$. Moreover, this space can be characterized by another simple norm. As an application, we prove the $h^p(\mathbb{R}^n)$ boundedness of inhomogeneous para-product operators.

Keywords Equivalent norms · Local Hardy spaces · Dual · Inhomogeneous para-product operators

Mathematics Subject Classification 42B35 · 42B30 · 42B25 · 42B20

1 Introduction

It is well known that Hardy space $H^p(\mathbb{R}^n)$ has much better functional properties than the space $L^p(\mathbb{R}^n)$ for $p < 1$ [5]. However, as Goldberg in [14] pointed out that $H^p(\mathbb{R}^n)$ space is well suited only to the Fourier analysis, and is not stable under multiplications by the Schwartz test functions. One reason is that $H^p(\mathbb{R}^n)$ does not contain $\mathcal{S}(\mathbb{R}^n)$, the space of the Schwartz test functions. To circumvent those drawbacks, Goldberg in [14] introduced the local Hardy spaces $h^p(\mathbb{R}^n)$, $0 < p < \infty$. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with

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$\int \Phi \neq 0$, $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$ and set

$$\mathcal{M}_\Phi(f)(x) = \sup_{0<t<1} |\Phi_t * f(x)|.$$

Then $h^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n), \mathcal{M}_\Phi(f) \in L^p(\mathbb{R}^n)\}$, where $\mathcal{S}'(\mathbb{R}^n)$ is the dual of $\mathcal{S}(\mathbb{R}^n)$.

Goldberg showed that the dual of $h^1(\mathbb{R}^n)$ is $bmo(\mathbb{R}^n)$, which is defined as the set of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$S_1(f) = \sup_{|Q|<1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \mathrm{d}x < \infty$$

and

$$S_2(f) = \sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |f(x)| \mathrm{d}x < \infty,$$

equipped with the norm $\|f\|_{bmo(\mathbb{R}^n)} = \max\{S_1(f), S_2(f)\}$, where f_Q is the mean of f over Q, i.e., $f_Q = \frac{1}{|Q|}\int_Q f(x)\mathrm{d}x$. For $0 < r < 1$, let $\Lambda_r = \{f \in L^\infty(\mathbb{R}^n) : \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^r} < \infty\}$. Then, it is well known that the dual of $h^p(\mathbb{R}^n)$ is $\Lambda_{n(\frac{1}{p}-1)}$ for $\frac{n}{n+1} < p < 1$. We refer the reader to [14] for more details of Λ_r when $r \geq 1$. It is convenient to denote $\Lambda_0 = bmo(\mathbb{R}^n)$. For more results about local Hardy spaces, we refer the reader to [1–3,18,19,25–29,31–33]. Some recent developments of multi-parameter local Hardy spaces can be seen in [4,10,11].

In [31], Rychkov obtained that $h^p(\mathbb{R}^n)$ can be characterized by a continuous Littlewood–Paley–Stein square function. More precisely, let

$$\varphi_0 \in \mathcal{D}(\mathbb{R}^n) \text{ with nonzero integral, and } \varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2}), \tag{1.1}$$

where $\mathcal{D}(\mathbb{R}^n)$ is the set of all smooth functions with compact support on $\mathbb{R}^n$, then for any $N \geq 0$, there exist two functions $\phi_0, \phi \in \mathcal{D}(\mathbb{R}^n)$ such that ϕ has vanishing moments up to order N (i.e., $\int x^\alpha \phi(x)\mathrm{d}x = 0$ for all multi-indices with $|\alpha| \leq N$) and

$$f(x) = \sum_{j\in\mathbb{N}} \phi_j * \varphi_j * f(x), \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{1.2}$$

where $\mathcal{D}'(\mathbb{R}^n)$ is dual space of $\mathcal{D}(\mathbb{R}^n)$, $\mathbb{N}$ is the set of all natural numbers. Here and the following, we use dyadic dilations defined by $g_j(x) = 2^{jn}g(2^jx)$ for $j \in \mathbb{N}$, $j \geq 1$, and $g_j(x)$ for $j = 0$ is just the value of a function g_0. For any $j \in \mathbb{Z}$, denote $\Pi_j = \{Q :$ Q are dyadic cubes in $\mathbb{R}^n$ with the side length $l(Q) = 2^{-j}$, and the left lower corners of Q are $x_Q = 2^{-j}\ell$, $\ell \in \mathbb{Z}^n\}$, and $\Pi = \cup_{j\in\mathbb{N}}\Pi_j$. At last, denote $\bar{\Pi} = \{Q : Q$ are cubes in $\mathbb{R}^n\}$. Using continuous local Calderón's identity (1.2), Rychkov proved that

$f \in h^p(\mathbb{R}^n)$ if and only if

$$\left\| \left(\sum_{j=0}^{\infty} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{1.3}$$

Moreover, the continuous local Calderón's identity (1.2) also holds in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$. We want to remark that using the fact pointed out in p160 of [31], the vanishing moments of φ in (1.3) can be up to any fixed order.

Recently, the authors in [9] obtained the discrete Littlewood–Paley–Stein characterization of $h^p(\mathbb{R}^n)$. More precisely, let $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\text{supp}\widehat{\psi_0} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}; \ \widehat{\psi_0}(\xi) = 1, \ \text{if } |\xi| \leq 1, \tag{1.4}$$

and

$$\text{supp}\widehat{\psi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}, \tag{1.5}$$

and

$$|\widehat{\psi_0}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{1.6}$$

then one has the following continuous Calderón reproducing formula

$$f(x) = \sum_{j=0}^{+\infty} \psi_j * \psi_j * f(x), \tag{1.7}$$

and the discrete Calderón reproducing formula is the following

$$f(x) = \sum_{j=0}^{+\infty} \sum_{Q \in \Pi_j} |Q|(\psi_j * f)(x_Q) \times \psi_j(x - x_Q), \tag{1.8}$$

where the both series converge in $L^2(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ [13]. It is proved that in [9] $f \in h^p(\mathbb{R}^n)$ if and only if

$$\left\| \Big(\sum_{j \in \mathbb{N}} \sum_{Q \in \Pi_j} |\psi_j * f(x_Q)|^2 \chi_Q(x) \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} < +\infty,$$

where ψ_0 and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are functions satisfying conditions (1.4)–(1.6). Moreover, they proved that $cmo^p(\mathbb{R}^n)$ defined as following is also the dual space of $h^p(\mathbb{R}^n)$.

Definition 1.1 Let $0 < p \leq 1$. Suppose that ψ_0 and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are functions satisfying conditions (1.4)–(1.6). $cmo^p(\mathbb{R}^n)$ is defined by

$$cmo^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|_{cmo^p(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{cmo^p(\mathbb{R}^n)} = \sup_{P\in\bar{\Pi}} \left(\frac{1}{|P|^{\frac{2}{p}-1}} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j, Q\subseteq P} (|\psi_j * f(x_Q)|^2 \chi_Q(x))\mathrm{d}x \right)^{1/2}. \tag{1.9}$$

Obviously, $cmo^p(\mathbb{R}^n)$ is coincident with $\Lambda_{n(\frac{1}{p}-1)}$ for $\frac{n}{n+1} < p \leq 1$ since they are all the dual spaces of $h^p(\mathbb{R}^n)$. A natural question is: can $cmo^p(\mathbb{R}^n)$ be coincident with $\Lambda_{n(\frac{1}{p}-1)}$ for $\frac{n}{n+1} < p \leq 1$ by norms? In this paper, we give a positive answer.

Theorem 1.1 *Suppose that ψ_0 and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are functions satisfying conditions* (1.4)–(1.6). *Then for $\frac{n}{n+1} < p \leq 1$, $f \in \Lambda_{n(\frac{1}{p}-1)}$ if and only if $f \in cmo^p(\mathbb{R}^n)$. Moreover, $\|f\|_{\Lambda_{n(\frac{1}{p}-1)}} \approx \|f\|_{cmo^p(\mathbb{R}^n)}$.*

Furthermore, we give another equivalent norm of $cmo^p(\mathbb{R}^n)$ which has a very simple form.

Definition 1.2 Let $0 < p \leq 1$. Suppose that ψ_0 and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are functions satisfying conditions (1.4)–(1.6). $Lip^p(\mathbb{R}^n)$ is defined by

$$Lip^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{Lip^p(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{Lip^p(\mathbb{R}^n)} = \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{jn(\frac{1}{p}-1)} |\psi_j * f(x)|. \tag{1.10}$$

We remark that using continuous local Calderón's identity (1.2), the test functions in (1.10) can be replaced by those given in (1.1).

Theorem 1.2 *Suppose that ψ_0 and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are functions satisfying conditions* (1.4)–(1.6). *Then for $0 < p \leq 1$, $f \in Lip^p(\mathbb{R}^n)$ if and only if $f \in cmo^p(\mathbb{R}^n)$. Moreover, $\|f\|_{Lip^p(\mathbb{R}^n)} \approx \|f\|_{cmo^p(\mathbb{R}^n)}$.*

Since $Lip^p(\mathbb{R}^n)$ is decreasing as p increases, we have the following corollary.

Corollary 1.3 *For $0 < p \leq 1$, $cmo^p(\mathbb{R}^n)$ is increasing as p increases. Precisely, for $0 < p_1 < p_2 \leq 1$, we have*

$$\|f\|_{cmo^{p_2}(\mathbb{R}^n)} \leq C\|f\|_{cmo^{p_1}(\mathbb{R}^n)},$$

where C is a constant independent of f.

The organization of this paper is as follows. In Sect. 2 we establish the equivalence between $cmo^p(\mathbb{R}^n)$ and $Lip^p(\mathbb{R}^n)$ for $0 < p \leq 1$. The proof of Theorem 1.1 is presented in Sect. 3. For this, we prove that $Lip^p(\mathbb{R}^n)$ identifies $\Lambda_{n(\frac{1}{p}-1)}$ with equivalent norms for $\frac{n}{n+1} < p < 1$ and Λ_0 equals to $cmo^1(\mathbb{R}^n)$ with equivalent norms, that is, Theorems 3.1 and 3.2, respectively. As an application of equivalent theorems, we discuss the boundedness of inhomogeneous para-product operators on $h^p(\mathbb{R}^n)$ in last section.

Finally, we make some conventions. Throughout the paper, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as C_1, do not change in different occurrences. We denote $f \le Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2 Equivalence Between $cmo^p(\mathbb{R}^n)$ and $Lip^p(\mathbb{R}^n)$

In this section, we prove the equivalence between $cmo^p(\mathbb{R}^n)$ and $Lip^p(\mathbb{R}^n)$ for $0 < p \le 1$, that is Theorem 1.2.

We first prove

$$\|f\|_{cmo^p(\mathbb{R}^n)} \lesssim \|f\|_{Lip^p(\mathbb{R}^n)}. \tag{2.1}$$

Firstly for any fixed $P \in \bar{\Pi}$,

$$\begin{aligned}&\frac{1}{|P|^{\frac{2}{p}-1}} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j, Q\subseteq P} (|\psi_j * f(x_Q)|^2 \chi_Q(x))\mathrm{d}x\\&= \frac{1}{|P|^{\frac{2}{p}-1}} \sum_{j\in N} \sum_{Q\in\Pi_j, Q\subseteq P} |Q||\psi_j * f(x_Q)|^2.\end{aligned}$$

We consider two cases $|P| < 1$ and $|P| \ge 1$. When $|P| < 1$ and $P \in \bar{\Pi}$, there exists $j_0 \in \mathbb{N}$ such that $2^{-j_0} \le \ell(P) < 2^{-j_0+1}$. Then

$$\begin{aligned}&\frac{1}{|P|^{\frac{2}{p}-1}} \sum_{j\in N} \sum_{Q\in\Pi_j, Q\subseteq P} |Q||\psi_j * f(x_Q)|^2\\&\le \|f\|^2_{Lip^p(\mathbb{R}^n)} |P|^{1-\frac{2}{p}} \sum_{j\ge j_0} \sum_{Q\in\Pi_j, Q\subseteq P} |Q| 2^{-2jn(\frac{1}{p}-1)} := A.\end{aligned}$$

Fix $\tau \in (0, 1)$, then for any $p \in (0, 1]$,

$$\begin{aligned}A &\le \|f\|_{Lip^p(\mathbb{R}^n)} |P|^{2-\tau-\frac{2}{p}} \sum_{j\ge j_0} 2^{-jn\tau} 2^{-2jn(\frac{1}{p}-1)}\\&\lesssim \|f\|_{Lip^p(\mathbb{R}^n)} |P|^{2-\tau-\frac{2}{p}} 2^{-j_0 n\tau} 2^{-2j_0 n(\frac{1}{p}-1)} \lesssim \|f\|^2_{Lip^p(\mathbb{R}^n)},\end{aligned}$$

If $|P| \ge 1$,

$$\frac{1}{|P|^{\frac{2}{p}-1}} \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j, Q\subseteq P} |Q||\psi_j * f(x_Q)|^2$$

$$= \frac{1}{|P|^{\frac{2}{p}-1}} \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j, Q\subseteq P} |Q|(2^{jn(\frac{1}{p}-1)}|\psi_j * f(x_Q)|)^2 2^{-2jn(\frac{1}{p}-1)}$$
$$\leq \|f\|^2_{Lip^p(\mathbb{R}^n)} |P|^{1-\tau-\frac{2}{p}} \sum_{j\in\mathbb{N}} 2^{-2jn\tau} 2^{-2jn(\frac{1}{p}-1)} \lesssim \|f\|^2_{Lip^p(\mathbb{R}^n)}.$$

Thus, we obtain (2.1).

Conversely, first we claim that

$$|\psi_{j'} * f(x)| \lesssim \|f\|_{cmo^p(\mathbb{R}^n)} \|\psi_{j'}(x-\cdot)\|_{h^p(\mathbb{R}^n)}. \tag{2.2}$$

Then, if we obtain that

$$\|\psi_{j'}(x-\cdot)\|_{h^p(\mathbb{R}^n)} \lesssim 2^{j'n(1-\frac{1}{p})}, \tag{2.3}$$

one can complete the proof. Now we prove (2.2). For any fixed $x \in \mathbb{R}^n$, set $h(t) = \psi_{j'}(x-t)$. Define

$$S(h)(\cdot) = \big(\sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j} |(\psi_j * h)(x_Q)|^2 \chi_Q(\cdot)\big)^{1/2}.$$

For any $i \in \mathbb{Z}$, let

$$\Omega_i = \left\{x \in \mathbb{R}^n : S(h)(x) > 2^i\right\}, \tilde{\Omega}_i = \left\{x \in \mathbb{R}^n : M(\chi_{\Omega_i})(x) > \frac{1}{(10)^n}\right\}$$

and

$$\mathcal{B}_i = \left\{Q : Q \in \cup_{j\geq 0}\Pi_j,\ |Q\cap\Omega_i| > \frac{1}{2}|Q|,\ \ |Q\cap\Omega_{i+1}| \leq \frac{1}{2}|Q|\right\}.$$

Then, $\cup_{Q\in\mathcal{B}_i} Q \subseteq \tilde{\Omega}_i$. By local reproducing formula (1.8), one has

$$\psi_{j'} * f(x) = \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_j} \psi_j * f(x_Q) \cdot \psi_j * \psi_{j'}(x-x_Q)$$
$$= \sum_{Q\in\Pi} \psi_Q * f(x_Q)\psi_Q * \psi_{j'}(x-x_Q)$$
$$= \sum_{i=-\infty}^{+\infty} \sum_{Q^*\in B_i} \sum_{Q\subseteq Q^*, Q\in B_i} |Q||\psi_Q * f(x_Q)||\psi_Q * \psi_{j'}(x-x_Q)|.$$

in $L^2(\mathbb{R}^n)$, where Q^* are the maximal dyadic cubes in $\mathcal{B}_i$, and φ_j are denoted by φ_Q.

Hence,

$$\begin{aligned}|\psi_{j'}*f(x)| &\leq \Big\{\sum_{i=-\infty}^{+\infty}\sum_{Q^*\in B_i}|Q^*|^{1-\frac{p}{2}}\Big(\sum_{Q\subseteq Q^*,Q\in B_i}|Q|(\psi_Q*\psi_{j'}(x-x_Q))^2\Big)^{\frac{p}{2}}\\ &\quad\cdot\Big(\frac{1}{|Q^*|^{\frac{2}{p}-1}}\sum_{Q\subseteq Q^*,Q\in B_i}|Q||\psi_Q*f(x_Q)|^2\Big)^{\frac{p}{2}}\Big\}^{\frac{1}{p}}\\ &\leq \|f\|_{cmo^p(\mathbb{R}^n)}\Big\{\sum_{i=-\infty}^{+\infty}\sum_{Q^*\in B_i}|Q^*|^{1-\frac{p}{2}}\\ &\quad\Big(\sum_{Q\subseteq Q^*,Q\in B_i}|Q||\psi_Q*\psi_{j'}(x-x_Q)|^2\Big)^{\frac{p}{2}}\Big\}^{\frac{1}{p}}.\end{aligned}$$

Since $Q\in\mathcal{B}_i$, one has $|Q|\leq 2|Q\cap\widetilde{\Omega}_i\backslash\Omega_{i+1}|$. Hence,

$$\begin{aligned}\sum_{Q\subseteq Q^*,Q\in\mathcal{B}_i}|Q||(\varphi_Q*h)(x_Q)|^2 &\leq 2\sum_{Q\in\mathcal{B}_i}|Q\cap\widetilde{\Omega}_i\backslash\Omega_{i+1}||(\varphi_Q*h)(x_Q)|^2\\ &=2\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}}\sum_{Q\in\mathcal{B}_i}|(\varphi_Q*h)(x_Q)|^2\chi_Q(x)dx\\ &\leq 2\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}}(S(h)(x))^2dx\lesssim 2^{2i}|\widetilde{\Omega}_i|,\end{aligned}$$

which implies that

$$\begin{aligned}|\psi_{j'}*f(x)| &\lesssim \|f\|_{cmo^p(\mathbb{R}^n)}\left\{\sum_{i=-\infty}^{+\infty}|\tilde{\Omega}_i|^{1-\frac{p}{2}}(2^{2i}|\tilde{\Omega}_i|)^{p/2}\right\}^{\frac{1}{p}}\\ &\lesssim \|f\|_{cmo^p(\mathbb{R}^n)}\{\sum_{i=-\infty}^{+\infty}2^{ip}|\Omega_i|\}^{\frac{1}{p}}\\ &\lesssim \|f\|_{cmo^p(\mathbb{R}^n)}\|h\|_{h^p(\mathbb{R}^n)}.\end{aligned}$$

It is well known that

$$\|S(h)\|_{L^p(\mathbb{R}^n)}\approx\|\Big(\sum_{j\in\mathbb{N}}|\psi_j*h(\cdot)|^2\Big)^{1/2}\|_{L^p(\mathbb{R}^n)}.$$

Hence,

$$\begin{aligned}\|\psi_{j'}(x-\cdot)\|^p_{h^p(\mathbb{R}^n)} &\approx\int\Big(\sum_{j\in\mathbb{N}}|\psi_j*[\psi_{j'}(x-\cdot)](y)|^2\Big)^{p/2}\mathrm{d}y\\ &=\int\Big(\sum_{j\in\mathbb{N}}|\psi_j*\psi_{j'}(y)|^2\Big)^{p/2}\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int \sum_{j\in\mathbb{N}} \frac{2^{(j\wedge j')np}2^{-|j-j'|Np}}{(1+2^{(j\wedge j')}|y|)^{Lp}}dy \\
&\lesssim \sum_{j\in\mathbb{N}} 2^{(j\wedge j')(np-n)} \cdot 2^{-|j-j'|Np} \\
&= 2^{j'n(p-1)} \sum_{j\in\mathbb{N}} 2^{(j\wedge j')n(p-1)} \cdot 2^{j'n(1-p)} \cdot 2^{-|j-j'|Np} \\
&= 2^{j'n(p-1)} \sum_{j\in\mathbb{N}} 2^{[(j'-j\wedge j')]n(1-p)} \cdot 2^{-|j-j'|Np}.
\end{aligned}
$$

Since$|j'-j\wedge j'| \le |j-j'|$, one has

$$
\begin{aligned}
\|\psi_{j'}(x-\cdot)\|^p_{h^p(\mathbb{R}^n)} &\lesssim 2^{j'n(p-1)} \sum_{j\in\mathbb{N}} 2^{|j-j'|n(1-p)} \cdot 2^{-|j-j'|Np} \\
&= 2^{j'n(p-1)} \sum_{j\in\mathbb{N}} 2^{-|j-j'|(Np-n(1-p))} \\
&= 2^{j'n(p-1)} \sum_{j\in\mathbb{N}} 2^{-|j-j'|p(N-n(\frac{1}{p}-1))}.
\end{aligned}
$$

Choosing N big enough such that $N-n(\frac{1}{p}-1)>0$, one obtain (2.3).

Thus we complete the proof. □

3 Equivalence Between $\boldsymbol{Lip^p}(\mathbb{R}^n)$ and $\boldsymbol{\Lambda_r}$

Theorem 3.1 *$f\in\Lambda_r, 0<r<1$ if and only if $f\in\mathcal{S}'(\mathbb{R}^n)$ and $\sup_{j\in\mathbb{N},x\in\mathbb{R}^n} 2^{jr}|\psi_j * f(x)|<+\infty$. Moreover, $\|f\|_{\Lambda_r}\approx \sup_{j\in\mathbb{N},x\in\mathbb{R}^n} 2^{jr}|\psi_j * f(x)|$.*

Proof Suppose that $f\in\Lambda_r, 0<r<1$, that is, f is a bounded and continuous function. Then it is easy to check that $f\in\mathcal{S}'(\mathbb{R}^n)$, $|\psi_0 * f(x)|\le C\|f\|_{L^\infty(\mathbb{R}^n)}$, and for $j\ge 1$, $|\psi_j * f(x)| = |\int\psi_j(x-y)f(y)\mathrm{d}y| = |\int\psi_j(x-y)[f(y)-f(x)]\mathrm{d}y|$ since $\int\psi_j(x)\mathrm{d}x=0$. Thus, $|\psi_j * f(x)|\le C2^{-rj}\|f\|_{\Lambda_r}$. These estimates imply that $\sup_{j\in\mathbb{N},x\in\mathbb{R}^n} 2^{jr}|\psi_j * f(x)|\le C\|f\|_{\Lambda_r}$. Conversely, if $f\in\mathcal{S}'(\mathbb{R}^n)$, by the continuous Calderón reproducing formula (1.7), $f(x)=\sum_{j=0}^{\infty}\psi_j*\psi_j*f(x)$ in $\mathcal{S}'(\mathbb{R}^n)$. Note that if $\sup_{j\in\mathbb{N},x\in\mathbb{R}^n} 2^{jr}|\psi_j * f(x)|\le C$, then $|\psi_j*\psi_j*f(x)|\le C2^{-rj}$ and hence, the series $\sum_{j=0}^{\infty}\psi_j*\psi_j*f(x)$ converges uniformly. This implies that $\sum_{j=0}^{\infty}\psi_j*\psi_j*f(x)$ is a bounded and continuous function, thus $f(x)=\sum_{j=0}^{\infty}\psi_j*\psi_j*f(x)$ for all $x\in\mathbb{R}^n$. To see that $f\in\Lambda_r, 0<r<1$, we only need to show that for any given x and y with $|x-y|\le 1$, $|f(x)-f(y)|\le C|x-y|^r\sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u)|$. To do this, let $j_0 \in \mathbb{N}$ such that $2^{-j_0} \leq |x-y| < 2^{1-j_0}$. Split the series into two parts: $I(x) = \sum_{j=0}^{j_0} \psi_j * \psi_j * f(x)$ and $II(x) = \sum_{j=j_0+1}^{\infty} \psi_j * \psi_j * f(x)$. Write $I(x) - I(y) = \sum_{j=0}^{j_0} \int [\psi_j(x-z) - \psi_j(y-z)]\psi_j * f(z)\mathrm{d}z$. Applying the estimate that $\int |[\psi_j(x-z) - \psi_j(y-z)]\psi_j * f(z)\mathrm{d}z| \leq C(2^j|x-y|)2^{-rj} \sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u)|$, we obtain that

$$\begin{aligned}|I(x) - I(y)| &\leq C2^{j_0(1-r)}|x-y| \sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u)| \\ &\leq C|x-y|^r \sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u)|.\end{aligned}$$

Applying the size condition, it follows that

$$\begin{aligned}|II(x) - II(y)| &\leq C2^{-rj_0} \sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u) \\ &\leq C|x-y|^r \sup_{j\in\mathbb{N},u\in\mathbb{R}^n} 2^{jr}|\psi_j * f(u)|.\end{aligned}$$

Thus, we complete the proof. □

Theorem 3.2 *$f \in bmo(\mathbb{R}^n)$ if and only if $f \in cmo(\mathbb{R}^n)$. Moreover, $\|f\|_{bmo(\mathbb{R}^n)} \approx \|f\|_{cmo(\mathbb{R}^n)}$.*

Note that, for convenience, here we denote $cmo(\mathbb{R}^n) = cmo^1(\mathbb{R}^n)$.

Proof Given any $f \in cmo(\mathbb{R}^n)$, we now prove

$$\|f\|_{bmo(\mathbb{R}^n)} \lesssim \|f\|_{cmo(\mathbb{R}^n)}.$$

To do this, define a linear functional on $h^1(\mathbb{R}^n)$ by

$$\mathcal{L}_f(g) = \langle f, g\rangle$$

for $g \in h^1(\mathbb{R}^n)$. By the above the duality argument, namely $(h^1(\mathbb{R}^n))^* = cmo(\mathbb{R}^n)$, we have

$$|\mathcal{L}_f(g)| \leq \|f\|_{cmo(\mathbb{R}^n)}\|g\|_{h^1(\mathbb{R}^n)}.$$

Now fix a $Q \in \bar{\Pi}$, and let L^2_Q denote the space of all square integrable functions supported in Q. If $\ell(Q) \geq 1$, it is easy to see that each $g \in L^2_Q$ is a multiple of an $(1, 2)$ atom of $h^1(\mathbb{R}^n)$ with $\|g\|_{h^1(\mathbb{R}^n)} \leq C|Q|^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)}$. Hence, $\mathcal{L}_f$ is a linear functional on L^2_Q with norm at most $C|Q|^{\frac{1}{2}}\|f\|_{cmo(\mathbb{R}^n)}$. Then, the Riesz representation theorem for Hilbert spaces L^2_Q tells us that there exists $F^Q \in L^2_Q$ such that

$$\mathcal{L}_f(g) = \langle f, g\rangle = \int_Q F^Q(x)g(x)\mathrm{d}x, \forall g \in L^2_Q$$

with $\|F^Q\|_{L^2_Q} = \|\mathcal{L}_f\| \le C|Q|^{\frac{1}{2}}\|f\|_{cmo(\mathbb{R}^n)}$, which yields that f must be a square integrable function on Q and $f = F^Q$ on Q. Therefore, for any cubes Q satisfying $\ell(Q) \ge 1$,

$$\left\{\frac{1}{|Q|}\int_Q |f(x)|^2\mathrm{d}x\right\}^{1/2} = \left\{\frac{1}{|Q|}\int_Q |F^Q(x)|^2\mathrm{d}x\right\}^{1/2} \le C\|f\|_{cmo(\mathbb{R}^n)},$$

which implies that

$$\sup_{|Q|\ge 1}\frac{1}{|Q|}\int_Q |f(x)|\mathrm{d}x \le C\|f\|_{cmo(\mathbb{R}^n)}.$$

On the other hand, if $\ell(Q) < 1$, let $L^2_{Q,0} = \{f \in L^2_Q, \int f = 0\}$. Recall that a function $a(x)$ supported in a cube Q is said to be a $(p, 2)$ atom of $h^p(\mathbb{R}^n)$, $0 < p \le 1$, if it satisfies a size condition $\|a\|_{L^2(\mathbb{R}^n)} \le |Q|^{\frac{1}{2}-\frac{1}{p}}$ and a cancellation condition $\int x^\alpha a(x)\mathrm{d}x = 0, \alpha \in \mathbb{N}^n, |\alpha| \le n(\frac{1}{p}-1)$, when $\ell(Q) < 1$ [14]. Then, each $g \in L^2_{Q,0}$ is a multiple of an $(1, 2)$ atom of $h^1(\mathbb{R}^n)$ with $\|g\|_{h^1(\mathbb{R}^n)} \le C|Q|^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)}$, and $\mathcal{L}_f$ is a linear functional on $L^2_{Q,0}$ with norm at most $C|Q|^{\frac{1}{2}}\|f\|_{cmo(\mathbb{R}^n)}$. By the Riesz representation theorem, there exists some $F^Q \in L^2_{Q,0}$ such that

$$\mathcal{L}_f(g) = \langle f, g\rangle = \int_Q F^Q(x)g(x)\mathrm{d}x, \forall g \in L^2_{Q,0}$$

with $\|F^Q\|_{L^2_Q} = \|\mathcal{L}_f\| \le C|Q|^{\frac{1}{2}}\|f\|_{cmo(\mathbb{R}^n)}$, which yields that f must be a square integrable function on Q and $f = F^Q + c_Q$ on Q for some constant c_Q since $\int g(x)\mathrm{d}x = 0$. Therefore, for any cube Q satisfying $\ell(Q) < 1$,

$$\left\{\frac{1}{|Q|}\int_Q |f(x) - c_Q|^2\mathrm{d}x\right\}^{1/2} = \left\{\frac{1}{|Q|}\int_Q |F^Q(x)|^2\mathrm{d}x\right\}^{1/2} \le C\|f\|_{cmo(\mathbb{R}^n)},$$

which implies that

$$\sup_{|Q|<1}\left\{\frac{1}{|Q|}\int_Q |f(x) - c_Q|^2\mathrm{d}x\right\}^{1/2} \le C\|f\|_{cmo(\mathbb{R}^n)}.$$

Hence

$$\sup_{|Q|<1}\Big\{\frac{1}{|Q|}\int_Q |f(x)-f_Q|^2 \mathrm{d}x\Big\}^{1/2} \le C\|f\|_{cmo(\mathbb{R}^n)}.$$

The estimates above imply that $\|f\|_{bmo(\mathbb{R}^n)} \lesssim \|f\|_{cmo(\mathbb{R}^n)}$.

We now prove

$$\|f\|_{cmo(\mathbb{R}^n)} \lesssim \|f\|_{bmo(\mathbb{R}^n)}.$$

By the definition of $bmo(\mathbb{R}^n)$, it is easy to see that,

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\mathrm{d}x \le 2\|f\|_{bmo(\mathbb{R}^n)},$$

where sup is taken over all cubes with sides parallel to the coordinate axes in $\mathbb{R}^n$.

For any $P\in\bar{\Pi}$, let $P^* = 3\sqrt{n}P$, the cube with the same center of P and side length $3\sqrt{n}\ell(P)$. Split f as $f = f_1+f_2+f_3$ with $f_1 = (f-f_{P^*})\chi_{P^*}$, $f_2 = (f-f_{P^*})\chi_{(P^*)^c}$ and $f_3 = f_{P^*}$. It is easy to see that

$$\begin{aligned}&\frac{1}{|P|}\int_P \sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j, Q\subseteq P}(|\psi_j * f(x_Q)|^2\chi_Q(x))\\ &\quad\lesssim \sum_{i=1}^{3}\int_P \sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j, Q\subseteq P}(|\psi_j * f_i(x_Q)|^2\chi_Q(x)).\end{aligned}$$

For f_1, by Littlewood–Paley theory,

$$\begin{aligned}&\frac{1}{|P|}\int_P \sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j, Q\subseteq P}(|\psi_j * f_1(x_Q)|^2\chi_Q(x))\mathrm{d}x\\ &= \frac{1}{|P|}\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j, Q\subseteq P}|Q||\psi_j * f_1(x_Q)|^2\\ &\le \frac{1}{|P|}\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j}|Q||\psi_j * f_1(x_Q)|^2\\ &\lesssim \frac{1}{|P|}\|f_1\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{|P|}\int_{P^*}|f(y)-f_{P^*}|^2\mathrm{d}y \lesssim \|f\|^2_{bmo(\mathbb{R}^n)}.\end{aligned}$$

For f_2, first of all, it is easy to see that $|\psi_j * f_2(x_Q)| = |\int \psi_j(x_Q-y)f_2(y)dy|$ is dominated by

$$\int_{(P^*)^c}\frac{2^{-j}}{(2^{-j}+|x_Q-y|)^{n+1}}|f_2(y)|\mathrm{d}y \approx \int_{(P^*)^c}\frac{2^{-j}}{(2^{-j}+|x_P-y|)^{n+1}}|f_2(y)|\mathrm{d}y,$$

where x_P is the center of P. Hence

$$
\begin{aligned}
&|\psi_j * f_2(x_Q)| \\
&\lesssim 2^{-j}\sum_{k=1}^{+\infty}\int_{2^{k-1}\ell(P)\le|x_P-y|<2^k\ell(P)}\frac{1}{(2^{-j}+|x_P-y|)^{n+1}}|f_2(y)|\mathrm{d}y \\
&\lesssim 2^{-j}\sum_{k=1}^{+\infty}\frac{1}{(2^k\ell(P))^{n+1}}\int_{|x_P-y|<2^k\ell(P)}|f_2(y)|\mathrm{d}y \\
&\lesssim 2^{-j}\sum_{k=1}^{+\infty}\frac{1}{(2^k\ell(P))^{n+1}}\left\{\int_{|x_P-y|<2^k\ell(P)}|f(y)\right. \\
&\quad \left. -f_{2^kP^*}|\mathrm{d}y+(2^k\ell(P))^n|f_{2^kP^*}-f_{P^*}|\right\},
\end{aligned}
$$

combining with classical result about $BMO(\mathbb{R}^n)$ functions, gives

$$|\psi_j * f_2(x_Q)| \lesssim 2^{-j}\ell(P)^{-1}\|f\|_{bmo(\mathbb{R}^n)}.$$

Note that there exists $j_0\in\mathbb{Z}$ such that $2^{-j_0}<\ell(P)\le 2^{-j_0+1}$. Then,

$$
\begin{aligned}
&\frac{1}{|P|}\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j,Q\subseteq P}|Q||\psi_j * f_2(x_Q)|^2 \\
&\quad\lesssim \frac{1}{|P|}\sum_{j=j_0}^{\infty}\sum_{Q\in\Pi_j,Q\subseteq P}|Q|2^{-2(j-j_0)}\|f\|^2_{bmo(\mathbb{R}^n)} \lesssim \|f\|^2_{bmo(\mathbb{R}^n)}.
\end{aligned}
$$

For the constant item f_3, note that $\int\psi_0=1$ and $\int\psi_j=0,\ j\ge 1$. Hence if $\ell(P)<1$,

$$\frac{1}{|P|}\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j,Q\subseteq P}|Q||\psi_j * f_3(x_Q)|^2=\frac{1}{|P|}\sum_{j\ge 1}\sum_{Q\in\Pi_j,Q\subseteq P}|Q||\psi_j * f_3(x_Q)|^2=0.$$

Otherwise, if $\ell(P)\ge 1$

$$\frac{1}{|P|}\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j,Q\subseteq P}|Q||\psi_j * f_3(x_Q)|^2\le|f_3|^2\le(S_2(f))^2\le\|f\|^2_{bmo(\mathbb{R}^n)},$$

since $f_3=f_{P^*}$ with $\ell(P^*)>1$.

Thus, we complete the proof. □

We want to point out that the results of this section can also be seen in [8]. For completeness, we give their proofs.

4 Applications

In this section, we discuss the boundedness of inhomogeneous para-product operators on $h^p(\mathbb{R}^n)$. Firstly, we recall some about non-convolution singular integral operators.

A locally integral function $\mathcal{K}(x, y)$ defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a Calderón–Zygmund kernel with regularity exponent $\varepsilon > 0$ if there exists a constant $C > 0$ such that

$$|\mathcal{K}(x, y)| \leq C\frac{1}{|x-y|^n}, \quad \text{for} \quad x \neq y, \tag{4.1}$$

and

$$|\mathcal{K}(x, y) - \mathcal{K}(x, y')| \leq C\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}}, \tag{4.2}$$

whenever $|y - y'| \leq \frac{1}{2}|x - y|$, and

$$|\mathcal{K}(x, y) - \mathcal{K}(x', y)| \leq C\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}, \tag{4.3}$$

whenever $|x-x'| \leq \frac{1}{2}|x-y|$. The operator T is said to be non-convolution Calderón–Zygmund singular integral if T is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle Tf, g\rangle = \int \mathcal{K}(x, y) f(y) g(x) \mathrm{d}x\mathrm{d}y$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with disjoint supports, where $\mathcal{K}$ is a Calderón–Zygmund kernel. The fundamental result for the third generation Calderón–Zygmund singular integrals is the Theorem $T1$ contained in [6]. See [7,16,17,20–22,30] for other versions of the $T1$ theorems for Hardy, Besov and Triebel–Lizorkin spaces. If the Calderón–Zygmund kernel $\mathcal{K}$ satisfies a restrictve size condition, namely, the condition (4.1) is replaced by

$$|\mathcal{K}(x, y)| \leq C \min\{\frac{1}{|x-y|^n}, \frac{1}{|x-y|^{n+\delta}}\}, \text{ for some } \delta > 0 \text{ and } x \neq y, \tag{4.4}$$

we then obtain an inhomogeneous Calderón–Zygmund kernel associate with regularity exponent $\varepsilon, \delta > 0$, and inhomogeneous Calderón–Zygmund singular integral associate with regularity exponent $\varepsilon, \delta > 0$, respectively. It is well known that each pseudo-differential operator $T_\sigma f(x) = \int \sigma(x, \xi) e^{2\pi i x\xi} \hat{f}(\xi) d\xi$ with $\sigma \in S^0_{1,0}$ is an inhomogeneous Calderón–Zygmund singular integral. For the boundedness of operators on local Hardy spaces, we refer the readers to the work in [14,26]

Using atomic decomposition, one has the following result.

Theorem 4.1 *Suppose that T is an inhomogeneous Calderón–Zygmund singular integral associate with regularity exponent* $\varepsilon, \delta > 0$. *Then, if* T *is bounded on* $L^2(\mathbb{R}^n)$, T *is bounded from* $h^p(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$ *if* $\max\{\frac{n}{n+\varepsilon}, \frac{n}{n+\delta}\} < p \leq 1$.

The proof of Theorem 4.1 is standard, and we refer the readers to [15] for non-convolution Calderón–Zygmund singular integral on $H^p(\mathbb{R}^n)$, to [12] for Journé's type of multi-parameter singular integral operators on multi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, to [10] for inhomogeneous Journé's type of multi-parameter singular integral operators on multi-parameter Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, so we omit its proof. We want to remark that it is enough to prove Theorem 4.1 under conditions (4.4) and (4.2).

Now we begin to define inhomogeneous para-product operators. In this section, we always suppose that ψ_0, ψ, φ belong to $\mathcal{D}(\mathbb{R}^n)$ supported on unit ball centered at origin satisfying that ψ has vanishing moments up to some proper order N and $\int \psi_0(x)\mathrm{d}x = \int \varphi(x)\mathrm{d}x = 1$. Set $\psi_j(x) = 2^{jn}\psi(2^j x)$ for $j \in \mathbb{N}, j \geq 1$, and $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ for $j \in \mathbb{N}$.

By Corollary 1.3, $bmo(\mathbb{R}^n) \supseteq cmo^p(\mathbb{R}^n)$ for all $0 < p \leq 1$. Fixed any $b \in bmo(\mathbb{R}^n)$, inhomogeneous para-product operators are defined as following:

$$\pi_b(f)(x) = \sum_{j\in N} \sum_{Q\in\Pi_j} |Q|\psi_j * b(x_Q) \cdot \psi_j(x - x_Q)\varphi_j * f(x_Q),$$

and its adjoint operator

$$\pi_b^*(f)(x) = \sum_{k\in N} \sum_{Q\in\Pi_k} |Q|\varphi_k(x - x_Q)\psi_k * b(x_Q) \cdot \psi_k * f(x_Q).$$

One can check that π_b, π_b^* are inhomogeneous Calderón–Zygmund singular integrals associate with regularity exponent $\varepsilon = \delta = 1$ and bounded on $L^2(\mathbb{R}^n)$. Hence π_b, π_b^* are bounded from $h^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if $\frac{n}{n+1} < p \leq 1$. Now we give the main result of this section.

Theorem 4.2 *Suppose that* $b \in Lip^{\frac{n}{n+1}}(\mathbb{R}^n)$. *Then,* π_b, π_b^* *are bounded on* $h^p(\mathbb{R}^n)$ *if* $\frac{n}{n+1} < p \leq 1$.

To prove Theorem 4.2, we will follow the approach in [23,24] by reducing the $h^p(\mathbb{R}^n)$ boundedness of π_b and π_b^* to $h^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ boundedness.

Since the proofs of the $h^p(\mathbb{R}^n)$ boundedness of π_b^* is similar as $\pi_b's$ but more difficult, we only give the proof of π_b^*. Obviously, $L^2(\mathbb{R}^n) \cap h^p(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n) \cap h^p(\mathbb{R}^n)$, using (1.3), one has

$$\|\pi_b^*(f)\|_{h^p(\mathbb{R}^n)} = \left\| \left\{ \sum_{k\in\mathbb{N}} |\psi_k * \pi_b^*(f)(x)|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

It is easy to see that $\{\sum_{k\in\mathbb{N}}|\psi_k*\pi_b^*(f)(x)|^2\}^{\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^n)$, since $b\in Lip^{\frac{n}{n+1}}(\mathbb{R}^n)\subseteq bmo(\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
\psi_k*\pi_b^*(f)(x)&=\int\psi_k(x-z)\pi_b^*(f)(z)\mathrm{d}z\\
&=\int\psi_k(x-z)\sum_{k'\in\mathbb{N}}\sum_{Q'\in\Pi_{k'}}|Q'|\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}*f(x_{Q'})\mathrm{d}z\\
&=\iint\sum_{k'\in\mathbb{N}}\sum_{Q'\in\Pi_{k'}}|Q'|\psi_k(x-z)\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)f(y)\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

Set

$$S_k(x,y)=\int\sum_{k'\in\mathbb{N}}\sum_{Q'\in\Pi_{k'}}|Q'|\psi_k(x-z)\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)\mathrm{d}z.$$

Then, the $h^p(\mathbb{R}^n)$ boundedness of π_b^* can be reduced to $h^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ boundedness of vector value inhomogeneous Calderón–Zygmund singular integral with kernel $\{S_k(x,y)\}$. We will finish the proof of Theorem 4.2 by proving that $\{S_k(x,y)\}$ satisfies vector value inhomogeneous Calderón–Zygmund kernel conditions, namely,

Lemma 4.3 *Let* $S_k(x,y)=\int\sum_{k'\in\mathbb{N}}\sum_{Q'\in\Pi_{k'}}|Q'|\psi_k(x-z)\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)dz$, *and suppose that* $b\in Lip^{\frac{n}{n+1}}(\mathbb{R}^n)$. *Then for every* $\varepsilon\in(0,1)$,

(i) $\{\sum_{k\in\mathbb{N}}|S_k(x,y)|^2\}^{1/2}\le C\min\{\frac{1}{|x-y|^n},\frac{1}{|x-y|^{n+1}}\}$ *if* $|x-y|>0$;

(ii) $\{\sum_{k\in\mathbb{N}}|S_k(x,y')-S_k(x,y)|^2\}^{1/2}\le C\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}}$, *if* $|y-y'|\le\frac{1}{2}|x-y|$.

Proof Split $S_k(x,y)$ as following

$$\begin{aligned}
S_k(x,y)&=\sum_{k'\ge k}\sum_{Q'\in\Pi_{k'}}|Q'|\int\psi_k(x-z)\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)\mathrm{d}z\\
&\quad+\sum_{k'<k}\sum_{Q'\in\Pi_{k'}}|Q'|\int\psi_k(x-z)\varphi_{k'}(z-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)\mathrm{d}z\\
&=S_k^1(x,y)+S_k^2(x,y).
\end{aligned}$$

We first prove size condition (i). For S_k^1, one has

$$\begin{aligned}
S_k^1=&\sum_{k'\ge k}\sum_{Q'\in\Pi_{k'}}|Q'|\int[\psi_k(x-z-x_{Q'})\\
&-\psi_k(x-x_{Q'})]\varphi_{k'}(z)\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)\mathrm{d}z
\end{aligned}$$

$$+\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'|\psi_k(x-x_{Q'})\psi_{k'}*b(x_{Q'})\psi_{k'}(x_{Q'}-y)=A+B.$$

We first show that

$$|B|\lesssim\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}. \tag{4.5}$$

If $|x-y|\leq 2\cdot 2^{-k}$,

$$\begin{aligned}|B| &\leq \sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'||\psi_k(x-x_{Q'})||\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|\\ &\leq \sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'|\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-x_{Q'}|)^{n+\varepsilon'}}|\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|\\ &\leq \frac{2^{-k}}{2^{-k(n+1)}}\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'||\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|\\ &\lesssim \frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'||\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|. \end{aligned}\tag{4.6}$$

When $|x-y|>2\cdot 2^{-k}$. Note that $|x_{Q'}-y|\leq 2^{-k'}\leq 2^{-k}$. Hence, $|x-x_{Q'}|\geq |x-y|-|y-x_{Q'}|\geq\frac{1}{2}|x-y|$, which also yields (4.6). Thus, (4.5) is obtained if we can prove

$$\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'||\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|\leq C.$$

Indeed, using $b\in Lip^{\frac{n}{n+1}}(\mathbb{R}^n)$, one has

$$\begin{aligned}&\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'||\psi_{k'}*b(x_{Q'})||\psi_{k'}(x_{Q'}-y)|\\ &\quad\leq \|b\|_{Lip^{\frac{n}{n+1}}(\mathbb{R}^n)}\sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'|2^{k'(1-\frac{n+1}{n})}|\psi_{k'}(x_{Q'}-y)|\\ &\quad\lesssim \sum_{k'\geq 0}2^{k'(1-\frac{n+1}{n})}\sum_{Q'\in\Pi_{k'}}\int_{Q'}\frac{2^{-k'}}{(2^{-k'}+|x_{Q'}-y|)^{n_1}}dz\\ &\quad\lesssim \int_{Q'}\frac{2^{-k'}}{(2^{-k'}+|z-y|)^{n_1}}dz\leq C,\end{aligned}$$

since $2^{-k'}+|x_{Q'}-y|\approx 2^{-k'}+|z-y|$ for any $z\in Q'$.

For A, firstly, by the fact that $\psi_{k'} * b(x_{Q'})$ is bounded uniformly, one has

$$|A| \lesssim \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| \int |\psi_k(x - z - x_{Q'}) - \psi_k(x - x_{Q'})||\varphi_{k'}(z)||\psi_{k'}(x_{Q'} - y)|\mathrm{d}z.$$

Hence, by the following classical result, for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} &|\psi_k(x - z - x_{Q'}) - \psi_k(x - x_{Q'})| \\ &\quad \lesssim \left(\frac{|z|}{2^{-k}}\right)^{\varepsilon} \left(\frac{2^{-k}}{(2^{-k} + |x - z - x_{Q'}|)^{n+1}} + \frac{2^{-k}}{(2^{-k} + |x - x_{Q'}|)^{n+1}}\right), \end{aligned}$$

and using the fact that the support of ψ is unit ball at origin, A is dominated by

$$\begin{aligned} &\sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| \\ &\quad \int \left(\frac{2^{-k'}}{2^{-k}}\right)^{\varepsilon} \left(\frac{2^{-k}}{(2^{-k} + |x - z - x_{Q'}|)^{n+1}} + \frac{2^{-k}}{(2^{-k} + |x - x_{Q'}|)^{n+1}}\right) \\ &\quad \frac{2^{-k'}}{(2^{-k'} + |z|)^{n+1}} |\psi_{k'}(x_{Q'} - y)|\mathrm{d}z \\ &= \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| 2^{-(k'-k)\varepsilon} |\psi_{k'}(x_{Q'} - y)| \\ &\quad \int \frac{2^{-k}}{(2^{-k} + |x - z - x_{Q'}|)^{n+1}} \frac{2^{-k'}}{(2^{-k'} + |z|)^{n+1}} \mathrm{d}z \\ &\quad + \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| 2^{-(k'-k)\varepsilon} \frac{2^{-k}}{(2^{-k} + |x - x_{Q'}|)^{n+1}} |\psi_{k'}(x_{Q'} - y)| \\ &\quad \int \frac{2^{-k'}}{(2^{-k'} + |z|)^{n+1}} \mathrm{d}z \\ &\lesssim \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| 2^{-(k'-k)\varepsilon} |\psi_{k'}(x_{Q'} - y)| \frac{2^{-k}}{(2^{-k} + |x - x_{Q'}|)^{n+1}} \\ &\lesssim \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| 2^{-(k'-k)\varepsilon} \frac{2^{-k'}}{(2^{-k'} + |y - x_{Q'}|)^{n+1}} \frac{2^{-k}}{(2^{-k} + |x - x_{Q'}|)^{n+1}} \\ &= \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} 2^{-(k'-k)\varepsilon'} \int_{Q'} \frac{2^{-k'(1-\varepsilon')}}{(2^{-k'} + |y - x_{Q'}|)^{n+1-\varepsilon'}} \frac{2^{-k(1-\varepsilon')}}{(2^{-k} + |x - x_{Q'}|)^{n+1-\varepsilon'}} \mathrm{d}z \\ &\approx \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} 2^{-(k'-k)\varepsilon'} \int_{Q'} \frac{2^{-k'(1-\varepsilon')}}{(2^{-k'} + |y - z|)^{n+1-\varepsilon'}} \frac{2^{-k(1-\varepsilon')}}{(2^{-k} + |x - z|)^{n+1-\varepsilon'}} \mathrm{d}z, \end{aligned}$$

since $2^{-k'} + |y - x_{Q'}| \approx 2^{-k'} + |y - z|$, $2^{-k} + |x - x_{Q'}| \approx 2^{-k} + |x - z|$ if $z \in Q'$. Then

$$|A| \lesssim \sum_{k' \geq k} 2^{-(k'-k)\varepsilon'} \frac{2^{-k}}{(2^{-k} + |x - y|)^{n+1}} \lesssim \frac{2^{-k}}{(2^{-k} + |x - y|)^{n+1}}. \tag{4.7}$$

By estimates (4.5) and (4.7), one has

$$|S_k^1(x, y)| \lesssim \frac{2^{-k}}{(2^{-k} + |x - y|)^{n+1}}. \tag{4.8}$$

For S_k^2, using the cancelation condition of ψ, one has

$$\begin{aligned} & S_k^2(x, y) \\ & = \sum_{k' < k} \sum_{Q' \in \Pi_{k'}} |Q'| \int \psi_k(x - z)[\varphi_{k'}(z - x_{Q'}) \\ & \quad -\varphi_{k'}(x - x_{Q'})]\psi_{k'} * b(x_{Q'})\psi_{k'}(x_{Q'} - y)\mathrm{d}z. \end{aligned}$$

Then with a similar process to estimate A, one can see that (4.8) also holds for S_k^2. Thus, we can complete the proof of size condition (i).

To prove (ii), we only estimate the smoothness of $S_k^1(x, y)$ with the second variable y since the proof to obtain the smoothness of $S_k^2(x, y)$ is similar and easier. Set

$$\begin{aligned} & S_k^1(x, y) - S_k^1(x, y') \\ & = \int \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| \psi_k(x - z)\psi_{k'}(z - x_{Q'})\psi_{k'} * b(x_{Q'})[\psi_{k'}(x_{Q'} - y)] \\ & \quad -\psi_{k'}(x_{Q'} - y')]\mathrm{d}z \\ & = \int \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'|[\psi_k(x - z) - \psi_k(x - x_{Q'})]\varphi_{k'}(z - x_{Q'})\psi_{k'} * b(x_{Q'}) \\ & \quad \cdot[\psi_{k'}(x_{Q'} - y) - \psi_{k'}(x_{Q'} - y')]dz \\ & \quad + \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |Q'| \psi_k(x - x_{Q'})\psi_{k'} * b(x_{Q'}) \left[\psi_{k'}(x_{Q'} - y) - \psi_{k'}(x_{Q'} - y')\right] \\ & = A_k^1(x, y) + A_k^2(x, y). \end{aligned}$$

For any $\varepsilon \in (0, 1)$, using the support condition, it is easy to have

$$\begin{aligned} & |A_k^1(x, y)| \\ & \lesssim \int_{|z - x_{Q'}| \leq 2^{-k'}} \sum_{k' \geq k} \sum_{Q' \in \Pi_{k'}} |\, Q' \,| \left(\frac{|z - x_{Q'}|}{2^{-k}}\right)^{\varepsilon} \left[\frac{2^{-k}}{(2^{-k} + |\, x - z \,|)^{n+1}}\right. \end{aligned}$$

$$+\frac{2^{-k}}{(2^{-k}+\mid x-x_{Q'}\mid)^{n+1}}\Bigg]\frac{2^{-k'}}{(2^{-k'}+\mid z-x_{Q'}\mid)^{n+1}}\mid\big[\psi_{k'}(x_{Q'}-y)$$
$$-\psi_{k'}(x_{Q'}-y')\big]\mid \mathrm{d}z,$$

since $\mid \psi_{k'} * b(x_{Q'}) \mid\leq C$, uniformly for k' and $x_{Q'}$. Then,

$$\begin{aligned}
|A_k^1(x,y)| &\lesssim \sum_{k'\geq k} 2^{(k-k')\varepsilon}\sum_{Q'\in\Pi_{k'}}|Q'|\frac{2^{-k}}{(2^{-k}+|x-x_{Q'}|)^{n+1}}\left(\frac{|y-y'|}{2^{-k'}}\right)^{\varepsilon'}\\
&\quad\cdot\left[\frac{2^{-k'}}{(2^{-k'}+|x_{Q'}-y|)^{n+1}}+\frac{2^{-k'}}{(2^{-k'}+|x_{Q'}-y'|)^{n+1}}\right]\\
&\lesssim \sum_{k'\geq k} 2^{(k-k')\varepsilon}\int\frac{2^{-k}}{(2^{-k}+|x-u|)^{n+1}}\left(\frac{|y-y'|}{2^{-k'}}\right)^{\varepsilon'}\\
&\quad\cdot\left[\frac{2^{-k'}}{(2^{-k'}+|u-y|)^{n+1}}+\frac{2^{-k'}}{(2^{-k'}+|u-y'|)^{n+1}}\right]\mathrm{d}u\\
&\lesssim \sum_{k'\geq k} 2^{(k-k')\varepsilon}2^{(k'-k)\varepsilon'}(\frac{|y-y'|}{2^{-k}})^{\varepsilon'}\left[\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}\right.\\
&\quad\left.+\frac{2^{-k}}{(2^{-k}+|x-y'|)^{n+1}}\right]\\
&\lesssim \left(\frac{|y-y'|}{2^{-k}}\right)^{\varepsilon'}\left[\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}+\frac{2^{-k}}{(2^{-k}+|x-y'|)^{n+1}}\right]
\end{aligned}$$

provided $\varepsilon > \varepsilon'$. Hence, for any $\varepsilon' \in (0,1)$, when $|y-y'| \leq \frac{1}{2}|y-x|$,

$$\begin{aligned}
\sum_{k\in\mathbb{N}}|A_k^1(x,y)| &\lesssim \sum_{k\in\mathbb{N}}\left(\frac{\mid y-y'\mid}{2^{-k}}\right)^{\varepsilon'}\left[\frac{2^{-k}}{(2^{-k}+\mid x-y\mid)^{n+1}}+\frac{2^{-k}}{(2^{-k}+|x-y'|)^{n+1}}\right]\\
&\lesssim \frac{|y-y'|^{\varepsilon'}}{|x-y|^{n+\varepsilon'}}.
\end{aligned}$$

At last, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
|A_k^2(x,y)| &\lesssim \sum_{k'\geq k}\sum_{Q'\in\Pi_{k'}}|Q'|\frac{2^{-k}}{(2^{-k}+|x-x_{Q'}|)^{n+1}}|\psi_{k'}*b(x_{Q'})|\left(\frac{|y-y'|}{2^{-k'}}\right)^{\varepsilon}\\
&\quad\cdot\left[\frac{2^{-k'}}{(2^{-k'}+|x_{Q'}-y|)^{n+1}}+\frac{2^{-k'}}{\left(2^{-k'}+|x_{Q'}-y'|\right)^{n+1}}\right]\\
&\lesssim \sum_{k'>k}\sum_{Q'\in\Pi_{k'}}\int_{Q'}\frac{2^{-k}}{(2^{-k}+|x-u|)^{n+1}}|\psi_{k'}*b(x_{Q'})|(\frac{|y-y'|}{2^{-k'}})^{\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left[\frac{2^{-k'}}{(2^{-k'}+|u-y|)^{n+1}}+\frac{2^{-k'}}{(2^{-k'}+|u-y'|)^{n+1}}\right]du\\
&\lesssim \|b\|_{Lip^{\frac{n}{n+1}}(\mathbb{R}^n)}\sum_{k'>k}2^{-k'}\sum_{Q'\in\Pi_{k'}}\int_{Q'}\frac{2^{-k}}{(2^{-k}+|x-u|)^{n+1}}\left(\frac{|y-y'|}{2^{-k'}}\right)^{\varepsilon}\\
&\cdot\left[\frac{2^{-k'}}{(2^{-k'}+|u-y|)^{n+1}}+\frac{2^{-k'}}{(2^{-k'}+|u-y'|)^{n+1}}\right]\mathrm{d}u\\
&\lesssim \|b\|_{Lip^{\frac{n}{n+1}}(\mathbb{R}^n)}\sum_{k'>k}2^{-k'}(\frac{|y-y'|}{2^{-k'}})^{\varepsilon}\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}\\
&\lesssim \|b\|_{Lip^{\frac{n}{n+1}}(\mathbb{R}^n)}\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}|y-y'|^{\varepsilon}\sum_{k'>k}2^{-k'(1-\varepsilon)}\\
&\lesssim \|b\|_{Lip^{\frac{n}{n+1}}(\mathbb{R}^n)}\frac{2^{-k}}{(2^{-k}+|x-y|)^{n+1}}|y-y'|^{\varepsilon}.
\end{aligned}
$$

Thus, for any $\varepsilon\in(0,1)$,

$$
\sum_{k\in\mathbb{N}}|A_k^2(x,y)|\lesssim\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}.
$$

Then, we complete the proof. □

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