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Ergodic Shadowing Properties of Iterated Function Systems

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Abstract

In this paper, we generalize the notion of the ergodic shadowing property to the iterated function systems and prove some related theorems on this notion. In addition, we give an example to show that there is an iterated function system which has the ergodic shadowing property but not weakly mixing. Moreover, we show that ergodic shadowing property implies the average shadowing property for iterated function systems.

Keywords Ergodic shadowing · Average shadowing · Iterated function system

Mathematics Subject Classification Primary: 37C50; Secondary 37C15

1 Introduction

Throughout this paper, (S, X, ϕ) denotes a *topological dynamical system* (or *dynamical system* for short), where S is a topological semigroup, (X, ρ) is a compact metric space and

$$\phi : S \times X \to X, (s, x) \mapsto sx$$

is a continuous action. So $t(sx) = (ts)x$ for all $x \in X$, $t, s \in S$. Sometimes, the dynamical system is denoted as a pair (S, X). Let $\mathscr{F} = \{f_0, f_1, \ldots, f_{m-1}\}$ be a family of continuous maps on X. The *iterated function system* IFS($\mathscr{F}$) is the action of the semigroup generated by $\{f_0, f_1, \ldots, f_{m-1}\}$ on X. If $\mathscr{F} = \{f\}$, then it is the

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classical dynamical system (it also is called a *cascade*). We use the standard notation: (X, f). In this paper, let $\mathbb{Z}_+$ be the set of nonnegative integers, and let $\mathbb{N}$ be the set of positive integers, respectively. For any $n \in \mathbb{N}$ and $n \geq 2$, let $X^n = X \times X \times \cdots \times X$ (n times). The action of S on X^n is defined by $s(x_1, x_2, \ldots, x_n) = (sx_1, sx_2, \ldots, sx_n)$, for all $s \in S$ and $(x_1, x_2, \ldots, x_n) \in X^n$.

In the general qualitative theory of dynamical systems, the shadowing property plays an important role (cf. [1]). At the end of 1980s, the average shadowing property was introduced and studied by Blank [3,4] in cascades. Later, some new notions of shadowing were introduced in cascades. For example, Dastjerdi introduced $\underline{d}$-shadowing [5] and Fakhari and Ghane introduced the ergodic shadowing [7]. Recently, we observed an increasing interest in shadowing property for iterated function systems (cf. [2,6,8–11,13]). Some important notions in cascades were extended to iterated function systems, such as chain transitivity [2], shadowing [8] and the average shadowing [10], etc.

In cascades, the notion of ergodic shadowing property was introduced in [7], and the following result was proved (see Theorem A in [7]).

Theorem 1.1 ([7], Theorem A) *Let f be continuous onto map of a compact metric space X. For the dynamical system (X, f), the following properties are equivalent:*

1. *ergodic shadowing;*
2. *shadowing and chain mixing;*
3. *shadowing and topologically mixing;*
4. *pseudo-orbital specification.*

In this paper, we generalize the notion of the ergodic shadowing property to the iterated function systems. The following theorems are main results of this paper.

Theorem 1.2 *For an iterated function system IFS($\mathscr{F}$), where one of $\mathscr{F}$ is surjective, the following properties are equivalent:*

1. *ergodic shadowing;*
2. *shadowing and chain mixing;*
3. *shadowing and topologically I-mixing;*
4. *pseudo-orbital specification.*

Theorem 1.3 *Let IFS($\mathscr{F}$) be an iterated function system, where one of $\mathscr{F}$ is surjective. If IFS($\mathscr{F}$) has the ergodic shadowing property, then IFS($\mathscr{F}$) has the average shadowing property.*

By Theorem 1.1, we know that the ergodic shadowing implies mixing in cascades. However, we find out that the ergodic shadowing may not imply weakly mixing for iterated function systems (see Example 5.12). In [11], another notion of the "ergodic shadowing" was introduced to iterated function systems and we call it as "concordant ergodic shadowing" in this paper. If IFS($\mathscr{F}$) has the concordant ergodic shadowing property, and one of $\mathscr{F}$ is surjective, then IFS($\mathscr{F}$) is weakly mixing (see Proposition 7.2).

The present work is inspired by the notions and results from the papers mentioned above and is organized as follows. In Sect. 2, we review some notions to be used in this paper. In Sect. 3, we prove that if an IFS($\mathscr{F}$) has the concordant shadowing property, then so does IFS($\mathscr{F}^k$) for any $k \in \mathbb{N}$, where IFS($\mathscr{F}^k$) is k-fold composition of IFS($\mathscr{F}$); if IFS($\mathscr{F}^k$) for $k \in \mathbb{N}$ has the concordant shadowing property, then so does IFS($\mathscr{F}$) (see Proposition 3.3). In Sect. 4, we point out that the two notions of mixing and I-mixing are different (see Example 4.2). In Sect. 5, Theorem 1.2 is proved. In Sect. 6, Theorem 1.3 is shown. Finally, we study some properties of the concordant ergodic shadowing.

2 Preliminaries

Firstly, we introduce some basic notations. For any $A \subset \mathbb{Z}_+$, the cardinal number of A is denoted $|A|$. The upper density of A is defined by $\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, \ldots, n-1\}|$; The lower density of A is defined by $\underline{d}(A) = \liminf_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, \ldots, n-1\}|$; If $\overline{d}(A) = \underline{d}(A) = a$, then the density of A is defined by $d(A) = a$.

Let (X, ρ) be a compact metric space, and let $x \in X$. For $\varepsilon > 0$, let $B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$.

Let

$$\Sigma_m = \{\omega = \omega_0\omega_1 \cdots \omega_i \cdots : \omega_i \in \{0, 1, \ldots, m-1\}\}.$$

For $n \in \mathbb{Z}_+$, let

$$[i_0 i_1 \cdots i_n] = \{x = x_0 x_1 \cdots \cdot x_n \cdots \in \Sigma_m : x_0 = i_0, x_1 = i_1, \ldots, x_n = i_n\}.$$

Let IFS($\mathscr{F}$) be an iterated function system, and let $\omega = \omega_0\omega_1 \cdots \in \Sigma_m$. Put $f_\omega^0 = id_X$. For $n \in \mathbb{N}$, let

$$f_\omega^n = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}.$$

For $k \in \mathbb{N}$, let

$$\mathscr{F}^k = \{f_{w_{k-1}} \circ f_{w_{k-2}} \circ \cdots \circ f_{w_0} : w_0, \ldots, w_{k-1} \in \{0, 1, \ldots, m-1\}\}.$$

Then, IFS($\mathscr{F}^k$) is also an iterated function system. Let

$$\Sigma_{m^k} = \{t_0 t_1 \cdots t_i \cdots : t_i = w_{i_1} w_{i_2} \cdots w_{i_k}, w_{i_j} \in \{0, 1, \ldots, m-1\}, j = 1, 2, \ldots, k\}.$$

Then, $\Sigma_{m^k} = \Sigma_m$.

1. *Step-skew product*
   For an iterated function system IFS($\mathscr{F}$) and the shift map $\sigma : \Sigma_m \mapsto \Sigma_m$, we consider the *step-skew product*

$$F : \Sigma_m \times X \to \Sigma_m \times X, (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)).$$

A metric ρ on $\Sigma_m \times X$ is defined as follows:

$$\rho((\omega, x), (\lambda, y)) = \max\{\rho_1(\omega, \lambda), \rho_2(x, y)\}$$

for $(\omega, x), (\lambda, y) \in \Sigma_m \times X$, where ρ_1 and ρ_2 are metrics on Σ_m and X, respectively. The metric ρ_1 on Σ_m is defined by $\rho_1(\omega, \lambda) = \frac{1}{2^k}$, where $k = \min\{i : \omega_i \neq \lambda_i\}$.
Let $k \in \mathbb{N}$. Then,

$$F^k : \Sigma_m \times X \to \Sigma_m \times X, (\omega, x) \mapsto (\sigma^k \omega, f_\omega^k(x)).$$

2. *Pseudo-orbit and shadowing*
A sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ in X is called an *orbit* of IFS($\mathscr{F}$) if there is $\omega \in \Sigma_m$ such that $f_{\omega_i}(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}_+$. Meanwhile, the sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ is also called the *ω-orbit of* x_0.
Let $\delta > 0$. A sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ in X is called a *δ-pseudo-orbit* of IFS($\mathscr{F}$) if there is $\omega \in \Sigma_m$ such that $\rho(f_{\omega_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}_+$. Meanwhile, the sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ is also called a *ω–δ-pseudo-orbit* of IFS($\mathscr{F}$).
Let $\lambda \in \Sigma_m$, and let $\varepsilon > 0$. A sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ of IFS($\mathscr{F}$) is *ε-shadowed* by a λ-orbit $\{z_i\}_{i \in \mathbb{Z}_+}$ of IFS($\mathscr{F}$) if $\rho(z_i, x_i) < \varepsilon$ for all $i \in \mathbb{Z}_+$. In this case, one says that $\{z_i\}_{i \in \mathbb{Z}_+}$ or z_0, *ε-shadows* the $\{x_i\}_{i \in \mathbb{Z}_+}$.
An IFS($\mathscr{F}$) has the *concordant shadowing property* (cf. [8]) if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-pseudo-orbit of IFS($\mathscr{F}$) can be ε-shadowed by some ω-orbit in X, where $\omega \in \Sigma_m$.
An IFS($\mathscr{F}$) has the *shadowing property* (cf. [8]) if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-pseudo-orbit of IFS($\mathscr{F}$) can be ε-shadowed by some λ-orbit in X, where $\omega, \lambda \in \Sigma_m$.
3. *Chain transitivity*
Assume $x_0 = x, x_1, \ldots, x_n = y \in X$ and $\delta > 0$. If for any $i \in \{0, 1, \ldots, n-1\}$ there is $\omega_i \in \{0, 1, \ldots, m-1\}$ such that $\rho(f_{\omega_i}(x_i), x_{i+1}) < \delta$, the sequence $x_0, x_1, \ldots, x_n$ is called a *δ-chain* of IFS($\mathscr{F}$) with length n from x to y. An IFS($\mathscr{F}$) is called *chain transitive* (cf. [2]) if for any two points $x, y \in X$ and any $\delta > 0$, there is a δ-chain from x to y. An IFS($\mathscr{F}$) is called *chain mixing* (cf. [2]) if for any two points $x, y \in X$ and any $\delta > 0$, there is a positive integer N such that for any $n \geq N$ there is a δ-chain with length n from x to y.

We need the following result (see Theorem 2.3 in [13]).

Proposition 2.1 *Let IFS($\mathscr{F}$) be an iterated function system. Then, IFS($\mathscr{F}$) is chain mixing if and only if IFS($\mathscr{F}^k$) is chain transitive for all $k \in \mathbb{N}$.*

3 Concordant Shadowing Property

Bahabadi showed the following result (see Theorem 1.3 in [2]):

Proposition 3.1 *Let IFS($\mathscr{F}$) be an iterated function system, and let F be the step-skew product map corresponding to the IFS($\mathscr{F}$). Then, F has the shadowing property if and only if IFS($\mathscr{F}$) has the concordant shadowing property.*

Proposition 3.2 *Let IFS($\mathscr{F}$) be an iterated function system, and let F be the step-skew product map corresponding to the IFS($\mathscr{F}$), and let $k \in \mathbb{N}$. Then, F^k has the shadowing property if and only if IFS($\mathscr{F}^k$) has the concordant shadowing property.*

Proof ($\Rightarrow$). Suppose that F^k has the shadowing property. We show that IFS($\mathscr{F}^k$) has the concordant shadowing property. Given $0 < \varepsilon < \frac{1}{2^k}$, let $0 < \delta < \varepsilon$ be an ε modulus shadowing for F^k. Assume that $\{x_i\}_{i\in\mathbb{Z}_+}$ is a η-δ-pseudo-orbit of IFS($\mathscr{F}^k$), where $\eta = \eta_0\eta_1\eta_2\cdots \in \Sigma_{m^k}$. Then, $\rho_2(g_{\eta_i}(x_i), x_{i+1})) < \delta$, where $g_{\eta_i} \in \mathscr{F}^k$.

Suppose

$$\eta_0 = \omega_0\omega_1\omega_2\cdots\omega_{k-1};$$
$$\eta_1 = \omega_k\omega_{k+1}\omega_{k+2}\cdots\omega_{2k-1};$$
$$\cdots$$

Then, $\eta = \omega_0\omega_1\omega_2\cdots\omega_{k-1}\omega_k\omega_{k+1}\omega_{k+2}\cdots\omega_{2k-1}\cdots \in \Sigma_m$. Put $\omega^0 = \eta$, $\omega^1 = \sigma^k(\omega^0)$, $\omega^2 = \sigma^{2k}(\omega^0)$, $\cdots$. Therefore, $\rho_1(\sigma^k(\omega^i), \omega^{i+1}) = 0$ and

$$\rho_2(f^k_{\omega^i}(x_i), x_{i+1}) = \rho_2(g_{\eta_i}(x_i), x_{i+1}) < \delta$$

for all $i \in \mathbb{Z}_+$. So $\{(\omega^i, x_i)\}_{i\in\mathbb{Z}_+}$ is a δ-pseudo-orbit of F^k. Since F^k has the shadowing property, there exists $z \in X$ such that $\rho(F^{ki}(\eta, z), (\omega^i, x_i)) < \varepsilon$ for all $i \in \mathbb{Z}_+$. This implies that $\rho_2(f^{ik}_\eta(z), x_i) = \rho_2(g^i_\eta(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}_+$, so IFS($\mathscr{F}^k$) has the concordant shadowing property.

($\Leftarrow$). Suppose that IFS($\mathscr{F}^k$) has the concordant shadowing property. We show that F^k has the shadowing property. Given $0 < \varepsilon < \frac{1}{2^k}$, let $0 < \delta < \varepsilon$ be an ε modulus shadowing for IFS($\mathscr{F}^k$). Let $\{(\omega^i, x_i)\}_{i\in\mathbb{Z}_+}$ be a δ-pseudo-orbit of F^k, where $\omega^i \in \Sigma_m$ and $x_i \in X$ for all $i \in \mathbb{Z}_+$. Then, $\rho(F^k(\omega^i, x_i), (\omega^{i+1}, x_{i+1})) < \delta$ for all $i \in \mathbb{Z}_+$. This implies that $\rho_1(\sigma^k(\omega^i), \omega^{i+1}) < \delta$ and $\rho_2(f^k_{\omega^i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}_+$. Let

$$\lambda_0 = \omega^0_0\omega^0_1\cdots\omega^0_{k-1}$$
$$\lambda_1 = \omega^1_0\omega^1_1\cdots\omega^1_{k-1}$$
$$\cdots$$
$$\lambda_n = \omega^n_0\omega^n_1\cdots\omega^n_{k-1}$$
$$\cdots$$

Considering $\lambda = \lambda_0\lambda_1\cdots\lambda_n\ldots$, then $\lambda \in \Sigma_{m^k} \subset \Sigma_m$.

Thus, $\{x_i\}_{i\in\mathbb{Z}_+}$ is a λ-δ-pseudo orbit of IFS($\mathscr{F}^k$). Since IFS($\mathscr{F}^k$) has the concordant shadowing property, there is a λ-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}^k$) such that $\rho_2(z_i, x_i) < \varepsilon$ for all $i \in \mathbb{Z}_+$. In addition, for this λ, we can get $\rho_1(\sigma^{ik}(\lambda), \omega^i) < \varepsilon$ for all $i \in \mathbb{Z}_+$. Therefore, we have $\rho(F^{ki}(\lambda, z_0), (\omega^i, x_i)) < \varepsilon$ for all $i \in \mathbb{Z}_+$. This implies F^k has the shadowing property. □

By Theorem 4.3 and Theorem 4.5 in [1], we know that F has the shadowing property, then so does for F^k for any $k \in \mathbb{N}$; if F^k for $k \in \mathbb{N}$ has the shadowing property, then so does for F. So by Proposition 3.1 and Proposition 3.2, the following result holds.

Proposition 3.3 *Let IFS($\mathscr{F}$) be an iterated function system.*

1. *If IFS($\mathscr{F}$) has the concordant shadowing property, then so does IFS($\mathscr{F}^k$) for any $k \in \mathbb{N}$;*
2. *If IFS($\mathscr{F}^k$) for $k \in \mathbb{N}$ has the concordant shadowing property, then so does IFS($\mathscr{F}$).*

4 Mixing

Let (S, X) be a dynamical system, and let $U, V \subset X$. We denote $N(U, V) = \{s \in S : sU \cap V \neq \emptyset\}$. A dynamical system (S, X) is *(topologically) transitive* if for every pair of nonempty open subsets U, V in X, we have $N(U, V) \neq \emptyset$. A dynamical system (S, X) is *(topologically) weakly mixing* if $(S, X \times X)$ is transitive. A dynamical system (S, X) is *(topologically) mixing*, if $N(U, V)$ is confinite (that is, $S \setminus N(U, V)$ is finite) for every pair of nonempty open subsets U, V in X.

Let IFS($\mathscr{F}$) be an iterated function system, let $\omega \in \Sigma_m$ and let $U, V \subset X$. Put $M_\omega(U, V) = \{n \in \mathbb{Z}_+ : f_\omega^n(U) \cap V \neq \emptyset\}$, and $M(U, V) = \bigcup_{\omega \in \Sigma_m} M_\omega(U, V)$. An IFS($\mathscr{F}$) is *(topologically) transitive*, if there is $\omega \in \Sigma_m$ such that $M_\omega(U, V)$ is nonempty. An IFS($\mathscr{F}$) is *(topologically) I-weakly mixing* if $M(U_1, V_1) \cap M(U_2, V_2) \neq \emptyset$ for any nonempty open subsets U_1, U_2, V_1, V_2 in X. An IFS($\mathscr{F}$) is *(topologically) I-mixing* if $M(U, V)$ is confinite for every pair of nonempty open subsets U, V in X.

Remark 4.1 In [2,9,11], the notion of I-mixing is called directly "mixing", the I-weakly mixing is called "weakly mixing", but the following example shows that I-mixing and mixing are different, and I-weakly mixing and weakly mixing are different.

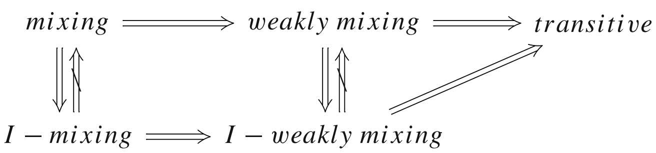
Example 4.2 There is an IFS($\mathscr{F}$) which is I-mixing but not weakly mixing.

Proof We define two continuous maps f_0, f_1 on Σ_2 as follows: for all $x = x_0 x_1 \cdots \in \Sigma_2$,

$$f_0(x) = 0x_0x_1\cdots; \quad f_1(x) = 1x_0x_1\cdots.$$

1. IFS($\mathscr{F}$) is not weakly mixing.
   Assume that IFS($\mathscr{F}$) is weakly mixing. Take open subsets $U_1 = [1]$, $U_2 = [0]$, $U_3 = [0]$, $U_4 = [1]$ of Σ_2. Then, $N(U_1, U_2) \cap N(U_3, U_4) \neq \emptyset$, so there are $\omega = \omega_0\omega_1 \cdots\cdots \in \Sigma_2$ and $n \in \mathbb{N}$ such that
   $$f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}(U_1) \cap U_2 \neq \emptyset, \ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}(U_3) \cap U_4 \neq \emptyset.$$
   This implies that $\omega_{n-1} = 0$ and $\omega_{n-1} = 1$, a contradiction.
2. IFS($\mathscr{F}$) is I-mixing.
   Take open subsets $U = [x_0x_1 \cdots x_n]$ and $V = [y_0y_1 \cdots y_n]$ of Σ_2. It is clear that $M(U, V)$ is confinite subset of $\mathbb{Z}_+$. So IFS($\mathscr{F}$) is I-mixing. □

The following implications are true.



It is easy to see that the following two propositions hold.

Proposition 4.3 *Let IFS($\mathscr{F}$) be an iterated function system.*

1. *If IFS($\mathscr{F}$) is transitive, then IFS($\mathscr{F}$) is chain transitive.*
2. *If IFS($\mathscr{F}$) is I-mixing, then IFS($\mathscr{F}$) is chain mixing.*

Proposition 4.4 *Let IFS($\mathscr{F}$) be an iterated function system. Suppose that IFS($\mathscr{F}$) has the shadowing property (it is not necessary to have the concordant shadowing property).*

1. *If IFS($\mathscr{F}$) is chain transitive, then IFS($\mathscr{F}$) is transitive.*
2. *If IFS($\mathscr{F}$) is chain mixing, then IFS($\mathscr{F}$) is I-mixing.*

A map $f : X \to X$ is *semi-open* if for any nonempty open subset U of X, $f(U)$ has nonempty interior. By Theorem 2.1 and Theorem 2.4 in [13], Lemma 3.3 and Lemma 3.4 in [9], the following proposition holds.

Proposition 4.5 *Let IFS($\mathscr{F}$) be an iterated function system, and let F be the step-skew product map corresponding to the IFS($\mathscr{F}$). Then, the following results hold:*

1. *If F is transitive (resp. weakly mixing, mixing), then IFS($\mathscr{F}$) is transitive (resp. I-weakly mixing, I-mixing).*
2. *If every $f \in \mathscr{F}$ is semi-open and IFS($\mathscr{F}$) is transitive (resp. I-weakly mixing, I-mixing), then F is transitive (resp. weakly mixing, mixing).*

The following example shows that I-mixing may not imply shadowing.

Example 4.6 There is an IFS($\mathscr{F}$) which is I-mixing but without shadowing.

Proof Consider the map $f_0(x) \equiv 0$, and the tent map $f_1(x)$ on $X = [0, 1]$ (that is, $f_1(x) = 2x, x \in [0, \frac{1}{2}]$; $f_1(x) = 2 - 2x, x \in [\frac{1}{2}, 1]$). Since the cascade (X, f_1) is mixing, then the IFS(f_0, f_1) is I-mixing. Example 1.5 in [2] shows that the IFS(f_0, f_1) does not have the concordant shadowing property. In fact, the IFS(f_0, f_1) does not have the shadowing property too. □

5 Ergodic Shadowing Property and Sub-shadowing Property

Let IFS($\mathscr{F}$) be an iterated function system. Let $\omega = \omega_0\omega_1 \cdots \in \Sigma_m$ and $\delta > 0$, an infinite sequence $\xi = \{x_i\}_{i \in \mathbb{Z}_+}$ in X is called a *ω–δ -ergodic pseudo-orbit* of IFS($\mathscr{F}$) if

$$d(\{i \in \mathbb{Z}_+ : \rho(f_{\omega_i}(x_i), x_{i+1}) < \delta\}) = 1.$$

An IFS($\mathscr{F}$) has the *ergodic shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$) can be *ε-ergodic-shadowed* by some η-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ where $\omega, \eta \in \Sigma_m$, that is, $d(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) = 1$. In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, *ε-ergodic-shadows* the $\{x_i\}_{i\in\mathbb{Z}_+}$.

Let $q \in [0, 1)$. An IFS($\mathscr{F}$) has the *$\underline{q}$-ergodic shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$) can be *ε-$\underline{q}$-shadowed* by some η-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ where $\omega, \eta \in \Sigma_m$, that is, $\underline{d}(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) > q$. In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, ε -$\underline{\text{q}}$-*shadows* the $\{x_i\}_{i\in\mathbb{Z}_+}$.

Let $p \in [0, 1)$. An IFS($\mathscr{F}$) has the *$\overline{p}$-ergodic shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$) can be *ε-$\overline{p}$-shadowed* by some η-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ where $\omega, \eta \in \Sigma_m$, that is, $\overline{d}(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) > p$. In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, *ε-$\overline{p}$-shadows* the $\{x_i\}_{i\in\mathbb{Z}_+}$.

We need the following two lemmas (see Lemma 4.1 and Lemma 4.2 in [12]).

Lemma 5.1 *Let $A, B \subset \mathbb{Z}_+$. If $\overline{d}(A) + \underline{d}(B) > 1$, then $\overline{d}(A \cap B) > 0$.*

Lemma 5.2 *Let $A, B \subset \mathbb{Z}_+$. Then, $\underline{d}(A \cap B) \geq \underline{d}(A) + \underline{d}(B) - \overline{d}(A \cup B)$.*

Proposition 5.3 *Let IFS($\mathscr{F}$) be an iterated function system, where one of $\mathscr{F}$ is surjective. Then, the following results hold:*

1. *If IFS($\mathscr{F}$) has the $\underline{0}$-ergodic shadowing property, then IFS($\mathscr{F}$) is chain transitive.*
2. *If IFS($\mathscr{F}$) has the $\overline{\frac{1}{2}}$-ergodic shadowing property, then IFS($\mathscr{F}$) is chain transitive.*

Proof We only prove the (1) holds, since the proof of (2) is similar to (1). Fix any $\varepsilon > 0$ and any $x, y \in X$. Next, we will show there exists an ε-chain from x to y. Let $n_0 = 2$. For any $i \in \mathbb{N}$, let $n_i = i! \cdot n_0$. For convenience, we will also use $f_0^{-1}(y)$ to represent an element of $f_0^{-1}(y)$.

Let

$$\begin{aligned}
\xi_1 &= x, f_0(x), \ldots, f_0^{n_1}(x) \\
\xi_2 &= f_0^{-(n_2-n_1)+1}(y), \ldots, f_0^{-1}(y), y \\
\xi_3 &= x, f_0(x), \ldots, f_0^{n_3-n_2-1}(x) \\
\xi_4 &= f_0^{-(n_4-n_3)+1}(y), \ldots, f_0^{-1}(y), y \\
&\cdots
\end{aligned}$$

Put $\eta = \{x_i\}_{i\in\mathbb{Z}_+} = \xi_1\xi_2\xi_3\cdots$. For any $\delta > 0$, η is a δ-ergodic pseudo-orbit of f_0. For $\varepsilon > 0$, we find a ω-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ for some $\omega \in \Sigma_m$, which ε-$\underline{0}$-shadows η. Let $E = \{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}$. Then, $\underline{d}(E) > 0$. Let $M_1 = \{i \in \mathbb{Z}_+ : x_i \in orb(x, f_0) = \{f_0^n(x) : n \in \mathbb{Z}_+\}\}$, and let $M_2 = \mathbb{Z}_+ \setminus M_1$. Clearly, $\overline{d}(M_1) = \overline{d}(M_2) = 1$. By Lemma 5.1, we have $\overline{d}(E \cap M_1) > 0$ and $\overline{d}(E \cap M_2) > 0$.

We can choose nonnegative integers s, r, i_r, i_s such that $r < s - 1$, $d(f_0^{i_r}(x), z_r) < \varepsilon$ and $d(f_0^{-i_s}(y), z_s) < \varepsilon$.

Then, the sequence

$$x, f_0(x), \ldots, f_0^{i_r-1}(x), z_r, z_{r+1}, \ldots, z_{s-1}, f_0^{-i_s}(y), f_0^{-i_s+1}(y), \ldots,$$

is an ε-chain from x to y. Thus, IFS($\mathscr{F}$) is chain transitive. □

Remark 5.4 In Proposition 5.3, we assume that one of $\mathscr{F}$ is surjective. We can replace this assumption by the following:

$$\bigcup_{i=0}^{m-1} f_i(X) = X.$$

So the condition of "one of $\mathscr{F}$ is surjective" in Theorem 1.2 can be replaced by it.

Proposition 5.5 *Let IFS($\mathscr{F}$) be an iterated function system, where one of $\mathscr{F}$ is surjective. If IFS($\mathscr{F}$) has the ergodic shadowing property, then IFS($\mathscr{F}$) has the shadowing property.*

Proof Given $\varepsilon > 0$, let $\delta > 0$ be ε modulus of ergodic shadowing property. Let $\omega \in \Sigma_m$, and let $\{x_i\}_{i\in\mathbb{Z}_+}$ be a ω–δ-pseudo-orbit. Suppose $\xi_n = x_0, x_1, \ldots, x_n$. Then, ξ_n is a δ-chain from x_0 to x_n. By Proposition 5.3, we choose a δ-chain $\gamma = x_n, y_1, \ldots, y_k, x_0$ from x_n to x_0. Then, $\eta = \xi_n y_1 \cdots y_k \xi_n y_1 \cdots y_k \xi_n \cdots$ is a δ-pseudo-orbit. So it can be ε-ergodic shadowed by a λ-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$. Hence, at least one ξ_n is entirely ε-shadowed by a piece of the $\{z_i\}_{i\in\mathbb{Z}_+}$.

Next, we prove that $\{x_i\}_{i\in\mathbb{Z}_+}$ is ε-shadowed by an orbit. As every ξ_n is ε-shadowed by a piece $\{z^n_{n_0}, z^n_{n_1}, \cdots, z^n_{n_n}\}$ of an orbit. Since X is compact, we may assume $s_0 = \lim_{n\to\infty} z^n_{n_0}, s_1 = \lim_{n\to\infty} z^n_{n_1}, \cdots, s_n = \lim_{n\to\infty} z^n_{n_n}$. Then, there is a θ-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ which ε-shadows $\{x_i\}_{i\in\mathbb{Z}_+}$, where $\theta \in \Sigma_m$. □

Proposition 5.6 *Let IFS($\mathscr{F}$) be an iterated function system. If IFS($\mathscr{F}$) has the ergodic shadowing property, then IFS($\mathscr{F}^k$) has the $\underline{0}$-ergodic shadowing property for any positive integer k.*

Proof Give a positive integer k. Suppose that IFS($\mathscr{F}$) has the ergodic shadowing property. Given $\varepsilon > 0$, let $\delta > 0$ be ε modulus of ergodic shadowing property. Let $\eta = \{u_i\}_{i\in\mathbb{Z}_+}$ be a ω–δ-ergodic pseudo-orbit of IFS($\mathscr{F}^k$), where $\omega \in \Sigma_{m^k}$. Then, for all $i \in \mathbb{Z}_+$, we have

$$d(\{i \in \mathbb{Z}_+ : \rho(g_{\omega_i}(u_i), u_{i+1}) < \delta\}) = 1.$$

where $g_{\omega_i} = f_{t^i_{k-1}} \circ \cdots \circ f_{t^i_1} \circ f_{t^i_0}$, and $t^i_0, t^i_1, \ldots, t^i_{k-1} \in \{0, 1, \ldots, m-1\}$.

Let $\xi = \{x_i\}_{i\in\mathbb{Z}_+}$ be

$$\begin{aligned}
&u_0, f_{t^0_0}(u_0), f_{t^0_1} \circ f_{t^0_0}(u_0), \ldots, f_{t^0_{k-2}} \circ \cdots \circ f_{t^0_1} \circ f_{t^0_0}(u_0);\\
&u_1, f_{t^1_0}(u_1), f_{t^1_1} \circ f_{t^1_0}(u_1), \ldots, f_{t^1_{k-2}} \circ \cdots \circ f_{t^1_1} \circ f_{t^1_0}(u_1);\\
&\quad\cdots
\end{aligned}$$

For every $l \in \mathbb{Z}_+$ and every $j \in \{1, \ldots, k-1\}$, we have $x_{lk+j} = f_{t^l_{j-1}} \circ \cdots \circ f_{t^l_0}(u_l)$. In particular, for any $l \in \mathbb{Z}_+$, we have $x_{lk} = u_l$. Then, ξ is a δ-ergodic pseudo-orbit of IFS($\mathscr{F}$), so there is an orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$) such that

$$\underline{d}(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) = 1.$$

As $\underline{d}(\{i \in \mathbb{Z}_+ : x_i \in \{u_i\}_{i\in\mathbb{Z}_+}\}) = \frac{1}{k}$, by Lemma 5.2 we have

$$\underline{d}[(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) \cap (\{i \in \mathbb{Z}_+ : x_i \in \{u_i\}_{i\in\mathbb{Z}_+}\})] \geq 1 + \frac{1}{k} - 1 > 0.$$

That is, $\underline{d}(\{i \in \mathbb{Z}_+ : \rho(z_{ki}, u_i) < \varepsilon\}) > 0$. Note that $\{z_{ki}\}_{i\in\mathbb{Z}_+}$ is an orbit of IFS($\mathscr{F}^k$). So IFS($\mathscr{F}^k$) has $\underline{0}$-ergodic shadowing property. □

Corollary 5.7 *Let IFS($\mathscr{F}$) be an iterated function system, where one of $\mathscr{F}$ is surjective. If IFS($\mathscr{F}$) has the ergodic shadowing property, then IFS($\mathscr{F}$) is chain mixing*

Proof By Proposition 5.6, for all positive integer k, IFS($\mathscr{F}^k$) has the $\underline{0}$-ergodic shadowing property. By Proposition 5.3, IFS($\mathscr{F}^k$) is chain transitive for all positive integer k. By Proposition 2.1, IFS($\mathscr{F}$) is chain mixing. □

The following definition is different from Definition 2.3 in [11].

Definition 5.8 An IFS($\mathscr{F}$) has the *pseudo-orbital specification property*, if for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $K = K(\varepsilon) > 0$ such that for any $\omega \in \Sigma_m$ and given nonnegative integers $a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_n \leq b_n$ with $a_{i+1} - b_i \geq K$ for all $i = \{0, 1, \ldots, n-1\}$, for any δ-chains $\xi_0, \xi_1, \ldots, \xi_n$ with $\xi_i = \{x^i_j\}$ for all $j \in [a_i, b_i] \subset \mathbb{Z}_+$ and $0 \leq i \leq n$ (that is, $\rho(f_{\omega_j}(x^i_j), x^i_{j+1}) < \delta$ for all $j \in [a_i, b_i]$ and $0 \leq i \leq n$), there exists an orbit $\{z_j\}_{j\in\mathbb{Z}_+}$ such that $\rho(z_j, x^i_j) < \delta$ for all $j \in [a_i, b_i]$ and $0 \leq i \leq n$.

Remark 5.9 If IFS($\mathscr{F}$) is I-mixing, then for any $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that for any two points x and y in X,

$$M(B(x, \varepsilon), B(y, \varepsilon)) \supseteq \{N(\varepsilon), N(\varepsilon) + 1, \cdots\}.$$

Indeed, given $\varepsilon > 0$, assume $X = \bigcup_{i=0}^{l} B(x_i, \frac{\varepsilon}{2})$ and put

$$N(\varepsilon) = \max_{1\leq i,j\leq l} \min\{n : \text{for all } k \in M\left(B\left(x_i, \frac{\varepsilon}{2}\right), B\left(x_j, \frac{\varepsilon}{2}\right)\right) \text{ we have } k \geq n\}.$$

Lemma 5.10 *Let IFS($\mathscr{F}$) be an iterated function system. If IFS($\mathscr{F}$) has the shadowing property and is I-mixing, then IFS($\mathscr{F}$) has the pseudo-orbital specification property.*

Proof Let $\varepsilon > 0$ be given and δ be an ε modulus of shadowing for IFS($\mathscr{F}$). Since X is compact, then there exists $0 < \eta < \delta$ such that for any $a, b \in X$ and $\rho(a, b) < \eta$, we have $\rho(f_i(a), f_i(b)) < \delta$ for all $f_i \in \mathscr{F}$. As IFS($\mathscr{F}$) is I-mixing, for $\eta > 0$ we choose $N(\eta)$ by Remark 5.9. Put $K(\varepsilon) = N(\eta)$ and let nonnegative integer intervals $[a_0, b_0], [a_1, b_1], \ldots, [a_n, b_n]$ with $a_{i+1} - b_i \geq K(\varepsilon)$ for all $i \in \{0, 1, \ldots, n-1\}$.

Suppose that $\xi_0, \xi_1, \ldots, \xi_n$ are δ-chains such that $\xi_i = \{x^i_j\}$ and $\rho(f_{\omega_j}(x^i_j), x^i_{j+1}) < \delta$ for all $j \in [a_i, b_i]$ and $0 \leq i \leq n$, where $\omega \in \Sigma_m$. By choice of $K(\varepsilon)$, there exists a true orbit θ_i with $a_i - b_{i-1} - 1$ elements which begin at θ^i_1 and end at $\theta^i_{a_i - b_{i-1} - 1}$ with $\rho(f_{\omega_{b_{i-1}}}(x^{i-1}_{b_{i-1}}), \theta^i_1) < \delta$ and $\rho(\theta^i_{a_i - b_{i-1} - 1}, f^{-1}_{\omega_{a_{i-1}}}(x^i_{a_i})) < \eta$ for any $i \in \{1, 2, \ldots, n\}$. So

$$\rho(f_{\omega_{a_{i-1}}}(\theta^i_{a_i - b_{i-1} - 1}), x^i_{a_i}) < \delta$$

for any $i \in \{1, 2, \ldots, n\}$. Thus, $\xi_0, \theta_1, \xi_1, \theta_2, \ldots, \xi_n$ is a piece of δ-pseudo-orbit which can be ε-shadowed. So IFS($\mathscr{F}$) has the pseudo-orbital specification property. □

Lemma 5.11 *IFS($\mathscr{F}$) has the pseudo-orbital specification property, then IFS($\mathscr{F}$) has the ergodic shadowing property.*

Proof Suppose that IFS($\mathscr{F}$) has the pseudo-orbital specification property. Given $\varepsilon > 0$, put $\delta > 0$ and K be two modulus according to the ε as in definition of the pseudo-orbital specification property. Let $\xi = \{x_i\}_{i \in \mathbb{Z}_+}$ be a ω–δ-ergodic pseudo orbit, where $\omega \in \Sigma_m$. Let $A = \{i \in \mathbb{Z}_+ : \rho(f_{\omega_i}(x_i), x_{i+1}) < \delta\}$ and choose a sequence

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

of nonnegative integers with the following properties:

1. for any n, $[a_n, b_n] \subseteq A$;
2. for any n, $a_{n+1} - b_n \geq K$;
3. $\lim_{n\to\infty} \sum_{k=1}^{n} (a_{k+1} - b_k)/b_n \to 0$.

For any n, there exists orbit $\{z^n_i\}_{i \in [0, b_n]}$ which ε-shadows n pieces of the ergodic pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}_+}$ corresponding to the n intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n].$$

Without loss of generality, suppose $\lim_{n\to\infty} z_n \to y_0$, then there exists an orbit $\{y_i\}_{i \in \mathbb{Z}_+}$ which ε-ergodic shadows ξ. So IFS($\mathscr{F}$) has the ergodic shadowing property. □

Proof of Theorem 1.2 It is sufficient to assemble the obtained results in Sect. 5. (1) ⇒ (2) is Proposition 5.5 and Corollary 5.7, (2) ⇒ (3) is Proposition 4.4, (3) ⇒ (4) is Lemma 5.10 and finally, (4) ⇒ (1) is Lemma 5.11. □

In cascades, ergodic shadowing implies mixing (see Theorem 1.1). However, for iterated function systems, this implication may not be true.

Example 5.12 There is an iterated function system IFS($\mathscr{F}$) which has the ergodic shadowing but not weakly mixing.

Proof Let IFS($\mathscr{F}$) be defined as in Example 4.2. Then, IFS($\mathscr{F}$) is I-mixing but not weakly mixing. By Example 1.2 in [2], IFS($\mathscr{F}$) has the concordant shadowing property. Since $f_0(X) \cup f_1(X) = X$, by Theorem 1.2 and Remark 5.4, IFS($\mathscr{F}$) has the ergodic shadowing property. □

6 Average Shadowing Property

Given $\delta > 0$, we say that an infinite sequence $\{x_i\}_{i\in\mathbb{Z}_+}$ in X is a *δ-average pseudo-orbit* of IFS($\mathscr{F}$) if there are $\omega \in \Sigma_m$ and positive integer $N = N(\delta) > 0$ such that for every integer $n \geq N(\delta)$ and every nonnegative integer k, the following condition is satisfied:

$$\frac{1}{n}\sum_{i=0}^{n-1} \rho(f_{\omega_{i+k}}(x_{i+k}), x_{i+k+1}) < \delta.$$

An IFS($\mathscr{F}$) has the *average shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that every δ-average pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ is *ε-shadowed in average* by some orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$), that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \rho(z_i, x_i) < \varepsilon.$$

In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, ε-shadows in average $\{x_i\}_{i\in\mathbb{Z}_+}$.

An IFS($\mathscr{F}$) has the has *$\underline{q}$-average shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that any *δ-average pseudo-orbit* of IFS($\mathscr{F}$) can be *ε-$\underline{q}$-shadowed* by some orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$), that is, $\underline{d}(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) > q$. In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, ε-$\underline{q}$-shadows $\{x_i\}_{i\in\mathbb{Z}_+}$.

Lemma 6.1 *Let IFS($\mathscr{F}$) be a chain mixing iterated function system. Then, for every $\varepsilon > 0$, there is $k(\varepsilon) \in \mathbb{N}$, such that for any $x, y \in X$ and any $n \geq k(\varepsilon)$, there are points $z_1, z_2, \cdots, z_n \in X$ such that the sequence*

$$x, z_1, z_2, \cdots, z_n, y$$

is an ε-chain.

Proof As IFS($\mathscr{F}$) is chain mixing, then for any $\varepsilon > 0$ and any $x, y \in X$ there exists $k_{xy}(\varepsilon) \in \mathbb{N}$, such that for any $n \geq k_{xy}(\varepsilon)$ there exist some points $z_1, z_2, \ldots, z_n \in X$ such that $x, z_1, z_2, \ldots, z_n, y$ is an $\frac{\varepsilon}{2}$-chain. Since X is compact, there exists $0 < \delta < \frac{\varepsilon}{2}$ such that for any two points $u, v \in X$ with $\rho(u, v) < \delta$, we have $\rho(f(u), f(v)) < \frac{\varepsilon}{2}$ for all $f \in \mathscr{F}$.

Claim For any two points $u \in B(x, \delta)$, $v \in B(y, \delta)$, the sequence $u, z_1, z_2, \ldots, z_n, v$ is an ε-chain.

Assume $\rho(f_i(x), z_1) < \frac{\varepsilon}{2}$ and $\rho(f_j(z_n), y) < \frac{\varepsilon}{2}$ for $f_i, f_j \in \mathscr{F}$. Then,

$$\rho(f_i(u), z_1) \leq \rho(f_i(u), f_i(x)) + \rho(f_i(x), z_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$
$$\rho(f_j(z_n), v) \leq \rho(f_j(z_n), y) + \rho(y, v) < \frac{\varepsilon}{2} + \delta < \varepsilon.$$

Therefore, the claim holds.

Since $\{B(x,\delta)\times B(y,\delta) : (x,y)\in X\times X\}$ is an open cover of $X\times X$, then there exists a finite subcover $\{(B(x_1,\delta)\times B(y_1,\delta)), (B(x_2,\delta)\times B(y_2,\delta))\cdots, (B(x_m,\delta)\times B(y_m,\delta))\}$. Let $k(\varepsilon)=\max\{k_{x_i y_i}(\varepsilon), i=1,2,\ldots,m\}$. Then, for any $x, y\in X$ and any $n\geq k(\varepsilon)$, there exists an ε-chain with length $n+1$ from x to y. □

Theorem 6.2 *Let IFS($\mathscr{F}$) be an iterated function system. Then, the following conditions are equivalent:*

1. *IFS($\mathscr{F}$) has the average shadowing property;*
2. *IFS($\mathscr{F}$) has the $\underline{q}$-average shadowing property for all $q\in[0,1)$.*

Proof (1) $\Rightarrow$ (2). Let $q\in[0,1)$. Fix any $\varepsilon>0$, let $\gamma=(1-q)\cdot\varepsilon$. Taking any δ-average pseudo-orbit $\xi=\{x_i\}_{i\in\mathbb{Z}_+}$, then there exists an orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ which γ-shadows ξ in average. Denote $A=\{j\in\mathbb{Z}_+ : \rho(z_j,x_j)<\varepsilon\}$ and observe

$$\begin{aligned}\gamma &> \limsup_{n\to\infty}\frac{\sum_{i=0}^{n-1}\rho(z_i,x_i)}{n}\\ &\geq \limsup_{n\to\infty}\varepsilon\cdot\frac{n-|A\cap\{0,1,\ldots,n-1\}|}{n}\\ &\geq \varepsilon-\varepsilon\cdot\underline{d}(A).\end{aligned}$$

But $\gamma=(1-q)\cdot\varepsilon$, thus $\underline{d}(A)>q$ and so IFS($\mathscr{F}$) has the $\underline{q}$-average shadowing property.

(2) $\Rightarrow$ (1). Without loss of generality, we assume $diam(X)=1$. Give $\varepsilon\in(0,1)$. Let $q>1-\frac{\varepsilon}{3}$. As IFS($\mathscr{F}$) has the $\overline{q}$-average shadowing property, there is $\delta>0$ such that any δ-average pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ can be $\frac{\varepsilon}{3}$-$\underline{q}$-shadowed by some orbit $\{z_i\}_{i\in\mathbb{Z}_+}$. Let $E=\{i\in\mathbb{Z}_+ : \rho(z_i,x_i)\geq\frac{\varepsilon}{3}\}$. Then, $\overline{d}(E)\leq 1-q\leq\frac{\varepsilon}{3}$. Let $E_n=E\cap\{0,1,\ldots,n-1\}$. Then,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho(z_i,x_i) &= \limsup_{n\to\infty}\frac{1}{n}\left(\sum_{i\in E_n}\rho(z_i,x_i)+\sum_{i\in E_n^c}\rho(z_i,x_i)\right)\\ &\leq \overline{d}(E)+\frac{\varepsilon}{3}<\varepsilon.\end{aligned}$$

So, IFS($\mathscr{F}$) has the average shadowing property. □

Proposition 6.3 *Let IFS($\mathscr{F}$) be an iterated function system. If IFS($\mathscr{F}$) has the shadowing property and is chain mixing, then IFS($\mathscr{F}$) has the $\underline{q}$-average shadowing property for all $q\in[0,1)$.*

Proof Give $\varepsilon>0$ ($\varepsilon<3(1-q)$). Let γ be an ε modulus of shadowing. By Lemma 6.1, there is a positive integer M such that for any $x,y\in X$, there is a γ-chain of length M from x to y.

Without loss of generality, we assume $diam(X)=1$. Let $\delta=\frac{\varepsilon\cdot\gamma}{3M}$. Let $\xi=\{x_i\}_{i\in\mathbb{Z}_+}$ be a ω–δ-average pseudo-orbit where $\omega\in\Sigma_m$. Then, there is N such that for any $n\geq$

N and $k \geq 0$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\rho(f_{\omega_{i+k}}(x_{i+k}), x_{i+k+1}) < \delta.$$

We may assume $N > M$.

For any $j \in \mathbb{Z}_+$, let

$$y_{jN}, y_{jN+1}, \dots, y_{jN+N} = x_{jN}, x_{jN+1}, \dots, x_{jN+N}.$$

Next, if there is $jN \leq i < jN + N$ such that $\rho(f_{\omega_i}(y_i), y_{i+1}) \geq \gamma$, we choose k so that $jN \leq k < i < k + M \leq jN + N$ and replace $y_k, y_{k+1}, \dots, y_{k+M}$ with any γ-chain of length M from y_k to y_{k+M}. We repeat these replacements until $y_{jN}, y_{jN+1}, \dots, y_{jN+N}$ becomes a γ-chain. Therefore $\{y_i\}_{i\in\mathbb{Z}_+}$ is a γ-pseudo-orbit. Then, there is an orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ which ε-shadows $\{y_i\}_{i\in\mathbb{Z}_+}$.

Observe that for any $j \in \mathbb{Z}_+$, we have

$$M \cdot |\{i \in [jN, jN + N) : \rho(f_{\omega_i}(x_i), x_{i+1}) \geq \gamma\}| \geq |\{i \in [jN, jN + N) : x_i \neq y_i\}|,$$

and

$$\begin{aligned}\delta \cdot M &> \frac{M}{N}\sum_{i=0}^{N-1}\rho(f_{\omega_{jN+i}}(x_{jN+i}), x_{jN+i+1})\\ &\geq \frac{M}{N}\cdot\gamma|\{i \in [jN, jN + N) : \rho(f_{\omega_i}(x_i), x_{i+1}) \geq \gamma\}|.\end{aligned}$$

So

$$\begin{aligned}&\frac{\gamma}{N}|\{i \in [jN, jN + N) : x_i \neq y_i\}|\\ &\quad\leq \frac{\gamma}{N}\cdot M|\{i \in [jN, jN + N) : \rho(f_{\omega_i}(x_i), x_{i+1}) \geq \gamma\}\\ &\quad< \delta \cdot M.\end{aligned}$$

It follows

$$\frac{|\{i \in [jN, jN + N) : x_i \neq y_i\}|}{N} < \frac{\delta \cdot M}{\gamma} = \frac{\varepsilon}{3} < 1 - q.$$

Let $E = \{i \in \mathbb{Z}_+ : x_i = y_i\}$. For any $n \in \mathbb{N}$, let $n = kN + l$, $0 \le l \le N - 1$. Then,

$$\begin{aligned}\limsup_{n\to\infty} \frac{|E^c \cap \{0, 1, \dots, n-1\}|}{n} &\le \limsup_{k\to\infty} \frac{|E^c \cap \{0, 1, \dots, N(k+1)-1\}|}{kN} \\ &\le \limsup_{k\to\infty} \frac{N(k+1)}{kN} \cdot \frac{\varepsilon}{3} \\ &= \frac{\varepsilon}{3} < 1 - q.\end{aligned}$$

So $\underline{d}(E) > q$. This implies $\underline{d}(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) > q$. □

Proof of Theorem 1.3 By Theorem 1.2, Propositions 6.2 and 6.3, the theorem holds. □

Next, we give two iterated function systems which have average shadowing property.

Example 6.4 There is an IFS($\mathscr{F}$) with average shadowing but not chain transitive.

Proof Let X be a compact metric space with more than two elements, and let a, b be two different points of X. Consider the constant maps $f_0(x) = a$, $f_1(x) = b$. The IFS(f_0, f_1) has the average shadowing property but it is not chain transitive (see Example 2.2 in [2]). □

Example 6.5 There is an IFS($\mathscr{F}$) with average shadowing and concordant shadowing but not transitive.

Proof Consider the map $f_0(x) = \frac{1}{3}x$, and the map $f_1(x) = \frac{1}{4}x$ on $[0, 1]$. Then, the IFS(f_0, f_1) has the average shadowing property and the concordant shadowing property (see Theorem 3.2 in [10], and the proof of Theorem 2.1 in [8]), but IFS(f_0, f_1) is not transitive. □

7 Concordant Ergodic Shadowing Property

An iterated function system IFS($\mathscr{F}$) has the *concordant ergodic shadowing property* [11], if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ω–δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ of IFS($\mathscr{F}$) can be *ε-concordant ergodic-shadowed* by some η-orbit $\{z_i\}_{i\in\mathbb{Z}_+}$ where $\omega, \eta \in \Sigma_m$, that is, $d(\{i \in \mathbb{Z}_+ : \rho(z_i, x_i) < \varepsilon\}) = 1$ and $d(\{i \in \mathbb{Z}_+ : \eta_i = \omega_i\}) = 1$. In this case, one says that $\{z_i\}_{i\in\mathbb{Z}_+}$ or z_0, *ε-concordant ergodic-shadows* the $\{x_i\}_{i\in\mathbb{Z}_+}$.

Let IFS($\mathscr{F}$) and IFS($\mathscr{G}$) be two iterated function systems, where $\mathscr{F} = \{f_0, f_1, \dots, f_{m-1}\}$ is a family of continuous maps on X, and $\mathscr{G} = \{g_0, g_1, \dots, g_{m-1}\}$ is a family of continuous maps on Y. Let $\mathscr{F} \otimes \mathscr{G} = \{f_0 \times g_0, f_1 \times g_1, \dots, f_{m-1} \times g_{m-1}\}$. Then, IFS($\mathscr{F} \otimes \mathscr{G}$) is the action of the semigroup generated by $\mathscr{F} \otimes \mathscr{G}$ on $X \times Y$. Let $\mathscr{F} \times \mathscr{G} = \{f_i \times g_j : i, j = 0, 1, \dots, m-1\}$. Then, IFS($\mathscr{F} \times \mathscr{G}$) is the action of the semigroup generated by $\mathscr{F} \times \mathscr{G}$ on $X \times Y$.

Proposition 7.1 *Let IFS($\mathscr{F}$) and IFS($\mathscr{G}$) be two iterated function systems, where $\mathscr{F} = \{f_0, f_1, \dots, f_{m-1}\}$ and $\mathscr{G} = \{g_0, g_1, \dots, g_{m-1}\}$. If IFS($\mathscr{F}$) and IFS($\mathscr{G}$) have the concordant ergodic shadowing property, so do IFS($\mathscr{F} \otimes \mathscr{G}$) and IFS($\mathscr{F} \times \mathscr{G}$).*

Proof We only consider the case of IFS($\mathscr{F} \otimes \mathscr{G}$), since the case of IFS($\mathscr{F} \times \mathscr{G}$) has been proved (see Lemma 4.2 in [11]). Let $\{(x_i, y_i)\}$ be a ω–δ-ergodic pseudo-orbit of IFS($\mathscr{F} \otimes \mathscr{G}$) where $\omega \in \Sigma_m$. Then,

$$d(\{i \in \mathbb{Z}_+ : \rho(f_{\omega_i}(x_i), x_{i+1}) < \delta \text{ and } \rho(g_{\omega_i}(y_i), y_{i+1}) < \delta\}) = 1.$$

Thus, $\{x_i\}_{i\in\mathbb{Z}_+}$ and $\{y_i\}_{i\in\mathbb{Z}_+}$ are ergodic pseudo-orbits for IFS($\mathscr{F}$) and IFS($\mathscr{G}$), respectively. Since IFS($\mathscr{F}$) and IFS($\mathscr{G}$) have the concordant ergodic shadowing property, there are z_1 and z_2 of X, which ε-concordant ergodic-shadow δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ and $\{y_i\}_{i\in\mathbb{Z}_+}$, respectively. Obviously, $\{(z_1, z_2)\}$ ε-concordant ergodic-shadows $\{(x_i, y_i)\}$. □

Shabani showed that if IFS($\mathscr{F}$) has the concordant ergodic shadowing property, and $\cup_{f\in\mathscr{F}} f(X) = X$, then IFS($\mathscr{F}$) is I-weakly mixing (see Lemma 4.6 in [11]). If we enhanced the condition of $\mathscr{F}$ such that one of $\mathscr{F}$ is surjective, we have the following result.

Proposition 7.2 *Let IFS($\mathscr{F}$) be an iterated function system such that one of $\mathscr{F}$ is surjective. If IFS($\mathscr{F}$) has the concordant ergodic shadowing property, then IFS($\mathscr{F}$) is weakly mixing.*

Proof Without loss of generality, we assume that f_0 is surjective, then $f_0 \times f_0$ is also surjective from $X \times X$ to itself. Since IFS($\mathscr{F}$) has the concordant ergodic shadowing property, by Proposition 7.1, so does IFS($\mathscr{F} \otimes \mathscr{F}$). By Theorem 1.1 in [11], IFS($\mathscr{F} \otimes \mathscr{F}$) is transitive. Then, for any nonempty open subsets U_1, V_1, U_2, V_2 in X, there exist $\omega \in \Sigma_m$ and $n \in \mathbb{N}$ such that

$$f_\omega^n \times f_\omega^n(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset.$$

Therefore, IFS($\mathscr{F}$) is weakly mixing. □

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