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Sign-Changing Solutions for Chern–Simons–Schrödinger Equations with Asymptotically 5-Linear Nonlinearity

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Abstract

In this paper, we study the following Chern–Simons–Schrödinger equation

$$\begin{cases} -\Delta u + \omega u + \lambda\Big(\dfrac{h^2(|x|)}{|x|^2} + \displaystyle\int_{|x|}^{+\infty} \dfrac{h(s)}{s} u^2(s)\mathrm{d}s\Big)u = g(u) \quad \text{in } \mathbb{R}^2, \\ u \in H_r^1(\mathbb{R}^2), \end{cases}$$

where $\omega, \lambda > 0$ and $h(s) = \frac{1}{2}\int_0^s r u^2(r)\mathrm{d}r$. Since the nonlinearity g is asymptotically 5-linear at infinity, there would be a competition between g and the nonlocal term. By constrained minimization arguments and the quantitative deformation lemma, we prove the existence of least energy sign-changing radial solution, which changes sign exactly once. Further, we study the concentration of the least energy sign-changing radial solutions as $\lambda \to 0$.

Keywords Chern–Simons–Schrödinger equation · Asymptotically 5-linear · Variational methods · Least energy sign-changing radial solution · Concentration

Mathematics Subject Classification 35J91 · 35J20 · 35J60

1 Introduction and Main Results

In this paper, we are interested in the following type of Chern–Simons–Schrödinger equation

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$$\begin{cases} -\Delta u + \omega u + \lambda\Big(\dfrac{h^2(|x|)}{|x|^2} + \displaystyle\int_{|x|}^{+\infty} \dfrac{h(s)}{s} u^2(s)\mathrm{d}s\Big)u = g(u) \quad \text{in } \mathbb{R}^2, \\ u \in H_r^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $\omega, \lambda > 0$ and $h(s) = \frac{1}{2}\int_0^s ru^2(r)\mathrm{d}r$. As we all know, Eq. (1.1) derives from studying the standing wave solutions of the following nonlinear Schrödinger system

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + g(\phi) = 0, \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}\,(\bar{\phi} D_2\phi), \\ \partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}\,(\bar{\phi} D_1\phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\dfrac{1}{2}|\phi|^2, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ denotes the complex scalar field, $A_\nu : \mathbb{R}^{1+2} \to \mathbb{R}$ denotes the gauge field and $D_\nu = \partial_\nu + iA_\nu$ denotes the covariant derivative for $\nu = 0, 1, 2$.

System (1.2) was firstly proposed in [12,13], where (1.2) is usually called as Chern–Simons–Schrödinger system. The Chern–Simons–Schrödinger system defined in $\mathbb{R}^2$ is a non-relativistic quantum model describing the dynamics of a large number of particles in the plane, in which these particles interact directly through the spontaneous magnetic field. In addition, it describes an external uniform magnetic field, which is of great significance to the application of Chern–Simons theory in quantum Hall effect [20]. For more physical backgrounds of system (1.2), we refer readers to [10,18,19]. After these works, many mathematical scholars have been studying the existence of standing wave solutions for system (1.2). Especially, when $g(u) = \lambda|u|^{p-2}u$ with $p > 2$ and $\lambda > 0$, some interesting results are presented in [2,3,11,14,22]. When it comes to the standing wave solutions of system (1.2) with the form

$$\begin{aligned} &\phi(t, x) = u(|x|)e^{i\omega t} \quad \text{and} \quad A_0(t, x) = A_0(|x|), \\ &A_1(t, x) = \frac{x_2}{|x|^2}h(|x|) \quad \text{and} \quad A_2(t, x) = -\frac{x_1}{|x|^2}h(|x|), \end{aligned}$$

system (1.2) reduces to the following nonlocal equation

$$-\Delta u + (\omega + \xi)u + \Big(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s\Big)u = \lambda|u|^{p-2}u \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

where $h(s) = \frac{1}{2}\int_0^s ru^2(r)\mathrm{d}r$, $\xi \in \mathbb{R}$ is an integration constant of A_0. Here, A_0 has the expression

$$A_0(r) = \xi + \int_r^{+\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s.$$

Moreover, in Chern–Simons theory, system (1.2) is invariant under the gauge transformation

$$\phi \mapsto \phi e^{i\chi} \quad \text{and} \quad A_\nu \mapsto A_\nu - \partial_\nu \chi \quad \text{for any } \chi \in C^\infty(\mathbb{R}). \tag{1.4}$$

Then, for a given stationary solution, if taking $\chi = ct$ in (1.4), we may obtain another standing wave solution, in the sense that the functions $u(x)$, $A_1(x)$, $A_2(x)$ are unchanged, $\omega \mapsto \omega + c$ and $A_0(x) \to A_0(x) - c$. That is, the constant $\omega + \xi$ is a gauge invariant of the stationary solution to system (1.2). Due to the above discussion, we may take $\xi = 0$ below, then $\lim_{|x|\to\infty} A_0(x) = 0$. In this case, Eq. (1.3) becomes

$$-\Delta u + \omega u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s\right)u = \lambda |u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{1.5}$$

Recently, many scholars pay attention to Eq. (1.5) and obtained lots of results, see, for example, [2,3,10,11,14,15,21,22]. Especially, for the case of $p \in (2, 4)$ and $\omega > 0$, the standing wave solutions of Eq. (1.5) are found in [2] by the constrained minimization methods. After this, Pomponio and Ruiz [22] proved the existence and nonexistence of nontrivial solutions depending on the range of ω to Eq. (1.5). Furthermore, Pomponio and Ruiz [21] have studied the existence of positive solution for Eq. (1.5) on large ball. For $p = 4$, Byeon et al. [2] proved that Eq. (1.5) has no standing wave solutions if $\lambda \in (0, 1)$, has a family of weak solutions in $H_r^1(\mathbb{R}^2)$ if $\lambda = 1$ and has a standing wave solution if $\lambda > 1$. In addition, Li and Luo [15] proved the nonexistence of normalized solution by the constrained minimization method when $p = 4$. Meanwhile, they also considered the case of $p > 4$. For $p \in (4, 6)$, Byeon et al. [2] proved the existence of standing wave solutions of equation (1.5) by considering a minimization problem on a manifold of Pohožaev–Nehari type in $H_r^1(\mathbb{R}^2)$. Additionally, Huh [11] proved that Eq. (1.5) has infinitely many solutions for any $p > 6$. For more investigations on the Chern–Simons–Schrödinger equations, we refer the interested readers to [3,6,8,10,14,17,23,26,27] and references therein.

As far as we know, for the existence of sign-changing solutions to Eq. (1.5), there are few works presented in [9,16,25]. In [16], Li, Luo and Shuai proved the existence of least energy sign-changing radial solution which changes sign exactly once when $p > 6$. In [9], Deng, Peng and Shuai studied that Eq. (1.5) has multiple nodal solutions when $p > 6$. Xie and Chen [25] considered the more general type of nonlinearity g, and a least energy sign-changing radial solution with two exactly nodal domains is obtained when g is 5-superlinear.

Inspired by the above results, we consider the existence of sign-changing solutions for Eq. (1.1) under the following assumptions:

- (g_1) $g \in C(\mathbb{R}, \mathbb{R})$,
- (g_2) $\lim_{t\to 0} \frac{g(t)}{t} = 0$,
- (g_3) $\lim_{t\to\infty} \frac{g(t)}{t^5} = 1$ and $\frac{g(t)}{t^5} < 1$ for all $t \in \mathbb{R}\backslash\{0\}$,
- (g_4) the function $t \mapsto \frac{g(t)}{|t|^5}$ is strictly increasing for all $t \in \mathbb{R}\backslash\{0\}$.

Our main results of this paper read as follows:

Theorem 1.1 *Assume that $\lambda > 0$ and $(g_1) - (g_4)$ are satisfied. Then, Eq. (1.1) admits a least energy sign-changing radial solution u_λ, which changes sign exactly once.*

We further study the concentration of least energy sign-changing radial solutions as $\lambda \to 0$.

Theorem 1.2 *For any sequence $\{\lambda_n\} \subset (0, +\infty)$ such that $\lambda_n \to 0$ as $n \to \infty$, $\{u_{\lambda_n}\}$ strongly converges to u_0 in $H_r^1(\mathbb{R}^2)$ up to a subsequence, where u_0 changing sign exactly once is a least energy sign-changing radial solution of*

$$-\Delta u + \omega u = g(u) \quad in \ \mathbb{R}^2. \tag{1.6}$$

Remark 1.3 To the best of our knowledge, the sign-changing solutions of Eq. (1.1) were considered in [9,16,25], where the nonlinearity g is supposed as 5-superlinear at infinity. In the present work, we assume that the nonlinearity g is asymptotically 5-linear, which means that there would be a competition between the nonlocal term and the local nonlinearity. Hence, we will encounter the main difficulty in proving Theorem 1.1 that the sign-changing Nehari-type manifold for Eq. (1.1) is nonempty under our assumptions $(g_1) - (g_4)$. Additionally, we point out that there are many functions satisfying $(g_1) - (g_4)$, for example, $g(t) = \frac{t^7}{1+t^2}$ for all $t \in \mathbb{R}$.

The rest of this paper is organized as follows. In Sect. 2, we give some preliminary lemmas which are necessary for proving our results. Section 3 is devoted to proving Theorems 1.1 and 1.2.

Henceforth, we use the following notations:

- $L^p(\mathbb{R}^2)$ is the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^2} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}$ for all $p \in [1, +\infty)$.
- $H_r^1(\mathbb{R}^2)$ consists of the all radial functions in $H^1(\mathbb{R}^2)$ with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \omega u v)\mathrm{d}x \quad \text{and} \quad \|u\| = (u, u)^{\frac{1}{2}}.$$

- $\|\cdot\|_{H^{-1}}$ denotes the norm of the dual space $H_r^{-1}(\mathbb{R}^2)$ of $H_r^1(\mathbb{R}^2)$.
- For any $p \in [2, +\infty)$, there exists the constant S_p such that $|u|_p^p \le S_p \|u\|^p$ for all $u \in H_r^1(\mathbb{R}^2)$.
- "$\to$" and "$\rightharpoonup$" denote the strong and weak convergences in function spaces, respectively.
- $u^+(x) = \max\{u, 0\}$, $u^- = \min\{u, 0\}$; $C_i, i = 1, 2, \cdots$, denote positive constants.
- For any $r > 0$, $B_r := \{x \in \mathbb{R}^2 : |x| < r\}$.

2 Preliminaries

Firstly, we present some properties on the nonlinearity g and its primitive $G(t) = \int_0^t g(s)\mathrm{d}s$. Due to $(g_1) - (g_3)$, for any $\varepsilon > 0$ and $p > 6$, there exists some constant

$C_{\varepsilon,p} > 0$ such that

$$|g(t)| \leq \varepsilon|t| + C_{\varepsilon,p}|t|^{p-1} \quad \text{and} \quad |G(t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_{\varepsilon,p}}{p}|t|^p \quad \text{for all } t \in \mathbb{R}. \quad (2.1)$$

Lemma 2.1 *The function* $t \mapsto g(t)t - 6G(t)$ *is strictly decreasing in* $(-\infty, 0)$ *and strictly increasing in* $(0, +\infty)$. *Particularly, there holds that* $g(t)t - 6G(t) \geq 0$ *for all* $t \in \mathbb{R}$.

Proof Let $\mathcal{G}(t) := g(t)t - 6G(t)$ for any $t \in \mathbb{R}$. Taking $0 < s < r$, by (g_4) we have

$$\begin{aligned}
\mathcal{G}(r) - \mathcal{G}(s) &= 6\Big[\frac{1}{6}\big(g(r)r - g(s)s\big) - \big(G(r) - G(s)\big)\Big] \\
&= 6\Big(\int_0^r \frac{g(r)}{r^5}\tau^5 \mathrm{d}\tau - \int_0^s \frac{g(s)}{s^5}\tau^5 \mathrm{d}\tau - \int_s^r \frac{g(\tau)}{\tau^5}\tau^5 \mathrm{d}\tau\Big) \\
&= 6\left[\int_0^s \Big(\frac{g(r)}{r^5} - \frac{g(s)}{s^5}\Big)\tau^5 \mathrm{d}\tau + \int_s^r \Big(\frac{g(r)}{r^5} - \frac{g(\tau)}{\tau^5}\Big)\tau^5 \mathrm{d}\tau\right] > 0,
\end{aligned}$$

which implies that $\mathcal{G}$ is strictly increasing in $(0, +\infty)$. Since $\mathcal{G}(t)$ is even in t, then $\mathcal{G}$ is strictly decreasing in $(-\infty, 0)$. Specially, $\mathcal{G}(t) \geq \mathcal{G}(0) = 0$ for all $t \in \mathbb{R}$. Hence, the proof is completed. □

From now on, we fix $\lambda > 0$. The energy functional of Eq. (1.1) is defined as

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2} \big(|\nabla u|^2 + \omega u^2\big)\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}u^2(r)\mathrm{d}r\Big)^2 \mathrm{d}x - \int_{\mathbb{R}^2} G(u)\mathrm{d}x.$$

As in [2], by (2.1), it is standard to verify that $I_\lambda \in C^1(H_r^1(\mathbb{R}^2), \mathbb{R})$ and, for any $u, \varphi \in H_r^1(\mathbb{R}^2)$,

$$\begin{aligned}
\langle I'_\lambda(u), \varphi\rangle &= \int_{\mathbb{R}^2} \big(\nabla u \cdot \nabla\varphi + \omega u\varphi\big)\mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}u^2(r)\mathrm{d}r\Big)\Big(\int_0^{|x|} ru(r)\varphi(r)\mathrm{d}r\Big)\mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2}u\varphi \mathrm{d}x - \int_{\mathbb{R}^2} g(u)\varphi \mathrm{d}x.
\end{aligned}$$

Then, the critical points of I_λ are weak solutions of Eq. (1.1). For convenience, we introduce

$$\begin{aligned}
B(u) &:= \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}u^2(r)\mathrm{d}r\Big)^2 \mathrm{d}x, \\
B_1(u) &:= \int_{\mathbb{R}^2} \frac{|u^+|^2}{|x|^2}\Big(\int_0^{|x|} \frac{r}{2}|u^-|^2 \mathrm{d}r\Big)^2 \mathrm{d}x
\end{aligned}$$

$$+2\int_{\mathbb{R}^2}\frac{|u^-|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u^+|^2\mathrm{d}r\Big)\Big(\int_0^{|x|}\frac{r}{2}|u^-|^2\mathrm{d}r\Big)\mathrm{d}x,$$

$$B_2(u):=\int_{\mathbb{R}^2}\frac{|u^-|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u^+|^2\mathrm{d}r\Big)^2\mathrm{d}x$$
$$+2\int_{\mathbb{R}^2}\frac{|u^+|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u^+|^2\mathrm{d}r\Big)\Big(\int_0^{|x|}\frac{r}{2}|u^-|^2\mathrm{d}r\Big)\mathrm{d}x.$$

Through direct calculation, we can prove that, for all $u\in H_r^1(\mathbb{R}^2)$,

$$I_\lambda(u)=I_\lambda(u^+)+I_\lambda(u^-)+\frac{\lambda}{2}B_1(u)+\frac{\lambda}{2}B_2(u), \tag{2.2}$$

$$\langle I'_\lambda(u),u^+\rangle=\langle I'_\lambda(u^+),u^+\rangle+\lambda B_1(u)+2\lambda B_2(u), \tag{2.3}$$

$$\langle I'_\lambda(u),u^-\rangle=\langle I'_\lambda(u^-),u^-\rangle+2\lambda B_1(u)+\lambda B_2(u). \tag{2.4}$$

To find the sign-changing solutions of Eq. (1.1), we introduce the following constrained set:

$$\mathcal{M}_\lambda:=\Big\{u\in H_r^1(\mathbb{R}^2): u^\pm\neq 0\ \text{ and }\ \langle I'_\lambda(u),u^\pm\rangle=0\Big\}.$$

Obviously, the set $\mathcal{M}_\lambda$ contains all of the radial sign-changing solutions to Eq. (1.1). Define

$$\mathcal{S}_\lambda=\left\{u\in H_r^1(\mathbb{R}^2)\backslash\{0\}:\lambda H^\pm(u)-\int_{\mathbb{R}^2}g(u^\pm)u^\pm\mathrm{d}x<0\right\}, \tag{2.5}$$

where the functional $H^\pm: H_r^1(\mathbb{R}^2)\mapsto\mathbb{R}$ is defined for $u\in H_r^1(\mathbb{R}^2)$ by

$$H^\pm(u)=2\int_{\mathbb{R}^2}\frac{|u|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)\Big(\int_0^{|x|}\frac{r}{2}|u^\pm|^2\mathrm{d}r\Big)\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\frac{|u^\pm|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)^2\mathrm{d}x.$$

By simple calculation, we get

$$H^+(u)=3B(u^+)+B_1(u)+2B_2(u), \tag{2.6}$$

$$H^-(u)=3B(u^-)+2B_1(u)+B_2(u). \tag{2.7}$$

Firstly, we prove $\mathcal{S}_\lambda\neq\emptyset$, which plays an important role in verifying $\mathcal{M}_\lambda\neq\emptyset$.

Lemma 2.2 *The set $\mathcal{S}_\lambda$ is nonempty and $\mathcal{M}_\lambda\subset\mathcal{S}_\lambda$.*

Proof From (g_2), (g_3) and Lemma 2.1, it follows that there exist $R>0$ and $C_1>0$ satisfying

$$g(t)t\geq 6G(t)\geq 0\ \text{ for all } |t|>R\quad\text{ and }\quad |g(t)|\leq C_1|t|\ \text{ for all } |t|\leq R. \tag{2.8}$$

For any fixed $u \in H_r^1(\mathbb{R}^2)$ with $u^\pm \neq 0$, there exists some $\nu > 0$ such that $\{x \in \mathbb{R}^2 : |u^\pm(x)| > \nu\}$ has positive measure. Setting $u_t(\cdot) = u(t\cdot)$ for $t > 0$, we deduce from (2.8) that

$$\begin{aligned}
&\lambda H^\pm(tu_t) - \int_{\mathbb{R}^2} g(tu_t^\pm)tu_t^\pm \mathrm{d}x \\
&= 2\lambda t^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)\Big(\int_0^{|x|}\frac{r}{2}|u^\pm|^2\mathrm{d}r\Big)\mathrm{d}x \\
&\quad + \lambda t^2 \int_{\mathbb{R}^2} \frac{|u^\pm|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)^2\mathrm{d}x \\
&\quad - \frac{1}{t}\int_{\{x\in\mathbb{R}^2:|tu^\pm(x)|\le R\}} g(tu^\pm)u^\pm \mathrm{d}x - \frac{1}{t}\int_{\{x\in\mathbb{R}^2:|tu^\pm(x)|> R\}} g(tu^\pm)u^\pm \mathrm{d}x \\
&\le 2\lambda t^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)\Big(\int_0^{|x|}\frac{r}{2}|u^\pm|^2\mathrm{d}r\Big)\mathrm{d}x \\
&\quad + \lambda t^2 \int_{\mathbb{R}^2} \frac{|u^\pm|^2}{|x|^2}\Big(\int_0^{|x|}\frac{r}{2}|u|^2\mathrm{d}r\Big)^2\mathrm{d}x \\
&\quad + C_1\int_{\{x\in\mathbb{R}^2:|tu^\pm(x)|\le R\}} |u^\pm|^2 \mathrm{d}x - \frac{1}{t^2}\int_{\{x\in\mathbb{R}^2:|tu^\pm(x)|> R\}} 6G(tu^\pm)\mathrm{d}x.
\end{aligned}$$

By (2.8), the Fatou lemma and (g_3), one obtains

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{\lambda H^\pm(tu_t) - \int_{\mathbb{R}^2} g(tu_t^\pm)tu_t^\pm \mathrm{d}x}{t^4} \\
&\quad \le -\liminf_{t\to+\infty}\int_{\{x\in\mathbb{R}^2:|tu^\pm(x)|> R\}} \frac{6G(tu^\pm)}{t^6}\mathrm{d}x \\
&\quad \le -6\int_{\{x\in\mathbb{R}^2:|u^\pm(x)|> \nu\}} \liminf_{t\to+\infty}\frac{G(tu^\pm)}{(tu^\pm)^6}|u^\pm|^6\mathrm{d}x \\
&\quad < 0.
\end{aligned}$$

Thus, if taking $u_\infty = t_\infty u_{t_\infty}$ with $t_\infty > 0$ sufficiently large, we conclude $u_\infty \in \mathcal{S}_\lambda$. Moreover, by the definition of $\mathcal{M}_\lambda$, it is easy to see that $\mathcal{M}_\lambda \subset \mathcal{S}_\lambda$. Therefore, we finish the proof of this lemma. □

Lemma 2.3 *For any $u \in \mathcal{S}_\lambda$, there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.*

Proof Let $\psi_1(s,t) = \langle I'_\lambda(su^+ + tu^-), su^+\rangle$ and $\psi_2(s,t) = \langle I'_\lambda(su^+ + tu^-), tu^-\rangle$ for $s, t > 0$, namely,

$$\begin{aligned}
\psi_1(s,t) &= s^2\|u^+\|^2 + 3\lambda s^6 B(u^+) + \lambda s^2t^4B_1(u) + 2\lambda s^4t^2B_2(u) \\
&\quad - \int_{\mathbb{R}^2} g(su^+)su^+\mathrm{d}x, \\
\psi_2(s,t) &= t^2\|u^-\|^2 + 3\lambda t^6 B(u^-) + 2\lambda s^2t^4B_1(u) + \lambda s^4t^2B_2(u)
\end{aligned}$$

$$-\int_{\mathbb{R}^2} g(tu^-)tu^-\mathrm{d}x.$$

Using (g_4) and (2.6), for all $t = s$ and $s \geq 1$, we have

$$\begin{aligned}\psi_1(s,s) &= s^2\|u^+\|^2 + s^6\Big(3\lambda B(u^+) + \lambda B_1(u) + 2\lambda B_2(u)\Big) - \int_{\mathbb{R}^2} g(su^+)su^+\mathrm{d}x \\ &\leq s^2\|u^+\|^2 + s^6\Big(\lambda H^+(u) - \int_{\mathbb{R}^2} g(u^+)u^+\mathrm{d}x\Big).\end{aligned} \tag{2.9}$$

It follows from (g_4) and (2.7) that, for all $s = t$ and $t \geq 1$,

$$\begin{aligned}\psi_2(t,t) &= t^2\|u^-\|^2 + t^6\Big(3\lambda B(u^-) + 2\lambda B_1(u) + \lambda B_2(u)\Big) - \int_{\mathbb{R}^2} g(tu^-)tu^-\mathrm{d}x \\ &\leq t^2\|u^-\|^2 + t^6\Big(\lambda H^-(u) - \int_{\mathbb{R}^2} g(u^-)u^-\mathrm{d}x\Big).\end{aligned} \tag{2.10}$$

From (2.9) and (2.10), there exists $R > 0$ large enough such that $\psi_1(R,R) < 0$ and $\psi_2(R,R) < 0$. Besides, we deduce that, for all $t > 0$,

$$\psi_1(s,t) \geq s^2\|u^+\|^2 - \int_{\mathbb{R}^2} g(su^+)su^+\mathrm{d}x = s^2\Big(\|u^+\|^2 - \int_{\mathbb{R}^2} \frac{g(su^+)}{s}u^+\mathrm{d}x\Big). \tag{2.11}$$

For all $s > 0$,

$$\psi_2(s,t) \geq t^2\|u^-\|^2 - \int_{\mathbb{R}^2} g(tu^-)tu^-\mathrm{d}x = t^2\Big(\|u^-\|^2 - \int_{\mathbb{R}^2} \frac{g(tu^-)}{t}u^-\mathrm{d}x\Big). \tag{2.12}$$

Choose $\varepsilon \in (0, S_2^{-1})$, it follows from (2.1) and the Sobolev inequality that, for $\iota > 0$,

$$\|u^\pm\|^2 - \int_{\mathbb{R}^2} \frac{g(\iota u^\pm)}{\iota}u^\pm\mathrm{d}x \geq (1-\varepsilon S_2)\|u^\pm\|^2 - \iota^{p-2}C_{\varepsilon,p}S_p\|u^\pm\|^p.$$

By (2.11) and (2.12), there exists $r \in (0,R)$ small enough such that $\psi_1(r,r) > 0$ and $\psi_2(r,r) > 0$. Noting that the functions $\psi_1(s,\cdot)$ and $\psi_2(\cdot,t)$ are increasing in $(0,+\infty)$ for any fixed $s > 0$ and $t > 0$, respectively, we can conclude that

$$\begin{aligned}&\psi_1(r,t) > 0 \quad \text{and} \quad \psi_1(R,t) < 0 \quad \text{for all } t \in [r,R],\\ &\psi_2(s,r) > 0 \quad \text{and} \quad \psi_2(s,R) < 0 \quad \text{for all } s \in [r,R].\end{aligned}$$

Consequently, $(\psi_1,\psi_2) \neq (0,0)$ on the boundary of $(r,R)\times(r,R)$. Then, by [16, Lemma 2.4], there exists $(s_u,t_u) \in (r,R)\times(r,R)$ such that $\psi_1(s_u,t_u) = \psi_2(s_u,t_u) =$

0. That is, $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$. Further, we claim that such (s_u, t_u) is unique. Indeed, for any $u \in \mathcal{M}_\lambda$,

$$\|u^+\|^2 + 3\lambda B(u^+) + \lambda B_1(u) + 2\lambda B_2(u) = \int_{\mathbb{R}^2} g(u^+)u^+ \mathrm{d}x, \tag{2.13}$$

$$\|u^-\|^2 + 3\lambda B(u^-) + 2\lambda B_1(u) + \lambda B_2(u) = \int_{\mathbb{R}^2} g(u^-)u^- \mathrm{d}x. \tag{2.14}$$

We show that if $u \in \mathcal{M}_\lambda$, then $(s_u, t_u) = (1, 1)$. In fact, since $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$, we have

$$s_u^2\|u^+\|^2 + 3\lambda s_u^6 B(u^+) + \lambda s_u^2 t_u^4 B_1(u) + 2\lambda s_u^4 t_u^2 B_2(u) = \int_{\mathbb{R}^2} g(s_u u^+)s_u u^+ \mathrm{d}x, \tag{2.15}$$

$$t_u^2\|u^-\|^2 + 3\lambda t_u^6 B(u^-) + 2\lambda s_u^2 t_u^4 B_1(u) + \lambda s_u^4 t_u^2 B_2(u) = \int_{\mathbb{R}^2} g(t_u u^-)t_u u^- \mathrm{d}x. \tag{2.16}$$

Without loss of generality, we assume $0 < s_u \le t_u$. Then, from (2.15), we conclude

$$s_u^2\|u^+\|^2 + 3\lambda s_u^6 B(u^+) + \lambda s_u^6 B_1(u) + 2\lambda s_u^6 B_2(u) \le \int_{\mathbb{R}^2} g(s_u u^+)s_u u^+ \mathrm{d}x. \tag{2.17}$$

From (2.13) and (2.17), we have

$$(s_u^{-4} - 1)\|u^+\|^2 \le \int_{\mathbb{R}^2} \left(\frac{g(s_u u^+)}{s_u^5 |u^+|^5} \mathrm{d}x - \frac{g(u^+)}{|u^+|^5} \right) |u^+|^6 \mathrm{d}x.$$

Using (g_4), we get $1 \le s_u \le t_u$. Similarly, by (2.14) and (2.16), we obtain $t_u \le 1$. Therefore, $s_u = t_u = 1$. Moreover, if $u \in \mathcal{S}_\lambda \backslash \mathcal{M}_\lambda$, suppose that there exists another pair (s_u', t_u') of positive numbers such that $s_u' u^+ + t_u' u^- \in \mathcal{M}_\lambda$. Then, we get

$$\frac{s_u'}{s_u}(s_u u^+) + \frac{t_u'}{t_u}(t_u u^-) = s_u' u^+ + t_u' u^- \in \mathcal{M}_\lambda.$$

Hence, we obtain that $s_u' = s_u$ and $t_u' = t_u$. That is, such (s_u, t_u) is unique. This lemma is proved. □

Lemma 2.4 $m_\lambda = \inf_{\mathcal{M}_\lambda} I_\lambda \ge C_2 > 0$.

Proof Letting $\varepsilon = \frac{\omega}{2}$ in (2.1), we deduce that, for all $u \in \mathcal{M}_\lambda$,

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2)\mathrm{d}x + 3\lambda \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 \mathrm{d}x \le \frac{\omega}{2} \int_{\mathbb{R}^2} u^2 \mathrm{d}x + C_{\frac{\omega}{2},p} \int_{\mathbb{R}^2} |u|^p \mathrm{d}x. \tag{2.18}$$

Then, by the Sobolev inequality,

$$\frac{1}{2}\|u\|^2 < C_{\frac{\omega}{2},p}|u|_p^p \leq C_{\frac{\omega}{2},p}S_p\|u\|^p. \tag{2.19}$$

Since $u \neq 0$ and $p > 6$, $\inf_{u\in\mathcal{M}_\lambda}\|u\| \geq \big(2C_{\frac{\omega}{2},p}S_p\big)^{\frac{1}{2-p}}$. For any $u \in \mathcal{M}_\lambda$, by Lemma 2.1, one has

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{6}\big\langle I'_\lambda(u), u\big\rangle = \frac{1}{3}\|u\|^2 + \int_{\mathbb{R}^2}\frac{1}{6}g(u)u - G(u)\mathrm{d}x \geq \frac{1}{3}\|u\|^2. \tag{2.20}$$

Then, $m_\lambda = \inf_{u\in\mathcal{M}_\lambda} I_\lambda(u) \geq \frac{1}{3}(2C_{\frac{\omega}{2},p}S_p)^{\frac{1}{2-p}} := C_2 > 0$. Thus, this lemma is proved. □

Remark 2.5 It is clear that (2.18) still holds for all $u \in H_r^1(\mathbb{R}^2)$ satisfying $\langle I'_\lambda(u), u\rangle \leq 0$. Then, there exists $\rho > 0$ such that $|u|_p, \|u\| > \rho$ for all $u \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ with $\langle I'_\lambda(u), u\rangle \leq 0$, where $p > 6$.

Lemma 2.6 *The set $\mathcal{M}_\lambda$ is closed.*

Proof Let $\{u_n\} \subset \mathcal{M}_\lambda$ such that $u_n \to u$ in $H_r^1(\mathbb{R}^2)$. Due to [7, Proposition 7.2], the maps $v \mapsto v^\pm$ are continuous from $H_r^1(\mathbb{R}^2)$ to itself. Hence, we can verify that $\gamma_\pm(v) = \langle I'_\lambda(v), v^\pm\rangle$ are continuous in $H_r^1(\mathbb{R}^2)$, which implies $\gamma_\pm(u) = 0$. By (2.3) and (2.4), we have $\big\langle I'_\lambda(u_n^\pm), u_n^\pm\big\rangle \leq 0$. Using Remark 2.5, it follows that $|u_n^\pm|_p > \rho > 0$ for all n once $p > 6$. Moreover, since $\{u_n^\pm\}$ is bounded in $H_r^1(\mathbb{R}^2)$, by the compactness of $H_r^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, $u_n^\pm \to u^\pm$ in $L^p(\mathbb{R}^2)$ for any $p > 6$ up to a subsequence. From this fact, $|u^\pm|_p > 0$ and then $u \in \mathcal{M}_\lambda$. Thus, this lemma is proved. □

Lemma 2.7 *For any $u \in \mathcal{M}_\lambda$, $I_\lambda(su^+ + tu^-) < I_\lambda(u)$ for every $s, t \geq 0$ and $(s, t) \neq (1, 1)$.*

Proof Let $\Omega^\pm = \big\{x \in \mathbb{R}^2 : u^\pm(x) \neq 0\big\}$, it is clear that $|\Omega^\pm| > 0$. For any given $x \in \Omega^\pm$, we define

$$\zeta_\pm(t) = \Big(\frac{t^2}{2} - \frac{t^6}{6}\Big)\Big(|\nabla u^\pm|^2 + \omega|u^\pm|^2\Big) + \frac{t^6}{6}g(u^\pm)u^\pm - G(tu^\pm) \quad \text{for all } t \geq 0.$$

By direct computation, we obtain that, for any $t \geq 0$,

$$\zeta'_\pm(t) = t(1 - t^4)\Big(|\nabla u^\pm|^2 + \omega|u^\pm|^2\Big) + t^5(u^\pm)^6\Big[\frac{g(u^\pm)}{(u^\pm)^5} - \frac{g(tu^\pm)}{(tu^\pm)^5}\Big].$$

Clearly, $\zeta'_\pm(0) = \zeta'_\pm(1) = 0$. By (g_1) and (g_4), $\zeta'_\pm(t) > 0$ in $(0, 1)$ and $\zeta'_\pm(t) < 0$ in $(1, +\infty)$. Then,

$$\zeta_\pm(t) < \zeta_\pm(1) \quad \text{for every } t \in [0, 1) \cup (1, +\infty). \tag{2.21}$$

If $u \in \mathcal{M}_\lambda$, for all $s, t \geq 0$ and $(s, t) \neq (1, 1)$, we deduce from (2.2)–(2.4) and (2.21) that

$$\begin{aligned} I_\lambda(su^+ + tu^-) &= I_\lambda(su^+ + tu^-) - \frac{s^6}{6}\langle I'_\lambda(u), u^+\rangle - \frac{t^6}{6}\langle I'_\lambda(u), u^-\rangle \\ &= I_\lambda(su^+) + I_\lambda(tu^-) - \frac{s^6}{6}\langle I'_\lambda(u^+), u^+\rangle - \frac{t^6}{6}\langle I'_\lambda(u^-), u^-\rangle \\ &\quad - \frac{\lambda}{6}\big(s^2 - t^2\big)^2\big(s^2 + 2t^2\big)B_1(u) - \frac{\lambda}{6}\big(s^2 - t^2\big)^2\big(2s^2 + t^2\big)B_2(u) \\ &\leq I_\lambda(su^+) - \frac{s^6}{6}\langle I'_\lambda(u^+), u^+\rangle + I_\lambda(tu^-) - \frac{t^6}{6}\langle I'_\lambda(u^-), u^-\rangle \\ &< I_\lambda(u^+) - \frac{1}{6}\langle I'_\lambda(u^+), u^+\rangle + I_\lambda(u^-) - \frac{1}{6}\langle I'_\lambda(u^-), u^-\rangle \\ &= I_\lambda(u^+) + I_\lambda(u^-) + \frac{\lambda}{2}B_1(u) + \frac{\lambda}{2}B_2(u) \\ &= I_\lambda(u). \end{aligned}$$

Therefore, we complete the proof of this lemma. □

Lemma 2.8 *If $m_\lambda = \inf_{u\in\mathcal{M}_\lambda} I_\lambda(u)$ is attained by $u \in \mathcal{M}_\lambda$, then u is a critical point of I_λ.*

Proof Assume by contrary that $I'_\lambda(u) \neq 0$ in $H_r^{-1}(\mathbb{R}^2)$, there exist $\varepsilon_0 > 0$ and $\delta \in \big(0, \frac{\sqrt{\omega}\varrho}{4}\big)$ such that $\|I'_\lambda(v)\|_{H^{-1}} \geq \varepsilon_0$ for all $v \in H_r^1(\mathbb{R}^2)$ satisfying $\|v - u\| \leq 3\delta$, where $\varrho = \min\big\{|u^+|_2, |u^-|_2\big\}$. Setting $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$, we define the map $h : \bar{D} \to H_r^1(\mathbb{R}^2)$ for $(\alpha, \beta) \in \bar{D}$ by $h(\alpha, \beta) = \alpha u^+ + \beta u^-$. Due to Lemma 2.7, there holds

$$\bar{m} := \max_{\partial D} I_\lambda \circ h < m_\lambda. \tag{2.22}$$

Let $\varepsilon := \min\big\{\frac{m_\lambda - \bar{m}}{4}, \frac{\varepsilon_0\delta}{8}\big\}$ and $S := \big\{v \in H_r^1(\mathbb{R}^2) : \|v - u\| \leq \delta\big\}$. By the quantitative deformation lemma (see [24, Lemma 2.3]), there exists a deformation $\eta \in C\big([0, 1] \times H_r^1(\mathbb{R}^2), H_r^1(\mathbb{R}^2)\big)$ such that

(a) $\eta(1, v) = v$ if $v \notin I_\lambda^{-1}\big([m_\lambda - 2\varepsilon, m_\lambda + 2\varepsilon]\big) \cap S_{2\delta}$,
(b) $\eta(1, I_\lambda^{m_\lambda+\varepsilon} \cap S) \subset I_\lambda^{m_\lambda-\varepsilon}$, where $I_\lambda^{m_\lambda\pm\varepsilon} := \big\{v \in H_r^1(\mathbb{R}^2) : I_\lambda(v) \leq m_\lambda \pm \varepsilon\big\}$,
(c) $I_\lambda\big(\eta(1, v)\big) \leq I_\lambda(v)$ for all $v \in H_r^1(\mathbb{R}^2)$,
(d) $\|\eta(t, v) - v\| \leq \delta$ for all $t \in [0, 1]$ and $v \in H_r^1(\mathbb{R}^2)$.

Then, we conclude that

$$\max_{(\alpha,\beta)\in\bar{D}} I_\lambda\big(\eta(1, h(\alpha, \beta))\big) < m_\lambda. \tag{2.23}$$

Next, we end the proof by proving $\eta(1, h(D)) \cap \mathcal{M}_\lambda \neq \emptyset$, then (2.23) implies $m_\lambda < m_\lambda$, a contradiction. For $(\alpha, \beta) \in \bar{D}$, define $\phi(\alpha, \beta) = \eta(1, h(\alpha, \beta))$ and

$$\Psi_0(\alpha, \beta) = \left(\left\langle I'_\lambda(\alpha u^+ + \beta u^-), \alpha u^+\right\rangle, \left\langle I'_\lambda(\alpha u^+ + \beta u^-), \beta u^-\right\rangle\right),$$
$$\Psi_1(\alpha, \beta) = \left(\left\langle I'_\lambda(\phi(\alpha, \beta)), \phi^+(\alpha, \beta)\right\rangle, \left\langle I'_\lambda(\phi(\alpha, \beta)), \phi^-(\alpha, \beta)\right\rangle\right).$$

By Lemma 2.3, we deduce $\deg(\Psi_0, D, 0) = 1$. From (2.22) and (a), it follows that $h = \phi$ on ∂D. Thus, using the degree theory, we get $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Consequently, $\Psi_1(\alpha_0, \beta_0) = 0$ for some $(\alpha_0, \beta_0) \in D$. If $[\phi(\alpha_0, \beta_0)]^+ = 0$, by the Young inequality, we have

$$\begin{aligned}\|\phi(\alpha_0, \beta_0) - h(\alpha_0, \beta_0)\| &\geq \left(\int_{\mathbb{R}^2} \omega\left|\alpha_0 u^+ + \beta_0 u^- - [\phi(\alpha_0, \beta_0)]^-\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\geq \omega^{\frac{1}{2}} \alpha_0 |u^+|_2 > 2\delta,\end{aligned}$$

which is contrary to (d). That is, $[\phi(\alpha_0, \beta_0)]^+ \neq 0$. Similarly, we can prove $[\phi(\alpha_0, \beta_0)]^- \neq 0$. Therefore, $\eta(1, h(\alpha_0, \beta_0)) = \phi(\alpha_0, \beta_0) \in \mathcal{M}_\lambda$ and this lemma is proved. □

3 Proof of Theorems 1.1 and 1.2

Before starting the proof of Theorem 1.1, we verify that there exists some $\sigma > 0$ such that

$$\left\langle I'_\lambda(w^\pm), w^\pm\right\rangle \geq \frac{1}{4}\|w^\pm\|^2 \quad \text{for all } w \in H^1_r(\mathbb{R}^2) \text{ with } \|w^\pm\| \leq \sigma. \tag{3.1}$$

In fact, fix $\varepsilon \in (0, \frac{\omega}{2})$, by (2.1) and the Sobolev inequality, we infer that

$$\begin{aligned}\left\langle I'_\lambda(w^\pm), w^\pm\right\rangle &\geq \|w^\pm\|^2 - \int_{\mathbb{R}^2} |g(w^\pm)||w^\pm| \mathrm{d}x \\ &\geq \|w^\pm\|^2 - \varepsilon \int_{\mathbb{R}^2} |w^\pm|^2 \mathrm{d}x - C_{p,\varepsilon} \int_{\mathbb{R}^2} |w^\pm|^p \mathrm{d}x \\ &\geq \frac{1}{2}\|w^\pm\|^2 - S_p C_{p,\varepsilon}\|w^\pm\|^p.\end{aligned}$$

Hence, if choosing $\sigma \leq (4S_p C_{p,\varepsilon})^{\frac{1}{2-p}}$, we will conclude that (3.1) holds.

Proof of Theorem 1.1 Let $\{u_n\} \subset \mathcal{M}_\lambda$ be a minimizing sequence of the infimum m_λ. Similar to (2.20), we observed that $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^2)$. Then, $u_n \rightharpoonup u_\lambda$ in $H^1_r(\mathbb{R}^2)$, $u_n \to u_\lambda$ in $L^p(\mathbb{R}^2)$ for $p > 6$ and $u_n(x) \to u_\lambda(x)$ *a.e.* in $\mathbb{R}^2$ up to a subsequence. We will prove that, as $n \to \infty$,

$$\int_{\mathbb{R}^2} g(u_n)u_n \mathrm{d}x \to \int_{\mathbb{R}^2} g(u_\lambda)u_\lambda \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^2} G(u_n)\mathrm{d}x \to \int_{\mathbb{R}^2} G(u_\lambda)\mathrm{d}x. \tag{3.2}$$

Indeed, set $P_1(s) = g(s)s$, $P_2(s) = G(s)$ and $Q(s) = \varepsilon s^2 + C_{p,\varepsilon}|s|^p$ for $s \in \mathbb{R}$, where $p > 6$, we get

$$\lim_{s\to\infty} \frac{P_1(s)}{Q(s)} = \lim_{s\to 0} \frac{P_1(s)}{Q(s)} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{P_2(s)}{Q(s)} = \lim_{s\to 0} \frac{P_2(s)}{Q(s)} = 0.$$

It is clear that $\sup_n \int_{\mathbb{R}^2} |Q(u_n)|\mathrm{d}x < +\infty$ and $P_i(u_n(x)) \to P_i(u_\lambda(x))$ *a.e.* in $\mathbb{R}^2$ as $n \to \infty$, $i = 1, 2$. Then, by Theorem A.I and Radial Lemma A.II in [1], (3.2) holds. Due to [2, Lemma 3.2], we have

$$\lim_{n\to\infty} B(u_n) = B(u_\lambda), \tag{3.3}$$

$$\lim_{n\to\infty} \langle B'(u_n), u_n\rangle = \langle B'(u_\lambda), u_\lambda\rangle, \tag{3.4}$$

$$\lim_{n\to\infty} \langle B'(u_n), \varphi\rangle = \langle B'(u_\lambda), \varphi\rangle \quad \text{for all } \varphi \in H_r^1(\mathbb{R}^2). \tag{3.5}$$

Noting that $u_n^+ \rightharpoonup u_\lambda^+$ and $u_n^- \rightharpoonup u_\lambda^-$ in $H_r^1(\mathbb{R}^2)$ up to a subsequence, we will prove $u_n^\pm \to u_\lambda^\pm$ in $H_r^1(\mathbb{R}^2)$. For similarity, we just give the details of proving $u_n^+ \to u_\lambda^+$ in $H_r^1(\mathbb{R}^2)$. Assume by contradiction that $u_n^+ \nrightarrow u_\lambda^+$ in $H_r^1(\mathbb{R}^2)$ up to a subsequence, then $\|u_\lambda^+\| < \liminf_{n\to\infty} \|u_n^+\|$. For $\sigma > 0$ given by (3.1), choosing $\kappa \in (0, 1)$ such that $\kappa\|u_\lambda^\pm\| \le \sigma$, by (3.1), we get

$$\langle I_\lambda'(\kappa u_\lambda^+ + tu_\lambda^-), \kappa u_\lambda^+\rangle \ge \langle I_\lambda'(\kappa u_\lambda^+), \kappa u_\lambda^+\rangle \ge \frac{\kappa^2}{4}\|u_\lambda^+\|^2 > 0 \quad \text{for every } t \in (\kappa, 1),$$

$$\langle I_\lambda'(su_\lambda^+ + \kappa u_\lambda^-), \kappa u_\lambda^-\rangle \ge \langle I_\lambda'(\kappa u_\lambda^-), \kappa u_\lambda^-\rangle \ge \frac{\kappa^2}{4}\|u_\lambda^-\|^2 > 0 \quad \text{for every } s \in (\kappa, 1).$$

Besides, by (2.3), (3.2), (3.3) and the Fatou lemma, we have

$$\begin{aligned}
&\langle I_\lambda'(u_\lambda), u_\lambda^+\rangle \\
&= \|u_\lambda^+\|^2 + \lambda B_1(u_\lambda) + 2\lambda B_2(u_\lambda) + 3\lambda B(u_\lambda^+) - \int_{\mathbb{R}^2} g(u_\lambda^+)u_\lambda^+\mathrm{d}x \\
&< \liminf_{n\to\infty} \left(\|u_n^+\|^2 + \lambda B_1(u_n) + 2\lambda B_2(u_n) + 3\lambda B(u_n^+) - \int_{\mathbb{R}^2} g(u_n^+)u_n^+\mathrm{d}x\right) \\
&= \liminf_{n\to\infty} \langle I_\lambda'(u_n), u_n^+\rangle \\
&= 0.
\end{aligned} \tag{3.6}$$

Recalling $\|u_\lambda^-\| \le \liminf_{n\to\infty} \|u_n^-\|$, we deduce from (2.4), (3.2), (3.3) and the Fatou lemma that

$$\begin{aligned}
&\langle I_\lambda'(u_\lambda), u_\lambda^-\rangle \\
&= \|u_\lambda^-\|^2 + 2\lambda B_1(u_\lambda) + \lambda B_2(u_\lambda) + 3\lambda B(u_\lambda^-) - \int_{\mathbb{R}^2} g(u_\lambda^-)u_\lambda^-\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le \liminf_{n\to\infty}\left(\|u_n^-\|^2+2\lambda B_1(u_n)+\lambda B_2(u_n)+3\lambda B(u_n^-)-\int_{\mathbb{R}^2} g(u_n^-)u_n^-\mathrm{d}x\right)\\
&=\liminf_{n\to\infty}\langle I_\lambda'(u_n),u_n^-\rangle\\
&=0.
\end{aligned}
\tag{3.7}
$$

Then, it follows from (2.3) and (3.6) that, for every $t\in(\kappa,1)$,

$$
\begin{aligned}
&\left\langle I_\lambda'(u_\lambda^+ + tu_\lambda^-),u_\lambda^+\right\rangle\\
&\quad=\|u_\lambda^+\|^2+\lambda t^4B_1(u_\lambda)+2\lambda t^2B_2(u_\lambda)+3\lambda B(u_\lambda^+)-\int_{\mathbb{R}^2} g(u_\lambda^+)u_\lambda^+\mathrm{d}x\\
&\quad<\|u_\lambda^+\|^2+\lambda B_1(u_\lambda)+2\lambda B_2(u_\lambda)+3\lambda B(u_\lambda^+)-\int_{\mathbb{R}^2} g(u_\lambda^+)u_\lambda^+\mathrm{d}x\\
&\quad=\langle I_\lambda'(u_\lambda),u_\lambda^+\rangle<0.
\end{aligned}
$$

From (2.4) and (3.7), for every $s\in(\kappa,1)$, we conclude

$$
\begin{aligned}
&\left\langle I_\lambda'(su_\lambda^+ + u_\lambda^-),u_\lambda^-\right\rangle\\
&\quad=\|u_\lambda^-\|^2+2\lambda s^2B_1(u_\lambda)+\lambda s^4B_2(u_\lambda)+3\lambda B(u_\lambda^-)-\int_{\mathbb{R}^2} g(u_\lambda^-)u_\lambda^-\mathrm{d}x\\
&\quad<\|u_\lambda^-\|^2+2\lambda B_1(u_\lambda)+\lambda B_2(u_\lambda)+3\lambda B(u_\lambda^-)-\int_{\mathbb{R}^2} g(u_\lambda^-)u_\lambda^-\mathrm{d}x\\
&\quad=\langle I_\lambda'(u_\lambda),u_\lambda^-\rangle\le 0.
\end{aligned}
$$

Besides, it is easy to verify that $\left(\langle I_\lambda'(su_\lambda^+ + tu_\lambda^-),su_\lambda^+\rangle,\langle I_\lambda'(su_\lambda^+ + tu_\lambda^-),tu_\lambda^-\rangle\right)\neq(0,0)$ on the boundary of $(\kappa,1)\times(\kappa,1)$. Therefore, applying [16, Lemma 2.4], there exists $(\alpha,\beta)\in(\kappa,1)\times(\kappa,1)$ satisfying $\alpha u_\lambda^+ + \beta u_\lambda^-\in\mathcal{M}_\lambda$. Due to this fact, $I_\lambda(\alpha u_\lambda^+ + \beta u_\lambda^-)\ge m_\lambda$. From (3.2) and Lemma 2.1, there holds

$$
\begin{aligned}
I_\lambda(\alpha u_\lambda^+ + \beta u_\lambda^-)&=I_\lambda(\alpha u_\lambda^+ + \beta u_\lambda^-)-\frac{1}{6}\left\langle I_\lambda'(\alpha u_\lambda^+ + \beta u_\lambda^-),\alpha u_\lambda^+ + \beta u_\lambda^-\right\rangle\\
&=\frac{\alpha^2}{3}\|u_\lambda^+\|^2+\int_{\mathbb{R}^2}\frac{1}{6}g(\alpha u_\lambda^+)\alpha u_\lambda^+ - G(\alpha u_\lambda^+)\mathrm{d}x\\
&\quad+\frac{\beta^2}{3}\|u_\lambda^-\|^2+\int_{\mathbb{R}^2}\frac{1}{6}g(\beta u_\lambda^-)\beta u_\lambda^- - G(\beta u_\lambda^-)\mathrm{d}x\\
&\le\frac{1}{3}\|u_\lambda\|^2+\int_{\mathbb{R}^2}\frac{1}{6}g(u_\lambda)u_\lambda - G(u_\lambda)\mathrm{d}x\\
&<\liminf_{n\to\infty}\left(\frac{1}{3}\|u_n\|^2+\int_{\mathbb{R}^2}\frac{1}{6}g(u_n)u_n - G(u_n)\mathrm{d}x\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \liminf_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle\right)\\
&= \liminf_{n\to\infty} I_\lambda(u_n)\\
&= m_\lambda,
\end{aligned}
$$

which is a contradiction. Namely, $u_n^\pm \to u_\lambda^\pm$ in $H_r^1(\mathbb{R}^2)$. Hence, we obtain that $u_n \to u_\lambda$ in $H_r^1(\mathbb{R}^2)$. This result and Lemma 2.6 imply $u_\lambda \in \mathcal{M}_\lambda$. Clearly, $I_\lambda(u_\lambda) = m_\lambda$. In virtue of Lemma 2.8, u_λ is a critical point of I_λ. Then, u_λ is a least energy sign-changing radial solution of Eq. (1.1).

Furthermore, we prove that u_λ changes sign exactly once. Following the arguments in [4], we assume that $u_\lambda = u_1 + u_2 + u_3$ with $u_i \neq 0$, $u_1(x) \geq 0$, $u_2(x) \leq 0$ and $\operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset$ for $i \neq j$ and $i, j = 1, 2, 3$. For $\sigma > 0$ given by (3.1), taking $\kappa \in (0, 1)$ such that $\kappa\|v^\pm\| \leq \sigma$, we deduce

$$
\left\langle I'_\lambda(\kappa v^+ + t v^-), \kappa v^+\right\rangle \geq \left\langle I'_\lambda(\kappa v^+), \kappa v^+\right\rangle \geq \frac{\kappa^2}{4}\|v^+\|^2 > 0 \quad \text{for every } t \in (\kappa, 1), \tag{3.8}
$$

$$
\left\langle I'_\lambda(s v^+ + \kappa v^-), \kappa v^-\right\rangle \geq \left\langle I'_\lambda(\kappa v^-), \kappa v^-\right\rangle \geq \frac{\kappa^2}{4}\|v^-\|^2 > 0 \quad \text{for every } s \in (\kappa, 1). \tag{3.9}
$$

Set $v = u_1 + u_2$, then $v^+ = u_1$ and $v^- = u_2$, from $\langle I'_\lambda(u_\lambda), v^+\rangle = 0$ it follows that

$$
\begin{aligned}
\langle I'_\lambda(v), v^+\rangle = \langle I'_\lambda(u_\lambda), v^+\rangle &- \lambda\int_{\mathbb{R}^2}\frac{|u_1|^2}{|x|^2}\left(\int_0^{|x|}\frac{r}{2}|u_3|^2\mathrm{d}r\right)^2\mathrm{d}x\\
&- 2\lambda\int_{\mathbb{R}^2}\frac{|u_1|^2}{|x|^2}\left(\int_0^{|x|}\frac{r}{2}|v|^2\mathrm{d}r\right)\left(\int_0^{|x|}\frac{r}{2}|u_3|^2\mathrm{d}r\right)\mathrm{d}x\\
&- 2\lambda\int_{\mathbb{R}^2}\frac{v^2}{|x|^2}\left(\int_0^{|x|}\frac{r}{2}|u_1|^2\mathrm{d}r\right)\left(\int_0^{|x|}\frac{r}{2}|u_3|^2\mathrm{d}r\right)\mathrm{d}x\\
&- 2\lambda\int_{\mathbb{R}^2}\frac{|u_3|^2}{|x|^2}\left(\int_0^{|x|}\frac{r}{2}|u_\lambda|^2\mathrm{d}r\right)\left(\int_0^{|x|}\frac{r}{2}|u_1|^2\mathrm{d}r\right)\mathrm{d}x\\
&< 0.
\end{aligned}\tag{3.10}
$$

Then, one obtains that, for any $t \in (\kappa, 1)$,

$$
\begin{aligned}
&\left\langle I'_\lambda(v^+ + t v^-), v^+\right\rangle\\
&\quad = \|v^+\|^2 + \lambda t^4 B_1(v) + 2\lambda t^2 B_2(v) + 3\lambda B(v^+) - \int_{\mathbb{R}^2} g(v^+)v^+\mathrm{d}x\\
&\quad < \|v^+\|^2 + \lambda B_1(v) + 2\lambda B_2(v) + 3\lambda B(v^+) - \int_{\mathbb{R}^2} g(v^+)v^+\mathrm{d}x\\
&\quad = \left\langle I'_\lambda(v), v^+\right\rangle < 0.
\end{aligned}\tag{3.11}
$$

Similar to the proof of (3.10), we have $\langle I_\lambda'(v), v^-\rangle < 0$, which implies that

$$\langle I_\lambda'(sv^+ + v^-), v^-\rangle < 0 \quad \text{for all } s \in (\kappa, 1). \tag{3.12}$$

Clearly, we have $\big(\langle I_\lambda'(sv^+ + tv^-), sv^+\rangle, \langle I_\lambda'(sv^+ + tv^-), tv^-\rangle\big) \neq (0, 0)$ on the boundary of $(\kappa, 1) \times (\kappa, 1)$. Therefore, combining (3.8), (3.9), (3.11) and (3.12), by [16, Lemma 2.4] we deduce that there exists a pair $(\alpha', \beta') \in (\kappa, 1) \times (\kappa, 1)$ such that $\alpha' v^+ + \beta' v^- \in \mathcal{M}_\lambda$. Besides, we claim that

$$I_\lambda(v) \geq I_\lambda(sv^+ + tv^-) + \frac{1 - s^6}{6}\langle I_\lambda'(v), v^+\rangle + \frac{1 - t^6}{6}\langle I_\lambda'(v), v^-\rangle. \tag{3.13}$$

Indeed, from (g_4), we deduce that, for all $t \geq 0$ and $\xi \in \mathbb{R}$,

$$G(t\xi) - \frac{t^6}{6} g(\xi)\xi + \frac{1}{6} g(\xi)\xi - G(\xi) = \int_t^1 \Big[\frac{g(\xi)}{\xi^5} - \frac{g(s\xi)}{(s\xi)^5}\Big] s^5 \xi^6 \mathrm{d}s \geq 0. \tag{3.14}$$

Then, for all $s, t \geq 0$, we infer

$$\begin{aligned}
I_\lambda(v) &= I_\lambda(sv^+ + tv^-) + \frac{1 - s^6}{6}\langle I_\lambda'(v), v^+\rangle + \frac{1 - t^6}{6}\langle I_\lambda'(v), v^-\rangle \\
&\quad + \frac{\lambda(s^6 - 3s^2 + 2)}{6}\|v^+\|^2 + \frac{\lambda(t^6 - 3t^2 + 2)}{6}\|v^-\|^2 \\
&\quad + \frac{\lambda}{6}\big(s^2 - t^2\big)^2\big(s^2 + 2t^2\big) B_1(v) - \frac{\lambda}{6}\big(s^2 - t^2\big)^2\big(2s^2 + t^2\big) B_2(v) \\
&\quad + \int_{\mathbb{R}^2} G(sv^+) - \frac{s^6}{6} g(v^+)v^+ + \frac{1}{6} g(v^+)v^+ - G(v^+) \\
&\quad + \int_{\mathbb{R}^2} G(tv^-) - \frac{t^6}{6} g(v^-)v^- + \frac{1}{6} g(v^-)v^- - G(v^-) \\
&\geq I_\lambda(sv^+ + tv^-) + \frac{1 - s^6}{6}\langle I_\lambda'(v), v^+\rangle + \frac{1 - t^6}{6}\langle I_\lambda'(v), v^-\rangle.
\end{aligned}$$

Hence, based on (3.13), Lemmas 2.1 and 2.7, using the fact that $\langle I_\lambda'(v), v^\pm\rangle < 0$, we conclude that

$$\begin{aligned}
m_\lambda = I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{1}{6}\langle I_\lambda'(u_\lambda), u_\lambda\rangle \\
&= I_\lambda(v) + I_\lambda(u_3) - \frac{1}{6}\langle I_\lambda'(v), v\rangle - \frac{1}{6}\langle I_\lambda'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0}\Big(I_\lambda(sv^+ + tv^-) + \frac{1 - s^6}{6}\langle I_\lambda'(v), v^+\rangle + \frac{1 - t^6}{6}\langle I_\lambda'(v), v^-\rangle\Big) \\
&\quad + I_\lambda(u_3) - \frac{1}{6}\langle I_\lambda'(v), v\rangle - \frac{1}{6}\langle I_\lambda'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0} I_\lambda(sv^+ + tv^-) + \frac{1}{3}\|u_3\|^2
\end{aligned}$$

$$\geq m_\lambda + \frac{1}{3}\|u_3\|^2,$$

which implies $u_3 = 0$.

That is, u_λ changes sign exactly once. Thus, Theorem 1.1 is proved. □

Similar to [5], we prove that the infimum m_λ has a minimax characterization expressed by

$$m_\lambda = \inf_{u\in\mathcal{S}_\lambda} \max_{s,t\geq 0} I_\lambda(su^+ + tu^-). \tag{3.15}$$

In fact, for every $u \in \mathcal{M}_\lambda$, from Lemma 2.7, we have $\max_{s,t\geq 0} I_\lambda(su^+ + tu^-) \leq I_\lambda(u)$ which implies $\inf_{u\in\mathcal{S}_\lambda} \max_{s,t\geq 0} I_\lambda(su^+ + tu^-) \leq m_\lambda$. Additionally, for any $u \in \mathcal{S}_\lambda$, according to Lemma 2.3, there holds $\max_{s,t\geq 0} I_\lambda(su^+ + tu^-) \geq m_\lambda$. Hence, (3.15) is true.

As in Sect. 2, we denote the energy functional and the sign-changing Nehari-type manifold of Eq. (1.6) by I_0 and $\mathcal{M}_0$, where $I_0 = I_\lambda|_{\lambda=0}$ and $\mathcal{M}_0 = \mathcal{M}_\lambda|_{\lambda=0}$. Meanwhile, $m_0 = \inf_{u\in\mathcal{M}_0} I_0(u)$ denotes the least energy of sign-changing solutions to Eq. (1.6). Further, it is not difficult to verify $I_0 \in C^1(H_r^1(\mathbb{R}^2), \mathbb{R})$.

Proof of Theorem 1.2 Let $\{\lambda_n\} \subset (0, +\infty)$ such that $\lambda_n \xrightarrow{n} 0$, $\{u_{\lambda_n}\} \subset H_r^1(\mathbb{R}^2)$ be a sequence of least energy sign-changing solutions to Eq. (1.1), from Theorem 1.1 it follows that u_{λ_n} changes sign exactly once for every $n \in \mathbb{N}_+$.

Firstly, we claim that $\{u_{\lambda_n}\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Indeed, for any $w_0 \in \mathcal{S}_\lambda$, by (2.2), (2.6), (2.7) and (3.14) one obtains that, for any $\lambda > 0$,

$$\begin{aligned}
&\max_{s,t\geq 0} I_\lambda(sw_0^+ + tw_0^-)\\
&= \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|w_0^+\|^2 + \frac{s^6}{2}\lambda B(w_0^+) - \int_{\mathbb{R}^2} G(sw_0^+)\mathrm{d}x + \frac{s^2t^4}{2}\lambda B_1(w_0)\\
&\quad + \frac{t^2}{2}\|w_0^-\|^2 + \frac{t^6}{2}\lambda B(w_0^-) - \int_{\mathbb{R}^2} G(tw_0^-)\mathrm{d}x + \frac{s^4t^2}{2}\lambda B_2(w_0)\Bigg]\\
&\leq \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|w_0^+\|^2 + \frac{s^6}{6}\Big(3\lambda B(w_0^+) + \lambda B_1(w_0) + 2\lambda B_2(w_0) - \int_{\mathbb{R}^2} g(w_0^+)w_0^+\mathrm{d}x\Big)\\
&\quad + \frac{t^2}{2}\|w_0^-\|^2 + \frac{t^6}{6}\Big(3\lambda B(w_0^-) + 2\lambda B_1(w_0) + \lambda B_2(w_0) - \int_{\mathbb{R}^2} g(w_0^-)w_0^-\mathrm{d}x\Big)\\
&\quad + \frac{1}{6}\int_{\mathbb{R}^2} g(w_0^+)w_0^+ - 6G(w_0^+)\mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^2} g(w_0^-)w_0^- - 6G(w_0^-)\mathrm{d}x\\
&\quad - \frac{\lambda}{6}(s^2 - t^2)^2(s^2 + 2t^2)B_1(w_0) - \frac{\lambda}{6}(s^2 - t^2)^2(2s^2 + t^2)B_2(w_0)\Bigg]\\
&\leq \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|w_0^+\|^2 + \frac{s^6}{6}\Big(\lambda H^+(w_0) - \int_{\mathbb{R}^2} g(w_0^+)w_0^+\mathrm{d}x\Big)\\
&\quad + \frac{t^2}{2}\|w_0^-\|^2 + \frac{t^6}{6}\Big(\lambda H^-(w_0) - \int_{\mathbb{R}^2} g(w_0^-)w_0^-\mathrm{d}x\Big)
\end{aligned}$$

$$+\frac{1}{6}\int_{\mathbb{R}^2} g(w_0^+)w_0^+ - 6G(w_0^+)\mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^2} g(w_0^-)w_0^- - 6G(w_0^-)\mathrm{d}x\Big]$$
$$:= \Lambda_0.$$

By (3.15), we get $m_\lambda \leq \Lambda_0 \in (0, +\infty)$ for all $\lambda > 0$. In view of this fact, from Lemma 2.1, we obtain

$$\Lambda_0 \geq m_{\lambda_n} = I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{6}\langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle \geq \frac{1}{3}\|u_{\lambda_n}\|^2,$$

which means that $\{u_{\lambda_n}\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Then, there exists $u_0 \in H_r^1(\mathbb{R}^2)$ such that $u_{\lambda_n} \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^2)$, $u_{\lambda_n} \to u_0$ in $L^p(\mathbb{R}^2)$ for any $p > 6$ and $u_{\lambda_n}(x) \to u_0(x)$ *a.e.* in $\mathbb{R}^2$ up to a subsequence.

Secondly, we prove that $u_{\lambda_n} \to u_0$ in $H_r^1(\mathbb{R}^2)$ up to a subsequence, then u_0 is a sign-changing solution of Eq. (1.6) and changes sign exactly once. For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, by (3.5), we deduce that $\{\langle B'(u_{\lambda_n}), \varphi\rangle\}$ is bounded in $\mathbb{R}$. Then, by Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&\langle I'_0(u_0), \varphi\rangle \\
&= \int_{\mathbb{R}^2} \left(\nabla u_0 \cdot \nabla\varphi + \omega u_0\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^2} g(u_0)\varphi\mathrm{d}x \\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^2} \left(\nabla u_{\lambda_n} \cdot \nabla\varphi + \omega u_{\lambda_n}\varphi\right)\mathrm{d}x + \frac{\lambda_n}{2}\langle B'(u_{\lambda_n}), \varphi\rangle - \int_{\mathbb{R}^2} g(u_{\lambda_n})\varphi\mathrm{d}x\right) \\
&= \lim_{n\to\infty}\langle I'_{\lambda_n}(u_{\lambda_n}), \varphi\rangle = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^2).
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^2)$ is dense in $H_r^1(\mathbb{R}^2)$, we conclude that

$$\langle I'_0(u_0), \varphi\rangle = 0 \quad \text{for all } \varphi \in H_r^1(\mathbb{R}^2). \tag{3.16}$$

In view of (2.1), (3.4), (3.5) and the Hölder inequality, we obtain that, up to a subsequence,

$$\begin{aligned}
\|u_{\lambda_n} - u_0\|^2 &= \langle I'_{\lambda_n}(u_{\lambda_n}) - I'_0(u_0), u_{\lambda_n} - u_0\rangle - \frac{\lambda_n}{2}\langle B'(u_{\lambda_n}) - B'(u_0), u_{\lambda_n} - u_0\rangle \\
&\quad + \int_{\mathbb{R}^2} \left(g(u_{\lambda_n}) - g(u_0)\right)(u_{\lambda_n} - u_0)\mathrm{d}x \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows from (3.3) that $\{B(u_{\lambda_n})\}$ is bounded. By (3.2) and Lemma 2.4, we obtain

$$I_0(u_0) = \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) = \lim_{n\to\infty} m_{\lambda_n} \geq C_2 > 0. \tag{3.17}$$

Hence, by (3.16) and (3.17), $u_0 \neq 0$ is a weak solution of Eq. (1.6). Similar to (2.19), we obtain that $\|u_{\lambda_n}^\pm\|^2 < 2C_{\varepsilon,p}\int_{\mathbb{R}^2} |u_{\lambda_n}^\pm|^p\mathrm{d}x \leq 2C_{\varepsilon,p}S_p\|u_{\lambda_n}^\pm\|^p$. Then, $(2C_{\varepsilon,p})^{\frac{p}{2-p}}S_p^{\frac{2}{2-p}} < |u_{\lambda_n}^\pm|_p^p$. In addition, since $\{u_{\lambda_n}^\pm\}$ is bounded in $H_r^1(\mathbb{R}^2)$, we obtain

$u_{\lambda_n}^{\pm} \rightharpoonup u_0^{\pm}$ in $H_r^1(\mathbb{R}^2)$ and $u_{\lambda_n}^{\pm} \to u_0^{\pm}$ in $L^p(\mathbb{R}^2)$ for any $p > 6$ up to a subsequence. Then, we have $|u_0^{\pm}|_p^p > 0$, which implies $u_0^{\pm} \neq 0$. Therefore, u_0 is a radial sign-changing solution of Eq. (1.6). It is clear that u_0 changes sign exactly once.

Finally, we prove $I_0(u_0) = m_0$. Repeating the discussion in Sect. 2, we may prove that Eq. (1.1) has a least energy sign-changing radial solution when $\lambda = 0$. That is, there exists $z_0 \in \mathcal{M}_0$ such that $I_0'(z_0) = 0$ and $I_0(z_0) = m_0$. Since $\langle I_0'(z_0), z_0^{\pm}\rangle = 0$, we obtain $\int_{\mathbb{R}^2} g(z_0^{\pm})z_0^{\pm}\mathrm{d}x > \frac{1}{2}\|z_0^{\pm}\|^2$. We claim that $H^{\pm}(u) \leq C_7\|u\|^6$ for all $u \in H_r^1(\mathbb{R}^2)$. In fact, by the Hölder and Sobolev inequalities, for all $u \in H_r^1(\mathbb{R}^2)$, we get

$$\frac{\left(\int_0^{|x|} \frac{r}{2}u^2(r)\mathrm{d}r\right)^2}{|x|^2} = \frac{\left(\int_{B_{|x|}} \frac{u^2(y)}{4\pi}\mathrm{d}y\right)^2}{|x|^2} \leq C_3\|u\|^4 \quad \text{for all } x \in \mathbb{R}^2\backslash\{0\}. \tag{3.18}$$

From (3.18), we deduce $B(u^{\pm}) \leq C_3\|u^{\pm}\|^4 \int_{\mathbb{R}^2} |u^{\pm}|^2\mathrm{d}x \leq C_4\|u\|^6$ for all $u \in H_r^1(\mathbb{R}^2)$. Similarly, we can prove that $B_1(u) \leq C_5\|u\|^6$ and $B_2(u) \leq C_6\|u\|^6$ for all $u \in H_r^1(\mathbb{R}^2)$. Then, by (2.6) and (2.7), $H^{\pm}(u) \leq C_7\|u\|^6$ for all $u \in H_r^1(\mathbb{R}^2)$. Let $\Lambda_1 = \min\left\{(2C_7)^{-1}\|z_0^+\|^2\|z_0\|^{-6}, (2C_7)^{-1}\|z_0^-\|^2\|z_0\|^{-6}\right\}$, for all large n, we get $\int_{\mathbb{R}^2} \lambda_n H^{\pm}(z_0) - g(z_0^{\pm})z_0^{\pm}\mathrm{d}x < \Lambda_1 C_7\|z_0\|^6 - \frac{1}{2}\|z_0^{\pm}\|^2 \leq 0$, which shows $z_0 \in \mathcal{S}_{\lambda_n}$. For every large n, by Lemma 2.3, there exists a unique pair $(s_{\lambda_n}, t_{\lambda_n})$ of positive numbers such that $s_{\lambda_n}z_0^+ + t_{\lambda_n}z_0^- \in \mathcal{M}_{\lambda_n}$. We claim that $\{s_{\lambda_n}\}$ and $\{t_{\lambda_n}\}$ are bounded. If not, without loss of generality, we may assume that $\lim_{n\to\infty} s_{\lambda_n} = +\infty$ up to a subsequence. Using Lemma 2.4, (2.2), (2.6), (2.7) and (3.14), for n large enough, we get

$$\begin{aligned}
0 &< I_{\lambda_n}(s_{\lambda_n}z_0^+ + t_{\lambda_n}z_0^-)\\
&= \frac{s_{\lambda_n}^2}{2}\|z_0^+\|^2 + \frac{s_{\lambda_n}^6}{2}\lambda_n B(z_0^+) - \int_{\mathbb{R}^2} G(s_{\lambda_n}z_0^+)\mathrm{d}x + \frac{s_{\lambda_n}^2 t_{\lambda_n}^4}{2}\lambda_n B_1(z_0)\\
&\quad + \frac{t_{\lambda_n}^2}{2}\|z_0^-\|^2 + \frac{t_{\lambda_n}^6}{2}\lambda_n B(z_0^-) - \int_{\mathbb{R}^2} G(t_{\lambda_n}z_0^-)\mathrm{d}x + \frac{s_{\lambda_n}^4 t_{\lambda_n}^2}{2}\lambda_n B_2(z_0)\\
&\leq \frac{s_{\lambda_n}^2}{2}\|z_0^+\|^2 + \frac{s_{\lambda_n}^6}{6}\left(\lambda_n H^+(z_0) - \int_{\mathbb{R}^2} g(z_0^+)z_0^+\mathrm{d}x\right)\\
&\quad + \frac{t_{\lambda_n}^2}{2}\|z_0^-\|^2 + \frac{t_{\lambda_n}^6}{6}\left(\lambda_n H^-(z_0) - \int_{\mathbb{R}^2} g(z_0^-)z_0^-\mathrm{d}x\right)\\
&\quad + \frac{1}{6}\int_{\mathbb{R}^2} g(z_0^+)z_0^+ - 6G(z_0^+)\mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^2} g(z_0^-)z_0^- - 6G(z_0^-)\mathrm{d}x\\
&< 0,
\end{aligned}$$

a contradiction. Hence, both $\{s_{\lambda_n}\}$ and $\{t_{\lambda_n}\}$ are bounded. Then, up to a subsequence, there exist constants $s', t' \geq 0$ such that $(s_{\lambda_n}, t_{\lambda_n}) \to (s', t')$ as $n \to \infty$. Since $s_{\lambda_n}z_0^+ + t_{\lambda_n}z_0^- \in \mathcal{M}_{\lambda_n}$, we obtain

$$s_{\lambda_n}^2\|z_0^+\|^2 + 3\lambda_n s_{\lambda_n}^6 B(z_0^+) + \lambda_n s_{\lambda_n}^2 t_{\lambda_n}^4 B_1(z_0) + 2\lambda_n s_{\lambda_n}^4 t_{\lambda_n}^2 B_2(z_0)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} g(s_{\lambda_n} z_0^+) s_{\lambda_n} z_0^+ \mathrm{d}x, \\
&t_{\lambda_n}^2 \|z_0^-\|^2 + 3\lambda_n t_{\lambda_n}^6 B(z_0^-) + 2\lambda_n s_{\lambda_n}^2 t_{\lambda_n}^4 B_1(z_0) + \lambda_n s_{\lambda_n}^4 t_{\lambda_n}^2 B_2(z_0) \\
&= \int_{\mathbb{R}^2} g(t_{\lambda_n} z_0^-) t_{\lambda_n} z_0^- \mathrm{d}x.
\end{aligned}
$$

Then, we deduce from (g_4) and $\langle I_0'(z_0), z_0\rangle = 0$ that (s', t') equals $(1, 1)$, $(0, 0)$, $(0, 1)$ or $(1, 0)$. If $(s', t') = (0, 0)$, by (3.17), we have $0 < I_0(u_0) = \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) \le \lim_{n\to\infty} I_{\lambda_n}(s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^-) = 0$, a contradiction. If $(s', t') = (0, 1)$, we get

$$
I_0(z_0) \le I_0(u_0) = \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) \le \lim_{n\to\infty} I_{\lambda_n}(s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^-) = I_0(z_0^-).
$$

Since $I_0(z_0) = I_0(z_0^+) + I_0(z_0^-)$, we know $I_0(z_0^+) \le 0$. Then, using the fact $\langle I_0'(z_0), z_0\rangle = 0$ and Lemma 2.1, we deduce $0 \le \int_{\mathbb{R}^2} g(z_0^+) z_0^+ - 6G(z_0^+)\mathrm{d}x < 0$, a contradiction. Similarly, we can prove $(s', t') \neq (1, 0)$. Hence, $(s', t') = (1, 1)$. Then,

$$
I_0(z_0) \le I_0(u_0) = \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) \le \lim_{n\to\infty} I_{\lambda_n}(s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^-) = I_0(z_0),
$$

which implies that u_0 is a least energy sign-changing radial solution to equation (1.6). Thus, we complete the proof of Theorem 1.2. □

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