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# The Sandpile Group of a Family of Nearly Complete Graphs

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**Abstract**
Let $K_n - C_k$ denote the graph obtained from the complete graph $K_n$ by deleting $k$ edges of a cycle $C_k$, that is, a nearly complete graph. In this paper, we completely determine the structure of the sandpile group of $K_n - C_k$ for $3 \leq k \leq n-2$.

**Keywords** The sandpile group · The nearly complete graph · The Smith normal form

**Mathematics Subject Classification** 05C25 · 05C76

## 1 Introduction

*The sandpile group* is originated from the Abelian Sandpile Model in statistical physics [5]. In fact, the sandpile group pops up in many different fields under different names, such as *the critical group* in the chip-firing game [2–4], *the Picard group* or *the Jacobian group* in the divisor theory of graphs [1], *the group of components* on arithmetical graphs [8], etc.

In [7], Lorenzini studied the structure of the sandpile group of a complete graph with edges of two disjoint paths $P_a$ and $P_b$ deleted. He showed that if $a+b=n-1$ and $gcd(a,b)=1$, then $S(K_n - P_a - P_b)$ is cyclic, and at the same time, he suggested that the sandpile groups of $K_n - C_{n-1}$ and $K_n - C_n$ are never cyclic but with the problems open. Then, in 2011, Norine and Whalen [10] not only showed the other direction of the first result is also true, but settled the case $K_n - C_{n-1}$ by completely

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determining the structure of its sandpile group. Motivated by the above two papers, we shall consider the sandpile group of the graph $K_n - C_k$ for $3 \leq k \leq n-2$ in this paper.

Let $G = (V, E)$ be a finite loopless graph with $n$ vertices. Then, its Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $A(G)$ are the degree matrix and the adjacency matrix of $G$, respectively. Thinking of $L(G)$ as a linear map $\mathbb{Z}^n \to \mathbb{Z}^n$, its *cokernel* has the form

$$coker\, L(G) = \mathbb{Z}^n / L(G)\mathbb{Z}^n = \mathbb{Z} \oplus S(G),$$

where $S(G)$ is the sandpile group on $G$ in the sense of isomorphism.

For $v \in V(G)$, we define the reduced Laplacian $\tilde{L}_v$ as the submatrix of $L(G)$ obtained by omitting the row and column corresponding to $v$. Then, it is well known that for any $v \in V(G)$,

$$S(G) \cong \mathbb{Z}^{n-1} / \tilde{L}_v \mathbb{Z}^{n-1} = coker(\tilde{L}_v).$$

That is, $coker(\tilde{L}_v)$ is independent of the choice of vertex $v$. Thus, we simply write $\tilde{L}_v$ as $\tilde{L}$.

Recall that two matrices $A, B \in \mathbb{Z}^{m \times n}$ are *(unimodular) equivalent* (written $A \sim B$) [9] if there exist $P \in GL(m, \mathbb{Z})$, $Q \in GL(n, \mathbb{Z})$, such that $B = PAQ$. Equivalently, $B$ is obtained from $A$ by a sequence of integer row and column operations that are invertible over the ring $\mathbb{Z}$ of integers. For any square integer matrix $A$, it is equivalent to a unique diagonal matrix $S(A) = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ ( the Smith normal form) whose entries are nonnegative and $s_i$ divides $s_{i+1} (i = 1, \ldots, n-1)$. It can be seen easily that $A \sim B$ implies $coker\, A \cong coker\, B$. So if the Smith normal form of the reduced Laplacian $\tilde{L}$ is $\mathrm{diag}(s_1, s_2, \ldots, s_{n-1})$, where $1 = s_1 = \cdots = s_{r-1} < s_r \leq \cdots \leq s_{n-1}$, then the sandpile group of $G$

$$S(G) = \mathbb{Z}_{s_r} \oplus \mathbb{Z}_{s_{r+1}} \oplus \cdots \oplus \mathbb{Z}_{s_{n-1}},$$

where $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$. $s_r, \ldots, s_n$ are called *invariant factors* of $S(G)$, and $\mu(G) = n - r$ is *the minimum number of generators* of $S(G)$.

The rest of the paper is arranged as follows: In Sect. 2, we give several lemmas which are needed for the main results. In Sect. 3, we determine the explicit structure of the sandpile group of $K_n - C_k$,$(3 \leq k \leq n-2)$. In Sect. 4, as applications of the main results, we deduce more explicit results for the sandpile group of $K_n - C_k$ for $k = 3, 4, 5, 6, 7$ and prime $k$.

## 2 Preliminaries

In this section, we shall first give several lemmas.

**Lemma 2.1** [9] *Suppose that A is an $n \times n$ integer matrix, and its Smith normal form is diag$(s_1, s_2, \ldots, s_n)$. Then,*

$$s_i = \frac{\Delta_i}{\Delta_{i-1}}, \qquad i = 1, 2, \ldots, n,$$

*where $\Delta_i$ (called i-th determinant divisor) is the greatest common divisor of the $i \times i$ minors of A. By convention $\Delta_0 := 1$.*

Now let $a_k(t), b_k(t), c_k(t)$ and $d_k(t)$ be four sequences satisfying the same linear recurrence relation: $y_k(t) = ty_{k-1}(t) - y_{k-2}(t)$ with initial values as follows, respectively,

$$\begin{aligned} a_0(t) &= 1, & a_1(t) &= t;\\ b_0(t) &= 2, & b_1(t) &= t;\\ c_0(t) &= 1, & c_1(t) &= t+1;\\ d_0(t) &= 1, & d_1(t) &= t-1. \end{aligned}$$

Solving the linear recurrence relations, it is easy to obtain that

$$\begin{aligned} a_k(t) &= \frac{1}{\sqrt{t^2-4}}(\lambda_1^{k+1} - \lambda_2^{k+1});\\ b_k(t) &= \lambda_1^k + \lambda_2^k;\\ c_k(t) &= \frac{1}{t-2}(\lambda_1^k(\lambda_1 - 1) + \lambda_2^k(\lambda_2 - 1));\\ d_k(t) &= \frac{1}{t+2}(\lambda_1^k(\lambda_1 + 1) + \lambda_2^k(\lambda_2 + 1)); \end{aligned}$$

where

$$\lambda_1 = \frac{t + \sqrt{t^2-4}}{2}, \quad \lambda_2 = \frac{t - \sqrt{t^2-4}}{2}$$

are two roots of the characteristic equation $x^2 - tx + 1 = 0$.

**Lemma 2.2** *For $a_k(t), b_k(t), c_k(t)$ and $d_k(t)$ as defined above, we have*

(1) *if $k = 2m$, then*

$$\begin{cases} a_{2m}(t) = c_m(t)d_m(t);\\ a_{2m}(t) + 1 = a_m(t)b_m(t). \end{cases}$$

(2) *if $k = 2m - 1$, then*

$$\begin{cases} a_{2m-1}(t) = a_{m-1}(t)b_m(t);\\ a_{2m-1}(t) - 1 = c_{m-1}(t)d_m(t). \end{cases}$$

(3)

$$tb_m(t) - 2b_{m-1}(t) = (t^2-4)a_{m-1}(t);$$
$$tc_m(t) - 2c_{m-1}(t) = (t+2)d_m(t).$$

***Proof*** Using the facts $\lambda_1\lambda_2 = 1$, $\lambda_1 + \lambda_2 = t$, $\lambda_1 - \lambda_2 = \sqrt{t^2-4}$. It is easy to check the identities. □

Note that if $t$ is an integer, then $a_k(t)$, $b_k(t)$, $c_k(t)$ and $d_k(t)$ are integer sequences. Let $gcd(a, b)$ denote the greatest common divisor of $a$ and $b$. Then, we have the following results.

**Lemma 2.3** *Let $t$ be an integer, then for any $m > 1$,*

$$gcd(a_m(t), a_{m-1}(t)) = gcd(c_m(t), c_{m-1}(t)) = gcd(d_m(t), d_{m-1}(t)) = 1$$

*and*

$$gcd(b_m(t), b_{m-1}(t)) = gcd(2, t).$$

***Proof*** We only need to notice that for any sequence $x_n$ satisfying $x_n = tx_{n-1} - x_{n-2}$,

$$gcd(x_n, x_{n-1}) = gcd(x_{n-1}, x_{n-2}) = \cdots = gcd(x_1, x_0).$$

Thus, the results follow directly. □

Using the fact

$$\begin{vmatrix} t & 1 & 0 & \cdots & 0 \\ 1 & t & 1 & \cdots & 0 \\ 0 & 1 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & t \end{vmatrix}_{k\times k} = a_k(t)$$

for $k \geq 1$, we obtain the following result.

**Lemma 2.4** *For $k \geq 3$, let*

$$A_k(t) = \begin{pmatrix} t & 1 & 0 & \cdots & 1 \\ 1 & t & 1 & \cdots & 0 \\ 0 & 1 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & t \end{pmatrix}_{k\times k}$$

*then*

$$\det A_k(t) = \begin{cases} (t^2-4)a_{m-1}^2(t), & \text{if } k = 2m; \\ (t+2)d_m^2(t), & \text{if } k = 2m+1. \end{cases}$$

***Proof*** It is easy to see that

$$\det A_k(t) = ta_{k-1}(t) - 2(a_{k-2}(t) + (-1)^k).$$

(1) If $k = 2m$, then by Lemma 2.2

$$\begin{aligned}\det A_k(t) &= ta_{2m-1}(t) - 2(a_{2m-2}(t) + 1)\\ &= ta_{m-1}(t)b_m(t) - 2a_{m-1}(t)b_{m-1}(t))\\ &= a_{m-1}(t)(tb_m(t) - 2b_{m-1}(t))\\ &= (t^2 - 4)a_{m-1}^2(t).\end{aligned}$$

(2) If $k = 2m + 1$, then by Lemma 2.2 again

$$\begin{aligned}\det A_k(t) &= ta_{2m}(t) - 2(a_{2m-1}(t) - 1)\\ &= tc_m(t)d_m(t) - 2c_{m-1}(t)d_m(t)\\ &= d_m(t)(tc_m(t) - 2c_{m-1}(t))\\ &= (t + 2)d_m^2(t).\end{aligned}$$

Now the proof is completed. □

## 3 The Sandpile Group of $K_n - C_k$

After the preparatory work we have done in the above section, in this section, we shall give the explicit structure of the sandpile group of the graph $G = K_n - C_k$ for $3 \leq k \leq n - 2$. Let $V(G) = \{1, \ldots, n\}$, by symmetry, we may assume that $d_1 = d_2 = \cdots = d_k = n - 3$ and $d_{k+1} = \cdots = d_n = n - 1$. Then, the Laplacian matrix of $G$ has the form

$$L = \begin{pmatrix} n-3 & 0 & -1 & \cdots & 0 & -1 & \cdots & -1\\ 0 & n-3 & 0 & \cdots & -1 & -1 & \cdots & -1\\ -1 & 0 & n-3 & \cdots & -1 & -1 & \cdots & -1\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ 0 & -1 & -1 & \cdots & n-3 & -1 & \cdots & -1\\ -1 & -1 & -1 & \cdots & -1 & n-1 & \cdots & -1\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & n-1 \end{pmatrix}_{n\times n}$$

Deleting the last row and column from $L$ to obtain the reduced Laplacian matrix, $\tilde{L}$ of $G$. For $\tilde{L}$, first subtracting the last row of $\tilde{L}$ from all other rows, and then subtracting the last column from all other columns. We get

$$\tilde{L} \sim \begin{pmatrix} n-2 & 1 & 0 & \cdots & 1 & 0 \cdots 0 & 0 \\ 1 & n-2 & 1 & \cdots & 0 & 0 \cdots 0 & 0 \\ 0 & 1 & n-2 & \cdots & 0 & 0 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & n-2 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \cdots 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots 0 & 1 \end{pmatrix}_{(n-1)\times(n-1)} = \begin{pmatrix} A_k(n-2) & 0 \\ 0 & B \end{pmatrix},$$

where

$$B = \begin{pmatrix} n & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & n & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(n-1-k)\times(n-1-k)}.$$

That is,

$$S(K_n - C_k) \cong (\mathbb{Z}_n)^{n-k-2} \oplus coker(A_k(n-2)). \tag{1}$$

***Remark*** Note that $K_n - C_k$ can be viewed as the join of the graph $K_k - C_k$ with the complete graph $K_{n-k}$. The structure of the sandpile group of this family of graphs had been addressed in [6,11]. (1) is just the same result of Theorem 1 (1) in [6]. In the following, we first determine the explicit structure of $coker(A_k(n-2))$. Then, we determine all invariant factors of $S(K_n - C_k)$.

**Theorem 3.1** *For the Smith normal form of $A_k(n-2)$, we have*

$$A_k(n-2) = \begin{pmatrix} n-2 & 1 & 0 & \cdots & 1 \\ 1 & n-2 & 1 & \cdots & 0 \\ 0 & 1 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & n-2 \end{pmatrix}_{k\times k} \sim \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_1 & 0 \\ 0 & 0 & \cdots & 0 & s_2 \end{pmatrix}_{k\times k}$$

*where*

$$s_1 = \begin{cases} a_{m-1}(n-2)gcd(2,n), & \text{if } k = 2m; \\ d_m(n-2), & \text{if } k = 2m+1. \end{cases}$$

$$s_2 = \begin{cases} \frac{n(n-4)}{gcd^2(2,n)} s_1, & \text{if } k = 2m; \\ n s_1, & \text{if } k = 2m+1. \end{cases}$$

***Proof*** Note that there exists a $(k-2)\times(k-2)$ minor in $A_k(n-2)$ with value 1 (deleting the first and the $k-$th columns, the $k-1$-th and $k$-th rows, respectively). So

$\Delta_1 = \cdots = \Delta_{k-2} = 1$. Also $\Delta_k = \det A_k(n-2)$ is given in Lemma 2.4. The only remain is to compute $\Delta_{k-1}$. Since $A_k(n-2)$ is a circulant matrix, so does its adjoint matrix $adj(A_k(n-2))$. Hence,

$$\Delta_{k-1} = gcd(A_{ij}; i, j = 1, \ldots, k) = gcd(A_{1j}; j = 1, \ldots, k),$$

where $A_{ij}(1 \leq j \leq k)$ is the algebraic cofactor of $ij-$element of $A_k(n-2)$. For simplicity, let $(x_1, x_2, \ldots, x_k) = (A_{11}, A_{12}, \ldots, A_{1k})$. Note that $x_i = x_{k+2-i}(i = 2, 3, \ldots, k)$ since $A_k(n-2)$ is a symmetric circulant matrix. From

$$\begin{pmatrix} n-2 & 1 & 0 & \cdots & 1 \\ 1 & n-2 & 1 & \cdots & 0 \\ 0 & 1 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & n-2 \end{pmatrix}_{k\times k} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} \det A_k(n-2) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we obtain that

$$\begin{aligned} &\det A_k(n-2) = (n-2)x_1 + 2x_2 \\ &x_1 + (n-2)x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - (n-2)x_2 \\ &x_2 + (n-2)x_3 + x_4 = 0 \Rightarrow x_4 = -x_2 - (n-2)x_3 \\ &\vdots \\ &x_{k-2} + (n-2)x_{k-1} + x_k = 0 \Rightarrow x_k = -x_{k-2} - (n-2)x_{k-1}. \end{aligned}$$

That is, $x_3, \ldots, x_k$ can be represented linearly by $x_1, x_2$.

So

$$\begin{aligned} \Delta_{k-1} &= gcd(x_1, x_2, \ldots, x_k) = gcd(x_1, x_2). \\ x_1 = A_{11} &= \begin{vmatrix} n-2 & 1 & \cdots & 0 \\ 1 & n-2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n-2 \end{vmatrix}_{(k-1)\times(k-1)} = a_{k-1}(n-2) \\ x_2 = A_{12} = A_{1k} &= (-1)^{k+1} \cdot \begin{vmatrix} 1 & n-2 & 1 & \cdots & 0 \\ 0 & 1 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}_{(k-1)\times((k-1)} \\ &= (-1)^{k+1} \cdot (1 + (-1)^k a_{k-2}(n-2)) \\ &= (-1)^{k+1} - a_{k-2}(n-2), \end{aligned}$$

So by Lemma 2.1,

$$\begin{cases} s_1 = \Delta_{k-1} = gcd(a_{k-1}(n-2), a_{k-2}(n-2) + (-1)^k); \\ s_2 = \dfrac{\det A_k(n-2)}{s_1}. \end{cases}$$

(1) If $k = 2m$, then by Lemmas 2.2, 2.3 and 2.4

$$\begin{aligned} s_1 &= gcd(a_{2m-1}(n-2), a_{2m-2}(n-2) + 1) \\ &= gcd(a_{m-1}(n-2)b_m(n-2), a_{m-1}(n-2)b_{m-1}(n-2)) \\ &= a_{m-1}(n-2)gcd(b_m(n-2), b_{m-1}(n-2)) \\ &= a_{m-1}(n-2)gcd(2, n-2) \\ &= a_{m-1}(n-2)gcd(2, n), \\ s_2 &= \frac{(n^2-4n)a_{m-1}^2(n-2)}{a_{m-1}(n-2)gcd(2,n)} \\ &= \frac{n(n-4)}{gcd^2(2,n)} s_1. \end{aligned}$$

(2) If $k = 2m + 1$, then by Lemmas 2.2, 2.3 and 2.4

$$\begin{aligned} s_1 &= gcd(a_{2m}(n-2), a_{2m-1}(n-2) - 1) \\ &= gcd(c_m(n-2)d_m(n-2), c_{m-1}(n-2)d_m(n-2)) \\ &= d_m(n-2)gcd(c_m(n-2), c_{m-1}(n-2)) \\ &= d_m(n-2), \\ s_2 &= \frac{nd_m^2(n-2)}{d_m(n-2)} \\ &= ns_1. \end{aligned}$$

□

In order to give all invariant factors of $S(K_n - C_k)$, we need the following result about $a_k(n-2)$.

**Lemma 3.2**

$$a_k(n-2) = nf_k(n) + (-1)^k(k+1),$$

*where $f_k(n)$'s are all integers.*

***Proof*** We prove the result by induction on $k$.

First $a_0(n-2) = 1$, $a_1(n-2) = n-2$, so $f_0(n) = 0$, $f_1(n) = 1$ are integers.

Now suppose $a_{k-1}(n-2) = nf_{k-1}(n) + (-1)^{k-1}k$, $a_{k-2}(n-2) = nf_{k-2}(n) + (-1)^{k-2}(k-1)$, where $f_{k-1}(n)$ and $f_{k-2}(n)$ are integers. Then, by $a_k(n-2) =$

$(n-2)a_{k-1}(n-2) - a_{k-2}(n-2)$,

$$\begin{aligned} a_k(n-2) &= (n-2)(nf_{k-1}(n) + (-1)^{k-1}k) - (nf_{k-2}(n) + (-1)^{k-2}(k-1)) \\ &= n((n-2)f_{k-1}(n) - f_{k-2}(n) + (-1)^{k-1}k) + (-1)^k 2k - (-1)^k(k-1) \\ &= n((n-2)f_{k-1}(n) - f_{k-2}(n) + (-1)^{k-1}k) + (-1)^k(k+1) \\ &= nf_k(n) + (-1)^k(k+1), \end{aligned}$$

where $f_k(n) = (n-2)f_{k-1}(n) - f_{k-2}(n) + (-1)^{k-1}k$ is an integer since $f_{k-1}(n)$ and $f_{k-2}(n)$ are integers. This completes the proof. □

Now we are ready to give the main result.

**Theorem 3.3** *For the nearly complete graph $G = K_n - C_k$ $(3 \le k \le n-2)$, we have*

(1) *if $k = n-2$, then*

$$S(G) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2};$$

(2) *if $3 \le k \le n-3$, then*

$$S(G) \cong \begin{cases} \underbrace{\mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n}_{n-3-k} \oplus \mathbb{Z}_{ns_1} \oplus \mathbb{Z}_{s_2}, & \text{if } \gcd(n,k) = 1; \\ \mathbb{Z}_i \oplus \underbrace{\mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n}_{n-3-k} \oplus \mathbb{Z}_{\frac{n}{i}\cdot s_1} \oplus \mathbb{Z}_{s_2}, & \text{if } \gcd(n,k) = i, (1 < i \le k); \end{cases}$$

*where*

$$s_1 = \begin{cases} a_{m-1}(n-2)gcd(2,n), & \text{if } k = 2m; \\ d_m(n-2), & \text{if } k = 2m+1; \end{cases}$$

*and*

$$s_2 = \begin{cases} \frac{n(n-4)}{gcd^2(2,n)} s_1, & \text{if } k = 2m; \\ ns_1, & \text{if } k = 2m+1. \end{cases}$$

***Proof*** By Theorem 3.1, we have

$$\tilde{L} \sim \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & s_1 & & & \\ & & & & s_2 & & \\ & & & & & n & \\ & & & & & & \ddots \\ & & & & & & & n \end{pmatrix}_{(n-1)\times(n-1)}.$$

Obviously, $S(G) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2}$ for $k = n - 2$.

For $3 \le k \le n - 3$, let

$$A = \begin{pmatrix} s_1 & & & & \\ & s_2 & & & \\ & & n & & \\ & & & \ddots & \\ & & & & n \end{pmatrix}_{(n-k)\times(n-k)}.$$

Note that both $s_1$ and $n$ divide $s_2$, so it is not difficult to get that

$$\begin{aligned} &\Delta_i(A) = n^{i-1} gcd(s_1, n), \qquad i = 1, \ldots, n-k-2; \\ &\Delta_{n-k-1}(A) = n^{n-k-2} s_1, \qquad \Delta_{n-k}(A) = n^{n-k-2} s_1 s_2. \end{aligned}$$

Now we shall show that $gcd(s_1, n) = gcd(n, k)$. First, for $k = 2m$, by Lemma 3.2

$$\begin{aligned} s_1 &= a_{m-1}(n-2) gcd(2, n) \\ &= (n f_{m-1}(n) + (-1)^{m-1} m) gcd(2, n) \\ &= n f_{m-1}(n) gcd(2, n) + (-1)^{m-1} gcd(2, n) m, \end{aligned}$$

so $gcd(s_1, n) = gcd(gcd(2, n)m, n) = gcd(2m, n) = gcd(n, k)$. Then, for $k = 2m + 1$, by Lemma 3.2 again

$$\begin{aligned} s_1 &= d_m(n-2) \\ &= a_m(n-2) - a_{m-1}(n-2) \\ &= n f_m(n) + (-1)^m (m+1) - (n f_{m-1}(n) + (-1)^{m-1} m) \\ &= n(f_m(n) - f_{m-1}(n)) + (-1)^m (2m+1), \end{aligned}$$

so $gcd(s_1, n) = gcd(2m + 1, n) = gcd(n, k)$. That is, for any case, $gcd(s_1, n) = gcd(n, k)$.

So by Lemma 2.1, we have

$$A \sim \mathrm{diag}\left( gcd(n, k), n, \ldots, n, \frac{n}{gcd(n, k)} \cdot s_1, s_2 \right).$$

Now the results about $S(G)$ follow directly. □

Recall that the minimum number of generators of the sandpile group $S(G)$ is denoted by $\mu(G)$. From the above theorem, we immediately derive the following corollary.

**Corollary 3.4** *Let $G = K_n - C_k$ $(3 \le k \le n - 2)$, we have*

(1) *if $k = n - 2$, then $\mu(G) = 2$;*

(2) *if* $3 \le k \le n-3$, *then*

$$\mu(G) = \begin{cases} n-k-1, & \text{if } \gcd(n,k)=1; \\ n-k, & \text{if } \gcd(n,k)=i, (1 < i \le k). \end{cases}$$

***Remark*** It is worthy of pointing out that our method can be used to determine the sandpile group of any graph as $K_n - \cup_{i=1}^{t} C_{k_i}$, where $C_{k_i}$ are mutual disjoint cycles and satisfy $3 \le k_1 + k_2 + \cdots + k_t \le n-2$. It is also not difficult to deduce that the sandpile groups of these graphs are never cyclic.

## 4 More Explicit Results for Some Special *k*

In this section, as applications of Theorem 3.3, we give more explicit results for the sandpile group of $K_n - C_k$ for $k = 3, 4, 5, 6, 7$ and prime $k$.

**Theorem 4.1** *For the nearly complete graph* $G = K_n - C_3$, *we have*

(1) *if* $G = K_5 - C_3$, *then*

$$S(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{10};$$

(2) *if* $n \ge 6$, *then*

$$S(G) \cong \begin{cases} \mathbb{Z}_3 \oplus (\mathbb{Z}_n)^{n-6} \oplus \mathbb{Z}_{p(n-3)} \oplus \mathbb{Z}_{n(n-3)}, & \text{if } n = 3p; \\ (\mathbb{Z}_n)^{n-6} \oplus (\mathbb{Z}_{n(n-3)})^2, & \text{if } n = 3p+1 \text{ or } 3p+2. \end{cases}$$

**Theorem 4.2** *For the nearly complete graph* $G = K_n - C_4$, *we have*

(1) *if* $G = K_6 - C_4$, *then*

$$S(G) \cong \mathbb{Z}_8 \oplus \mathbb{Z}_{24};$$

(2) *if* $n \ge 7$, *then*

$$S(G) \cong \begin{cases} \mathbb{Z}_4 \oplus (\mathbb{Z}_n)^{n-7} \oplus \mathbb{Z}_{2p(n-2)} \oplus \mathbb{Z}_{2p(n-2)(n-4)}, & \text{if } n = 4p; \\ \mathbb{Z}_2 \oplus (\mathbb{Z}_n)^{n-7} \oplus \mathbb{Z}_{n(n-2)} \oplus \mathbb{Z}_{(2p+1)(n-2)(n-4)}, & \text{if } n = 4p+2; \\ (\mathbb{Z}_n)^{n-7} \oplus \mathbb{Z}_{n(n-2)} \oplus \mathbb{Z}_{n(n-2)(n-4)}, & \text{if } n = 4p+1 \text{ or } 4p+3. \end{cases}$$

**Theorem 4.3** *For the nearly complete graph* $G = K_n - C_5$, *we have*

(1) *if* $G = K_7 - C_5$, *then*

$$S(G) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{133};$$

(2) *if* $n \ge 8$, *then*

$$S(G) \cong \begin{cases} \mathbb{Z}_5 \oplus (\mathbb{Z}_n)^{n-8} \oplus \mathbb{Z}_{p(n-2)(n-3)-p} \oplus \mathbb{Z}_{n(n-2)(n-3)-n}, & \text{if } n = 5p; \\ (\mathbb{Z}_n)^{n-8} \oplus (\mathbb{Z}_{n(n-2)(n-3)-n})^2, & \text{otherwise.} \end{cases}$$

**Theorem 4.4** *For the nearly complete graph $G = K_n - C_6$, we have*

(1) *if $G = K_8 - C_6$, then*

$$S(G) \cong \mathbb{Z}_{70} \oplus \mathbb{Z}_{560};$$

(2) *if $n \geq 9$, then*

$$S(G) \cong \begin{cases} \mathbb{Z}_6 \oplus (\mathbb{Z}_n)^{n-9} \oplus \mathbb{Z}_{2p(n-1)(n-3)} \oplus \mathbb{Z}_{3p(n-1)(n-3)(n-4)}, & \text{if } n = 6p; \\ \mathbb{Z}_2 \oplus (\mathbb{Z}_n)^{n-9} \oplus \mathbb{Z}_{n(n-1)(n-3)} \oplus \mathbb{Z}_{(3p+1)(n-1)(n-3)(n-4)}, & \text{if } n = 6p+2; \\ \mathbb{Z}_3 \oplus (\mathbb{Z}_n)^{n-9} \oplus \mathbb{Z}_{(2p+1)(n-1)(n-3)} \oplus \mathbb{Z}_{n(n-1)(n-3)(n-4)}, & \text{if } n = 6p+3; \\ (\mathbb{Z}_n)^{n-9} \oplus \mathbb{Z}_{n(n-1)(n-3)} \oplus \mathbb{Z}_{n(n-1)(n-3)(n-4)}, & \text{if } n = 6p+1 \text{ } or \text{ } 6p+5. \end{cases}$$

**Theorem 4.5** *For the nearly complete graph $G = K_n - C_7$, we have*

(1) *if $G = K_9 - C_7$, then*

$$S(G) \cong \mathbb{Z}_{281} \oplus \mathbb{Z}_{2529};$$

(2) *if $n \geq 10$, then*

$$S(G) \cong \begin{cases} \mathbb{Z}_7 \oplus (\mathbb{Z}_n)^{n-10} \oplus \mathbb{Z}_{p(n-1)(n-2)(n-4)+p} \oplus \mathbb{Z}_{n(n-1)(n-2)(n-4)+n}, & \text{if } n = 7p; \\ (\mathbb{Z}_n)^{n-10} \oplus (\mathbb{Z}_{n(n-1)(n-2)(n-4)+n})^2, & \text{otherwise.} \end{cases}$$

In fact, for any odd prime $k = 2m + 1$, we have the following result.

**Theorem 4.6** *For the nearly complete graph $G = K_n - C_k$. If $k = 2m + 1$ is a prime, then*

(1) *if $G = K_{k+2} - C_k$, then*

$$S(G) \cong \mathbb{Z}_{d_m(k)} \oplus \mathbb{Z}_{(k+2)d_m(k)};$$

(2) *if $n \geq k + 3$, then*

$$S(G) \cong \begin{cases} \mathbb{Z}_k \oplus (\mathbb{Z}_n)^{n-k-3} \oplus \mathbb{Z}_{pd_m(n-2)} \oplus \mathbb{Z}_{nd_m(n-2)}, & \text{if } n = kp; \\ (\mathbb{Z}_n)^{n-k-3} \oplus (\mathbb{Z}_{nd_m(n-2)})^2, & \text{otherwise.} \end{cases}$$

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