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Refinements of Triangle-Like Inequalities in Lie's Framework

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Abstract

In this note, by using gradients of Gateaux differentiable G-increasing functions we prove some refinements of the following Tam's triangle-type inequality:

$$\mathfrak{t}_+(x+y) \le \mathfrak{t}_+(x) + \mathfrak{t}_+(y) \quad \text{for } x, y \in \mathfrak{g}$$

in the context of a compact connected Lie group G with Lie algebra $\mathfrak{g}$ and corresponding Weyl chamber $\mathfrak{t}_+$. We also establish refinements of Tam's inequality:

$$\mathfrak{a}_+(x+y) \le \mathfrak{a}_+(x) + \mathfrak{a}_+(y) \quad \text{for } x, y \in \mathfrak{p}$$

for a real semisimple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and its closed Weyl chamber $\mathfrak{a}_+$.

Keywords Compact connected Lie group · Semisimple Lie algebra · Triangle-like inequality

Mathematics Subject Classification 22E30 · 22E60

1 Motivation

We begin our presentation with notation and terminology quoted from [6].

Let G be a compact connected Lie group and $\mathfrak{g}$ be its Lie algebra. Assume T is a maximal torus of G and $\mathfrak{t}$ is the Lie algebra of T. By $\mathfrak{t}_+$, we denote a closed Weyl

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chamber in $\mathfrak{t}$. For a given element $x \in \mathfrak{g}$, the symbol $\mathfrak{t}_+(x)$ represents the unique element of the set $\mathfrak{t}_+ \cap Gx$, where $Gx = \{g \cdot x : g \in G\}$ and $g \cdot x = (\operatorname{Ad} g)x$.

Let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on $\mathfrak{g}$. The dual cone of $\mathfrak{t}_+$ is given by $\mathfrak{t}_+^* = \{x \in \mathfrak{g} : \langle x, v \rangle \geq 0 \text{ for all } v \in \mathfrak{t}_+\}$. This cone generates the preorder $\leq$ on $\mathfrak{g}$ by $y \leq x$ iff $x - y \in \mathfrak{t}_+^*$ for $x, y \in \mathfrak{g}$. In addition, a related preorder $\prec$ can be defined by $y \prec x$ iff $y \in \operatorname{conv} Gx$ for $x, y \in \mathfrak{g}$, where $\operatorname{conv} Gx$ is the convex hull of the G-orbit $\{g \cdot x : g \in G\}$ (cf. [3, Corollary B.3]). It is known that $\leq$ and $\prec$ coincide on $\mathfrak{t}_+$ [1, Proposition 18].

In [6, Theorem 7] T.-Y. Tam presented a triangle-type inequality for connected compact groups, as follows

$$\mathfrak{t}_+(x + y) \leq \mathfrak{t}_+(x) + \mathfrak{t}_+(y) \quad \text{for } x, y \in \mathfrak{g}, \tag{1}$$

where $\mathfrak{t}_+(z)$ denotes the unique element in $\mathfrak{t}_+ \cap Gz$ corresponding to an element $z \in \mathfrak{g}$.

A similar framework works in the context of real semisimple Lie algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p} \neq 0$. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$ and $\mathfrak{a}_+$ be a closed Weyl chamber in $\mathfrak{a}$.

The Killing form $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{p}$, so we can define the dual cone of $\mathfrak{a}_+$ by $\mathfrak{a}_+^* = \{x \in \mathfrak{p} : \langle x, v \rangle \geq 0 \text{ for all } v \in \mathfrak{p}\}$. Then we introduce the preorder $\leq$ on $\mathfrak{p}$ by $y \leq x$ iff $x - y \in \mathfrak{a}_+^*$ for $x, y \in \mathfrak{p}$.

Let K be the maximal compact subgroup in the adjoint group $\operatorname{Int}(\mathfrak{g})$. So, $\operatorname{Ad} K$ is maximal compact subgroup of $\operatorname{Ad} G$. We define preorder $\prec$ in the following manner: $y \prec x$ iff $y \in \operatorname{conv} Kx$ for $x, y \in \mathfrak{p}$, where $\operatorname{conv} Kx$ is the convex hull of the K-orbit $Kx = \{k \cdot x : k \in K\}$ with $k \cdot x = (\operatorname{Ad} k)x$ (cf. [3, Corollary B.3]). The preorders $\leq$ and $\prec$ coincide on $\mathfrak{a}_+$ [2, Lemma 3.2]. For an element $x \in \mathfrak{p}$, the symbol $\mathfrak{a}_+(x)$ represents the unique element of the set $\mathfrak{a}_+ \cap Kx$, where $Kx = \{k \cdot x : k \in K\}$ and $k \cdot x = (\operatorname{Ad} k)x$.

In [6, Theorem 2] T.-Y. Tam showed that

$$\mathfrak{a}_+(x + y) \leq \mathfrak{a}_+(x) + \mathfrak{a}_+(y) \quad \text{for } x, y \in \mathfrak{p}. \tag{2}$$

In this note, our purpose is to show refinements of inequalities (1) and (2) by employing gradients of differentiable real functions increasing with respect to corresponding preorder $\prec$ (cf. [6]).

2 Results for Compact Connected Lie Groups

In this section, we consider the Lie algebra $\mathfrak{g}$ of a compact connected Lie group G. We use the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ on $\mathfrak{g}$ generated by a G-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ by $\operatorname{Ad} G$. The preorder $\prec$ is defined in Sect. 1.

A function Φ defined on $\mathfrak{g}$ is called *G-invariant* if

$$\Phi(g \cdot x) = \Phi(x) \quad \text{for all } x \in \mathfrak{g} \quad \text{and} \quad g \in G.$$

We say that a function $\Phi : \mathfrak{g} \to \mathbb{R}$ is *G-increasing*, if for $x, y \in \mathfrak{g}$,

$$y \prec x \quad \text{implies} \quad \Phi(y) \le \Phi(x).$$

In order to state our results, we need to employ the convex cone $\mathcal{C}$ of all G-increasing real functions defined on $\mathfrak{g}$ as well as the cone preorder $\le_{\mathcal{C}}$ generated by $\mathcal{C}$, as follows: given two real functions $\varphi : \mathfrak{g} \to \mathbb{R}$ and $\psi : \mathfrak{g} \to \mathbb{R}$, we use the notation $\psi \le_{\mathcal{C}} \varphi$ whenever the difference function $\varphi - \psi$ is G-increasing on $\mathfrak{g}$.

Given two vectors $x, y \in \mathfrak{g}$, if there exists a $g \in G$ such that $x \in g\,\mathfrak{t}_+$ and $y \in g\,\mathfrak{t}_+$, then

$$\mathfrak{t}_+(x + y) = \mathfrak{t}_+(x) + \mathfrak{t}_+(y). \tag{3}$$

In this note, we take the convention that the Gateaux differentiability of a function $\Phi : \mathfrak{g} \to \mathbb{R}$ means the existence of the directional derivative

$$\nabla_h \Phi(y) = \lim_{t \to 0} \frac{\Phi(y + th) - \Phi(y)}{t} \tag{4}$$

at each point $y \in \mathfrak{g}$ and in each direction $h \in \mathfrak{g}$, and moreover that the map $\mathfrak{g} \ni h \to \nabla_h \Phi(y) \in \mathbb{R}$ is continuous and linear as a function of h. Consequently, there exists the gradient $\nabla \Phi(y) \in \mathfrak{g}$ satisfying the condition

$$\nabla_h \Phi(y) = \langle \nabla \Phi(y), h \rangle \quad \text{for all} \quad h \in \mathfrak{g}. \tag{5}$$

In accordance with [4, Theorem 2.1], a Gateaux differentiable G-increasing function $\Phi : \mathfrak{g} \to \mathbb{R}$ with continuous gradient $\nabla \Phi(\cdot)$ satisfies the condition

$$\nabla \Phi(g\,\mathfrak{t}_+(y)) \in g\,\mathfrak{t}_+ \quad \text{for all } g \in G \text{ and } y \in \mathfrak{g}. \tag{6}$$

Theorem 1 *Let φ and ψ be Gateaux differentiable real functions on $\mathfrak{g}$ with continuous gradients $\nabla\varphi(\cdot)$ and $\nabla\psi(\cdot)$, respectively. Suppose that*

$$0 \le_{\mathcal{C}} \psi \le_{\mathcal{C}} \varphi \le_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2, \tag{7}$$

i.e., the functions ψ, $\varphi - \psi$ and $\frac{1}{2}\|\cdot\|^2 - \varphi$ are G-increasing on $\mathfrak{g}$.

If $x, y \in \mathfrak{g}$ then

$$\mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)) \prec \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(y - \nabla\psi(y)). \tag{8}$$

Remark 1 Observe that for any points $x, y \in \mathfrak{g}$, statement (8) in Theorem 1 shows the *anti-isotonity* of the functional

$$\phi \to \mathfrak{t}_+(x + \nabla\phi(y)) + \mathfrak{t}_+(y - \nabla\phi(y))$$

with ϕ running over the set of all Gateaux differentiable real functions defined on $\mathfrak{g}$. Here the anti-isotonity is with respect to the pair $(\leq_{\mathcal{C}}, \prec)$ on the set

$$\left\{\phi : \mathfrak{g} \to \mathbb{R} : 0 \leq_{\mathcal{C}} \phi \leq_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2\right\}.$$

Proof By making use of (1), we obtain

$$\begin{aligned} \mathfrak{t}_+(x + \nabla\varphi(y)) &= \mathfrak{t}_+(x + \nabla\psi(y) + \nabla\varphi(y) - \nabla\psi(y)) \\ &\prec \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(\nabla\varphi(y) - \nabla\psi(y)). \end{aligned} \tag{9}$$

The preorder $\prec$ restricted to $\mathfrak{t}_+$ is a cone preorder on $\mathfrak{t}_+$, so we have

$$\mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)) \prec \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(\nabla\varphi(y) - \nabla\psi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)). \tag{10}$$

For the element $y \in \mathfrak{g}$, there exists a $g \in G$ such that $y = g \cdot \mathfrak{t}_+(y) \in g\,\mathfrak{t}_+$. Because of the G-increase in the functions ψ and $\varphi - \psi$, it follows from (6) that

$$\begin{aligned} &\nabla\psi(y) \in g\,\mathfrak{t}_+, \\ &\nabla\varphi(y) - \nabla\psi(y) = \nabla\left(\varphi - \psi\right)(y) \in g\,\mathfrak{t}_+ \end{aligned}$$

[see (6)]. This via (3) ensures that

$$\mathfrak{t}_+(\nabla\psi(y)) + \mathfrak{t}_+(\nabla\varphi(y) - \nabla\psi(y)) = \mathfrak{t}_+(\nabla\psi(y) + \nabla\varphi(y) - \nabla\psi(y)) = \mathfrak{t}_+(\nabla\varphi(y)).$$

In consequence, we get

$$\mathfrak{t}_+(\nabla\varphi(y) - \nabla\psi(y)) = \mathfrak{t}_+(\nabla\varphi(y)) - \mathfrak{t}_+(\nabla\psi(y)). \tag{11}$$

Since

$$\nabla\left(\frac{1}{2}\|\cdot\|^2\right)(y) = y$$

and the functions $\varphi = \psi + (\varphi - \psi)$ and $\nabla\frac{1}{2}\|\cdot\|^2 - \varphi$ are G-increasing, we deduce from (6) that

$$\begin{aligned} &\nabla\varphi(y) \in g\,\mathfrak{t}_+, \\ &y - \nabla\varphi(y) = \nabla\left(\frac{1}{2}\|\cdot\|^2\right)(y) - \nabla\varphi(y) = \nabla\left(\frac{1}{2}\|\cdot\|^2 - \varphi\right)(y) \in g\,\mathfrak{t}_+. \end{aligned}$$

and therefore by (3) we have

$$\mathfrak{t}_+(\nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)) = \mathfrak{t}_+(\nabla\varphi(y) + y - \nabla\psi(y)) = \mathfrak{t}_+(y),$$

and next

$$\mathfrak{t}_+(y - \nabla\varphi(y)) = \mathfrak{t}_+(y) - \mathfrak{t}_+(\nabla\varphi(y)). \tag{12}$$

Similarly, the function

$$\frac{1}{2}\|\cdot\|^2 - \psi = \left(\frac{1}{2}\|\cdot\|^2 - \varphi\right) + (\varphi - \psi)$$

is G-increasing, because ψ, $\varphi - \psi$ and $\frac{1}{2}\|\cdot\|^2 - \varphi$ are so. Hence, via (6) we have

$$\begin{aligned}
&\nabla\psi(y) \in g\,\mathfrak{t}_+,\\
&y - \nabla\psi(y) = \nabla\left(\frac{1}{2}\|\cdot\|^2\right)(y) - \nabla\psi(y) = \nabla\left(\frac{1}{2}\|\cdot\|^2 - \psi\right)(y) \in g\,\mathfrak{t}_+.
\end{aligned}$$

This and (3) yield

$$\mathfrak{t}_+(y - \nabla\psi(y)) = \mathfrak{t}_+(y) - \mathfrak{t}_+(\nabla\psi(y)). \tag{13}$$

Finally, by (11)–(13) and (10) we obtain

$$\begin{aligned}
&\mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y))\\
&\prec \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(\nabla\varphi(y)) - \mathfrak{t}_+(\nabla\psi(y)) + \mathfrak{t}_+(y) - \mathfrak{t}_+(\nabla\varphi(y))\\
&\quad = \mathfrak{t}_+(x + \nabla\psi(y)) - \mathfrak{t}_+(\nabla\psi(y)) + \mathfrak{t}_+(y) = \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(y - \nabla\psi(y)),
\end{aligned}$$

as claimed. □

As a corollary to Theorem 1, we now present a refinement of the triangle-type inequality (1).

Theorem 2 *Let φ be a Gateaux differentiable real function on $\mathfrak{g}$ with continuous gradient $\nabla\varphi(\cdot)$. Suppose that*

$$0 \le_{\mathcal{C}} \varphi \le_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2, \tag{14}$$

i.e., the functions φ and $\frac{1}{2}\|\cdot\|^2 - \varphi$ are G-increasing on $\mathfrak{g}$.

If $x, y \in \mathfrak{g}$ then

$$\mathfrak{t}_+(x + y) \prec \mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)) \prec \mathfrak{t}_+(x) + \mathfrak{t}_+(y). \tag{15}$$

Proof It follows from (14) that

$$0 \le_{\mathcal{C}} \psi \le_{\mathcal{C}} \varphi \le_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2$$

with $\psi = 0$. By applying Theorem 1 with $\nabla\psi(y) = 0$, we obtain

$$\begin{aligned}
&\mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)) \prec \mathfrak{t}_+(x + \nabla\psi(y)) + \mathfrak{t}_+(y - \nabla\psi(y))\\
&\quad = \mathfrak{t}_+(x) + \mathfrak{t}_+(y).
\end{aligned}$$

By virtue of (14), we get

$$0 \leq_{\mathcal{C}} \varphi \leq_{\mathcal{C}} \psi \leq_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2$$

with $\psi = \frac{1}{2}\|\cdot\|^2$. By using Theorem 1 with $\nabla\psi(y) = y$, we establish

$$\begin{aligned} \mathfrak{t}_+(x+y) = \mathfrak{t}_+(x + \nabla\psi(y)) \\ +\mathfrak{t}_+(y - \nabla\psi(y)) \prec \mathfrak{t}_+(x + \nabla\varphi(y)) + \mathfrak{t}_+(y - \nabla\varphi(y)). \end{aligned}$$

This completes the proof. □

Corollary 1 *Let* $0 \leq t \leq 1$. *If* $x, y \in \mathfrak{g}$ *then*

$$\mathfrak{t}_+(x+y) \prec \mathfrak{t}_+(x+ty) + \mathfrak{t}_+(y-ty) \prec \mathfrak{t}_+(x) + \mathfrak{t}_+(y). \tag{16}$$

Proof The function $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ is G-increasing on $\mathfrak{g}$, because it is convex and G-invariant. Therefore, the functions $\frac{1}{2}\|\cdot\|^2$, $\varphi = t\frac{1}{2}\|\cdot\|^2$ and $\frac{1}{2}\|\cdot\|^2 - \varphi = (1-t)\frac{1}{2}\|\cdot\|^2$ are G-increasing on $\mathfrak{g}$. That is,

$$0 \leq_{\mathcal{C}} \varphi \leq_{\mathcal{C}} \frac{1}{2}\|\cdot\|^2.$$

So, by Theorem 2 with $\nabla\varphi(y) = ty$, we infer that inequality (16) holds valid. □

3 Results for Real Semisimple Lie Algebras

In this section, we show corresponding results to those in Sect. 2 for a real semisimple Lie algebra $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p} \neq 0$, and with a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ equipped with a fixed closed Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$. The inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ is the restriction of Killing form, and as previously the norm is given by $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. The preorder $\prec$ is defined in Sect. 1.

The maximal compact subgroup in the adjoint group Int $(\mathfrak{g})$ is denoted by K. It acts on $\mathfrak{p}$ by Ad K, i.e., $k \cdot x = (\mathrm{Ad}\, k)x$ for $k \in K$ and $x \in \mathfrak{p}$.

A function Φ defined on $\mathfrak{p}$ is called *K-invariant* if

$$\Phi(k \cdot x) = \Phi(x) \quad \text{for all } x \in \mathfrak{p} \quad \text{and} \quad k \in K.$$

We say that a function $\Phi : \mathfrak{p} \to \mathbb{R}$ is *K-increasing*, if for $x, y \in \mathfrak{p}$,

$$y \prec x \quad \text{implies} \quad \Phi(y) \leq \Phi(x).$$

Here, by definition, the convex cone $\mathcal{C}$ consists of all K-increasing real functions defined on $\mathfrak{p}$. The cone preorder $\leq_{\mathcal{C}}$ induced by $\mathcal{C}$ is defined as follows: for any two real functions $\varphi : \mathfrak{p} \to \mathbb{R}$ and $\psi : \mathfrak{p} \to \mathbb{R}$, we write $\psi \le_C \varphi$ provided that the difference function $\varphi - \psi$ is K-increasing on $\mathfrak{p}$.

Theorem 3 *Let φ and ψ be Gateaux differentiable real functions on $\mathfrak{p}$ with continuous gradients $\nabla\varphi(\cdot)$ and $\nabla\psi(\cdot)$, respectively. Suppose that*

$$0 \le_C \psi \le_C \varphi \le_C \frac{1}{2}\|\cdot\|^2, \tag{17}$$

i.e., the functions ψ, $\varphi - \psi$ and $\frac{1}{2}\|\cdot\|^2 - \varphi$ are K-increasing on $\mathfrak{p}$.
If $x, y \in \mathfrak{p}$ then

$$\mathfrak{a}_+(x + \nabla\varphi(y)) + \mathfrak{a}_+(y - \nabla\varphi(y)) \prec \mathfrak{a}_+(x + \nabla\psi(y)) + \mathfrak{a}_+(y - \nabla\psi(y)). \tag{18}$$

Proof The proof of Theorem 3 is similar to that of Theorem 1, and therefore omitted. □

An analog of Theorem 2 is the following.

Theorem 4 *Let φ be a Gateaux differentiable real function on $\mathfrak{p}$ with continuous gradient $\nabla\varphi(\cdot)$. Suppose that*

$$0 \le_C \varphi \le_C \frac{1}{2}\|\cdot\|^2, \tag{19}$$

i.e., the functions φ and $\frac{1}{2}\|\cdot\|^2 - \varphi$ are K-increasing on $\mathfrak{p}$.
If $x, y \in \mathfrak{p}$ then

$$\mathfrak{a}_+(x + y) \prec \mathfrak{a}_+(x + \nabla\varphi(y)) + \mathfrak{a}_+(y - \nabla\varphi(y)) \prec \mathfrak{a}_+(x) + \mathfrak{a}_+(y). \tag{20}$$

Proof Use a similar method as in the proof of Theorem 2. □

Finally, we present an analog of Corollary 1.

Corollary 2 *Let $0 \le t \le 1$. If $x, y \in \mathfrak{p}$ then*

$$\mathfrak{a}_+(x + y) \prec \mathfrak{a}_+(x + ty) + \mathfrak{a}_+(y - ty) \prec \mathfrak{a}_+(x) + \mathfrak{a}_+(y). \tag{21}$$

Proof It follows easily from Theorem 4 applied to the function $\varphi = t\frac{1}{2}\|\cdot\|^2$ with $\nabla\varphi(y) = ty$. □

Compliance with Ethical Standards

Conflict of interest The author declares that he has no conflict of interest.

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