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Attractors of the 3D Magnetohydrodynamics Equations with Damping

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Abstract
The three-dimensional magnetohydrodynamics equation with damping is considered in this paper. Global attractor of the 3D magnetohydrodynamics equations with damping is proved for $4 \le \beta < 5$ with any $\alpha > 0$.

Keywords Magnetohydrodynamics equations · Damping · Global attractor

Mathematics Subject Classification 76W05 · 35B40 · 35B41

1 Introduction

In this paper, we consider the following three-dimensional magnetohydrodynamics (MHD) equations with damping:

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u - (b \cdot \nabla) b + \alpha |u|^{\beta-1} u + \nabla (p + \frac{|b|^2}{2}) = f_1(x), \\ \partial_t b - \kappa \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u = f_2(x), \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u|_{\partial D} = b|_{\partial D} = 0, \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, \end{cases} \tag{1.1}$$

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where $D \subseteq \mathbb{R}^3$ is a bounded domain with the boundary ∂D and $t > 0$. u, b are the fluid velocity and magnetic field, respectively. $f_1(x)$, $f_2(x)$ are the external body force. p is the pressure, $\beta \geq 1$ is constant, and α is the damping coefficient. The constants $\nu, \kappa \geq 0$ are kinematic viscosity and magnetic resistivity. For simplicity, we set $\nu = \kappa = 1$.

The magnetohydrodynamic model has been investigated by many authors. In [12], Sermange and Temam have proved the well-posedness of solutions for MHD system in a bounded and a periodic domain. At the same time, regularity properties and attractors were also obtained. In [1], the pullback attractors and well-posedness of solutions of 2D MHD system were proved by using the Galerkin method. Based on the well-posedness of strong solutions of 3D MHD system that is difficult problem, many authors in [3,6,18,19,21] have studied the attractors and invariant measures of solutions of 3D-modified MHD system. Our result improves the early results in [21].

The damping term is very important for proving the well-posedness of 3D MHD system. In previous years, well-posedness and regularity of solutions of 3D Navier–Stokes system with damping were proved in [2,20]. In [4,5,7–9,11], the global existence of strong solution for 3D Navier–Stokes system with damping was proved for $\beta > 3$ with any $\alpha > 0$ and $\alpha \geq \frac{1}{4}$ as $\beta = 3$. Moreover, the global well-posedness of the 3D magnetohydrodynamics equations with damping was proved for $\beta \geq 4$ with any $\alpha > 0$ in [16]. Based on [10], global well-posedness of the 3D magneto–micropolar equations with damping was proved for $\beta \geq 4$ with any $\alpha > 0$. The existence and regularity of the trajectory attractor of 3D modified Navier–Stokes equations were proved in [17].

To obtain the existence of attractors for the three-dimensional magnetohydrodynamics equations with damping, we overcome the main difficulty lies in dealing with the nonlinear term $(u \cdot \nabla)u$, $(u \cdot \nabla)b$, $(b \cdot \nabla)u$, $(b \cdot \nabla)b$ and $F(u) = \alpha|u|^{\beta-1}u$. C is a nonnegative constant which may change from line to line.

This paper is organized as follows. In Sect. 2, we give some preliminaries and main Theorem 2.1. In Sect. 3, the uniform estimate of solutions for system (1.1) is proved. In Sect. 4, the existence of a global attractor for system (1.1) is proved.

2 Preliminaries

In this paper, the inner products and norms are defined by

$$(u, v) = \int_D u \cdot v \mathrm{d}x, \quad \forall u, v \in H, \quad ((u, v)) = \int_D \nabla u \cdot \nabla v \mathrm{d}x, \quad \forall u, v \in V,$$

and $||\cdot||^2 = (\cdot, \cdot)$, $||\nabla \cdot||^2 = ((\cdot, \cdot))$, $\mathcal{V} = \{u \in (C_0^\infty(D))^3 : \mathrm{div} u = 0\}$, $H =$ the closure of $\mathcal{V}$ in $(L^2(D))^3$ and $V =$ the closure of $\mathcal{V}$ in $(H_0^1(D))^3$. The L^p-norm is given by $||\cdot||_p$. By using the Poincaré inequality, there exists a positive constant λ_1 such that

$$\sqrt{\lambda_1}(||u|| + ||b||) \leq ||\nabla u|| + ||\nabla b||, \quad \forall u, b \in V, \tag{2.1}$$

here, λ_1 represents the minimum of between the first eigenvalue of $-\Delta u$ and the first eigenvalue of $-\Delta b$. Let $F(u) = \alpha|u|^{\beta-1}u$ and $D(A) = H^2(D) \cap V$. Here, P is the orthogonal projection of $(L^2(D))^3$ onto H such that $Au = -P\Delta u$ and $Ab = -P\Delta b$. For $u, v \in V$, we define the bilinear form $B(u, v) = P((u \cdot \nabla)v)$. We define $g(u) = PF(u)$. Now, we rewrite system (1.1) as follows in the abstract form:

$$\begin{cases} \partial_t u + Au + B(u, u) - B(b, b) + g(u) = f_1, \\ \partial_t b + Ab + B(u, b) - B(b, u) = f_2, \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0. \end{cases} \tag{2.2}$$

Now, we introduce the main result as follows.

Theorem 2.1 *Assume that $4 \le \beta < 5$ with any $\alpha > 0$, $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$. The operator $\{S(t)\}_{t\ge0}$ of the three-dimensional MHD equations with damping system (2.2) satisfies*

$$S(t)(u_0, b_0) = (u(t), b(t)).$$

$\{S(t)\}_{t\ge0}$ is defined in the space $V \times V$. System (2.2) has a $(V \times V, H^2 \times H^2)-$global attractor that satisfies the following.

(i) The global attractor $\mathcal{A}$ is invariant and compact in $H^2 \times H^2$.
(ii) The global attractor $\mathcal{A}$ attracts bounded subset of $V \times V$ in relation to the norm topology of $H^2 \times H^2$.

3 Uniform Estimate

Firstly, we will prove the uniform estimates of strong solutions for system (2.2) as $t \to \infty$. We show the existence of attractors by using the following estimates.

Lemma 3.1 *Let $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$ for $4 \le \beta < 5$ with any $\alpha > 0$. There exists a constant t_0 such that*

$$||u(t)||^2 + ||b(t)||^2 \le C, \tag{3.1}$$

$$\int_t^{t+1} (||\nabla u(s)||^2 + ||\nabla b(s)||^2 + ||u(s)||_{\beta+1}^{\beta+1})\mathrm{d}s \le C. \tag{3.2}$$

Proof Multiplying the first equation of (2.2) by u and the second equation of (2.2) by b, integrating over D, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(||u(t)||^2 + ||b(t)||^2) + 2\left(||\nabla u||^2 + ||\nabla b||^2 + \alpha||u||_{\beta+1}^{\beta+1}\right) = 2(f_1, u) + 2(f_2, b)$$
$$\le ||\nabla u||^2 + ||\nabla b||^2 + \frac{1}{\lambda_1}(||f_1||^2 + ||f_2||^2).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(||u(t)||^2+||b(t)||^2)+||\nabla u||^2+||\nabla b||^2+\alpha||u||_{\beta+1}^{\beta+1}\leq\frac{1}{\lambda_1}(||f_1||^2+||f_2||^2), \tag{3.3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(||u(t)||^2+||b(t)||^2)+\lambda_1(||u||^2+||b||^2)\leq\frac{1}{\lambda_1}(||f_1||^2+||f_2||^2). \tag{3.4}$$

Applying the Gronwall inequality, it is easy to get

$$||u(t)||^2+||b(t)||^2\leq(||u_0||^2+||b_0||^2)e^{-\lambda_1 t}+\frac{1}{\lambda_1^2}(||f_1||^2+||f_2||^2). \tag{3.5}$$

Let $t_0=\max\{-\frac{1}{\lambda_1}\ln\frac{||f_1||^2+||f_2||^2}{\lambda_1^2(||u_0||^2+||b_0||^2)},0\}$. For any $t\geq t_0$,

$$||u(t)||^2+||b(t)||^2\leq\frac{2}{\lambda_1^2}(||f_1||^2+||f_2||^2)\leq C. \tag{3.6}$$

Integrating (3.3) on $[t,t+1]$ and applying above inequality (3.6), we get for any $t\geq t_0$,

$$\begin{aligned}\int_t^{t+1}(||\nabla u(s)||^2+||\nabla b(s)||^2+\alpha||u(s)||_{\beta+1}^{\beta+1})\mathrm{d}s&\leq||u(t)||^2+||b(t)||^2+\frac{1}{\lambda_1}(||f_1||^2+||f_2||^2)\\&\leq C.\end{aligned}$$

Lemma 3.2 *Let $(u_0,b_0)\in V\times V$ and $f_1,f_2\in H$ for $4\leq\beta<5$ with any $\alpha>0$. There exists a t_1 such that for every $t\geq t_1$,*

$$||\nabla u(t)||^2+||\nabla b(t)||^2+||u||_{\beta+1}\leq C. \tag{3.7}$$

Proof Inspired by [16], it is easy to get for $\beta\geq 4$ with any $\alpha>0$,

$$\begin{aligned}&||\nabla u(t)||^2+||\nabla b(t)||^2+\int_0^t(||\Delta u(s)||^2+||\Delta b(s)||^2)\mathrm{d}s\\&+\int_0^t(|||u|^{\frac{\beta-1}{2}}\nabla u||^2+||\nabla|u|^{\frac{\beta+1}{2}}||^2)\mathrm{d}s\leq C.\end{aligned} \tag{3.8}$$

Multiplying the L^2-inner product of the first equation of (1.1) by u_t, then we get

$$\begin{aligned}||u_t||^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\nabla u||^2+\frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{\beta+1}^{\beta+1}&=-\int_D(u\cdot\nabla)uu_t\mathrm{d}x+\int_D(b\cdot\nabla)bu_t\mathrm{d}x+(f_1,u_t)\\&\leq\frac{1}{2}||u_t||^2+||f_1||^2+C||u\cdot\nabla u||^2+C||b\cdot\nabla b||^2.\end{aligned} \tag{3.9}$$

It is easy to get that

$$||u_t||^2 + \frac{\mathrm{d}}{\mathrm{d}t}||\nabla u||^2 + \frac{\mathrm{d}}{\mathrm{d}t}||u||_{\beta+1}^{\beta+1} \leq C||f_1||^2 + C||u \cdot \nabla u||^2 + C||b \cdot \nabla b||^2. \quad (3.10)$$

For the second term on the right-hand side of inequality (3.10), inspired by Theorem 1.1 in [10], we deduce

$$\begin{aligned} C||u \cdot \nabla u||^2 &\leq C \int_D |u|^2 |\nabla u|^{\frac{4}{\beta-1}} |\nabla u|^{2-\frac{4}{\beta-1}} \mathrm{d}x \\ &\leq C(|||u|^{\frac{\beta-1}{2}} \nabla u||^2 + ||\nabla u||^2). \end{aligned} \quad (3.11)$$

Inspired by Theorem 1.2 in [16] and Theorem 1.1 in [7], we get for $\beta \geq 4$

$$||b(t)||_{3\frac{\beta+1}{\beta-1}} \leq C. \quad (3.12)$$

For the third term on the right-hand side of inequality (3.10), using (3.12), then we get

$$\begin{aligned} C||b \cdot \nabla b||^2 &\leq C||b||^2_{\frac{3(\beta+1)}{\beta-1}} ||\nabla b||^2_{\frac{6(\beta+1)}{\beta+5}} \\ &\leq C||b||^2_{\frac{3(\beta+1)}{\beta-1}} ||\nabla b||^{\frac{4}{\beta+1}} ||\Delta b||^{\frac{2(\beta-1)}{\beta+1}} \\ &\leq C(||\nabla b||^2 + ||\Delta b||^2). \end{aligned} \quad (3.13)$$

Integrating (3.10) on $[0, t]$ and applying inequalities (3.11) and (3.13) to get

$$||u||_{\beta+1} \leq C, \quad \forall t \geq t_0 + 1 \equiv t_1. \quad (3.14)$$

Lemma 3.3 *Let $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$ for $4 \leq \beta < 5$ with any $\alpha > 0$. There exists a t_2 such that for every $t \geq t_2$,*

$$\int_t^{t+1} (||\Delta u||^2 + ||\Delta b||^2 + |||u|^{\frac{\beta-1}{2}} \nabla u||^2) \mathrm{d}s \leq C. \quad (3.15)$$

Proof By (3.8), there exists a t_2 such that for any $t \geq t_2$

$$\int_t^{t+1} (||\Delta u||^2 + ||\Delta b||^2 + |||u|^{\frac{\beta-1}{2}} \nabla u||^2) \mathrm{d}s \leq C. \quad (3.16)$$

Lemma 3.4 *Let $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$ for $4 \leq \beta < 5$ with any $\alpha > 0$. There exists a t_3 such that for every $t \geq t_3$,*

$$||u_t||^2 + ||b_t||^2 \leq C. \quad (3.17)$$

Proof Multiplying the first equation of (1.1) by u_t and the second equation of (1.1) by b_t, integrating the result on D, then we have

$$\begin{aligned}
&||u_t||^2 + ||b_t||^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla u||^2 + ||\nabla b||^2) + \frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{\beta+1}^{\beta+1} \\
&\le \frac{||u_t||^2 + ||b_t||^2}{2} + ||f_1||^2 + ||f_2||^2 \\
&\quad + C||u\cdot\nabla u||^2 + C||b\cdot\nabla b||^2 + C||b\cdot\nabla u||^2 + C||u\cdot\nabla b||^2 \\
&= \frac{||u_t||^2 + ||b_t||^2}{2} + ||f_1||^2 + ||f_2||^2 + \sum_{i=1}^{4} I_i. \qquad (3.18)
\end{aligned}$$

For I_1, inspired by Theorem 1.1 in [10], we get

$$\begin{aligned}
I_1 &\le C\int_D |u|^2|\nabla u|^{\frac{4}{\beta-1}}|\nabla u|^{2-\frac{4}{\beta-1}}\mathrm{d}x \\
&\le C(|||u|^{\frac{\beta-1}{2}}\nabla u||^2 + ||\nabla u||^2). \qquad (3.19)
\end{aligned}$$

For I_2, by Sobolev inequality and (3.12), we get

$$\begin{aligned}
I_2 &\le C||b||^2_{\frac{3(\beta+1)}{\beta-1}}||\nabla b||^2_{\frac{6(\beta+1)}{\beta+5}} \\
&\le C||b||^2_{\frac{3(\beta+1)}{\beta-1}}||\nabla b||^{\frac{4}{\beta+1}}||\Delta b||^{\frac{2(\beta-1)}{\beta+1}} \\
&\le C(||\nabla b||^2 + ||\Delta b||^2). \qquad (3.20)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
I_3 &\le C||b||^2_{\frac{3(\beta+1)}{\beta-1}}||\nabla u||^{\frac{4}{\beta+1}}||\Delta u||^{\frac{2(\beta-1)}{\beta+1}} \\
&\le C(||\nabla u||^2 + ||\Delta u||^2). \qquad (3.21)
\end{aligned}$$

For I_4, applying the Sobolev inequality, we get

$$\begin{aligned}
I_4 &\le C||u||^2_{\beta+1}||\nabla b||^2_{\frac{2(\beta+1)}{\beta-1}} \\
&\le C||u||^2_{\beta+1}||\nabla b||^{\frac{2(\beta-2)}{\beta+1}}||\Delta b||^{\frac{6}{\beta+1}} \\
&\le C(||\nabla b||^2 + ||\Delta b||^2). \qquad (3.22)
\end{aligned}$$

Adding up (3.18)–(3.22), we get

$$
\begin{aligned}
&||u_t||^2 + ||b_t||^2 + \frac{\mathrm{d}}{\mathrm{d}t}(||\nabla u||^2 + ||\nabla b||^2) + \frac{\mathrm{d}}{\mathrm{d}t}||u||_{\beta+1}^{\beta+1} \\
&\leq C(||f_1||^2 + ||f_2||^2) + C(||\nabla b||^2 + ||\Delta b||^2 + ||\nabla u||^2 + ||\Delta u||^2 + |||u|^{\frac{\beta-1}{2}}\nabla u||^2).
\end{aligned} \tag{3.23}
$$

By Lemma 3.2–Lemma 3.3, integrating (3.23) in time from t to $t+1$, it is easy to get

$$
\int_t^{t+1} (||u_t(s)||^2 + ||b_t(s)||^2)\mathrm{d}s \leq C. \tag{3.24}
$$

We apply ∂_t to the first equation of (2.2) and multiply the L^2-inner product by u_t. Similarly, we apply ∂_t to the second equation of (2.2) and multiply the L^2-inner product by b_t. Then we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u_t||^2 + ||b_t||^2) + ||\nabla u_t||^2 + ||\nabla b_t||^2 \leq |\int_D u_t \nabla u u_t \mathrm{d}x| + |\int_D b_t \nabla b u_t \mathrm{d}x| \\
&+ |\int_D u_t \nabla b b_t \mathrm{d}x| + |\int_D b_t \nabla u b_t \mathrm{d}x| - \int_D F'(u) u_t u_t \mathrm{d}x = \sum_{i=5}^{9} I_i.
\end{aligned} \tag{3.25}
$$

For I_9, by Lemma 2.4 in [14], we have $I_9 \leq 0$.

For I_5, by using Sobolev inequality and Lemma 3.2, we get

$$
\begin{aligned}
I_5 &\leq C||u_t||^{\frac{1}{2}}||\nabla u_t||^{\frac{3}{2}}||\nabla u|| \\
&\leq \frac{1}{4}||\nabla u_t||^2 + C||u_t||^2||\nabla u||^4 \\
&\leq \frac{1}{4}||\nabla u_t||^2 + C||u_t||^2.
\end{aligned} \tag{3.26}
$$

For I_8, similarly, then we get

$$
I_8 \leq \frac{1}{4}||\nabla b_t||^2 + C||b_t||^2. \tag{3.27}
$$

For I_6 and I_7, by Gagliardo–Nirenberg inequality and Lemma 3.2, we get

$$
\begin{aligned}
I_6 + I_7 &\leq C||u_t||_4||b_t||_4||\nabla b|| \\
&\leq C||u_t||^{\frac{1}{4}}||\nabla u_t||^{\frac{3}{4}}||b_t||^{\frac{1}{4}}||\nabla b_t||^{\frac{3}{4}}||\nabla b|| \\
&\leq \frac{1}{4}||\nabla u_t||^2 + \frac{1}{4}||\nabla b_t||^2 + C(||u_t||^2 + ||b_t||^2)||\nabla b||^4 \\
&\leq \frac{1}{4}||\nabla u_t||^2 + \frac{1}{4}||\nabla b_t||^2 + C(||u_t||^2 + ||b_t||^2).
\end{aligned} \tag{3.28}
$$

Adding up (3.25)–(3.28), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(||u_t||^2 + ||b_t||^2) + ||\nabla u_t||^2 + ||\nabla b_t||^2 \le C(||u_t||^2 + ||b_t||^2). \tag{3.29}$$

Applying the uniform Gronwall's lemma, there exists a t_3 such that for any $s \ge t_3$

$$||u_t(s)||^2 + ||b_t(s)||^2 \le C. \tag{3.30}$$

Lemma 3.5 *Let $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$ for $4 \le \beta < 5$ with any $\alpha > 0$. There exists a t_4 such that for every $t \ge t_4$,*

$$||Au(t)|| + ||Ab(t)|| \le C. \tag{3.31}$$

Proof By using the Minkowski inequality, we deduce

$$\begin{aligned} ||Au|| + ||Ab|| \le\ & ||u_t|| + ||b_t|| + ||f_1|| + ||f_2|| + ||B(u,u)|| \\ & + ||B(b,b)|| + ||B(u,b)|| + ||B(b,u)|| + \alpha||u|u|^{\beta-1}|| \\ =\ & ||u_t|| + ||b_t|| + ||f_1|| + ||f_2|| + \sum_{i=1}^{5} J_i. \end{aligned} \tag{3.32}$$

By the Sobolev inequality, we have

$$J_1 \le C||u||_\infty ||\nabla u|| \le C||\nabla u||^{\frac{3}{2}}||Au||^{\frac{1}{2}} \le \frac{1}{4}||Au|| + C||\nabla u||^3. \tag{3.33}$$

For J_2, similarly, then we also have

$$J_2 \le \frac{1}{4}||Ab|| + C||\nabla b||^3. \tag{3.34}$$

For J_3, applying the Sobolev inequality, we deduce

$$J_3 \le C||u||_\infty ||\nabla b|| \le C||\nabla u||^{\frac{1}{2}}||Au||^{\frac{1}{2}}||\nabla b|| \le \frac{1}{8}||Au|| + C||\nabla u||^2 + C||\nabla b||^4. \tag{3.35}$$

For J_4, similarly, then we also have

$$J_4 \le \frac{1}{4}||Ab|| + C||\nabla b||^2 + C||\nabla u||^4. \tag{3.36}$$

For J_5, since $\frac{\beta-3}{2} < 1$ for $4 \le \beta < 5$, applying the Young's inequality, we get

$$J_5 = \alpha||u||_{2\beta}^{\beta} \le C||\Delta u||^{\frac{\beta-3}{2}}||\nabla u||^{\frac{\beta+3}{2}} \le \frac{1}{8}||Au|| + C||\nabla u||^{\frac{\beta+3}{5-\beta}}. \tag{3.37}$$

Substituting (3.33)–(3.37) into (3.32), it is easy to get that for any $t \geq t_4$,

$$||Au|| + ||Ab|| \leq C. \tag{3.38}$$

Lemma 3.6 *Let $(u_0, b_0) \in V \times V$ and $f_1, f_2 \in H$ for $4 \leq \beta < 5$ with any $\alpha > 0$. There exists a t_5 such that for every $t \geq t_5$,*

$$||\nabla u_t||^2 + ||\nabla b_t||^2 \leq C, \tag{3.39}$$

$$\int_t^{t+1} (||\nabla u_t(s)||^2 + ||\nabla b_t(s)||^2)\mathrm{d}s \leq C. \tag{3.40}$$

Proof We integrate inequality (3.29) from t to $t+1$ and use the Lemma 3.4 to get

$$\begin{aligned}\int_t^{t+1} (||\nabla u_t(s)||^2 + ||\nabla b_t(s)||^2)\mathrm{d}s &\leq ||u_t(t)||^2 + ||b_t(t)||^2 + C\int_t^{t+1} (||u_t||^2 + ||b_t||^2)\mathrm{d}s \\ &\leq C.\end{aligned} \tag{3.41}$$

By virtue of Lemma 3.5, then we deduce

$$||u(t)||_{D(A)} + ||b(t)||_{D(A)} \leq C.$$

Applying the Agmon inequality, it is easy to get

$$||u(t)||_\infty + ||b(t)||_\infty \leq C. \tag{3.42}$$

We apply ∂_t to the first equation of (2.2) and multiply the L^2-inner product by Au_t. Similarly, we apply ∂_t to the second equation of (2.2) and multiply the L^2-inner product by Ab_t. Then we also have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla u_t||^2 + ||\nabla b_t||^2) + ||Au_t||^2 + ||Ab_t||^2 \\ &\leq |\int_D u_t \nabla u Au_t \mathrm{d}x| + |\int_D u \nabla u_t Au_t \mathrm{d}x| + |\int_D b_t \nabla b Au_t \mathrm{d}x| + |\int_D b \nabla b_t Au_t \mathrm{d}x| \\ &+ |\int_D u_t \nabla b Ab_t \mathrm{d}x| + |\int_D u \nabla b_t Ab_t \mathrm{d}x| + |\int_D b_t \nabla u Ab_t \mathrm{d}x| + |\int_D b \nabla u_t Ab_t \mathrm{d}x| \\ &+ |\int_D F'(u) u_t Au_t \mathrm{d}x| = \sum_{i=1}^{9} K_i.\end{aligned} \tag{3.43}$$

For K_1 and K_2, applying the Sobolev inequality and Lemma 3.5, we get

$$\begin{aligned}K_1 &\leq C||\nabla u_t|| ||\nabla u||^{\frac{1}{2}} ||Au||^{\frac{1}{2}} ||Au_t|| \\ &\leq \frac{1}{16}||Au_t||^2 + C||\nabla u_t||^2,\end{aligned} \tag{3.44}$$

and

$$
\begin{aligned}
K_2 &\le C||\nabla u||||\nabla u_t||^{\frac{1}{2}}||Au_t||^{\frac{3}{2}} \\
&\le \frac{1}{16}||Au_t||^2 + C||\nabla u_t||^2.
\end{aligned} \tag{3.45}
$$

For K_3 and K_4, we get by using the similar method

$$
\begin{aligned}
K_3 &\le C||\nabla b_t||||\nabla b||^{\frac{1}{2}}||Ab||^{\frac{1}{2}}||Au_t|| \\
&\le \frac{1}{16}||Au_t||^2 + C||\nabla b_t||^2,
\end{aligned} \tag{3.46}
$$

and

$$
\begin{aligned}
K_4 &\le C||\nabla b||||\nabla b_t||^{\frac{1}{2}}||Ab_t||^{\frac{1}{2}}||Au_t|| \\
&\le \frac{1}{16}||Au_t||^2 + \frac{1}{4}||Ab_t||^2 + C||\nabla b_t||^2.
\end{aligned} \tag{3.47}
$$

Similarly, we deduce

$$
\begin{aligned}
\sum_{i=5}^{8} K_i &\le C||\nabla u_t||||\nabla b||^{\frac{1}{2}}||Ab||^{\frac{1}{2}}||Ab_t|| + C||\nabla u||||\nabla b_t||^{\frac{1}{2}}||Ab_t||^{\frac{3}{2}} \\
&\quad + C||\nabla b_t||||\nabla u||^{\frac{1}{2}}||Au||^{\frac{1}{2}}||Ab_t|| + C||\nabla b||||\nabla u_t||^{\frac{1}{2}}||Au_t||^{\frac{1}{2}}||Ab_t|| \\
&\le \frac{1}{8}||Au_t||^2 + \frac{1}{4}||Ab_t||^2 + C(||\nabla u_t||^2 + ||\nabla b_t||^2).
\end{aligned} \tag{3.48}
$$

For K_9, by (3.42), we have

$$
\begin{aligned}
K_9 &\le C||u||_\infty^{\beta-1}||u_t||||Au_t|| \\
&\le \frac{1}{8}||Au_t||^2 + C||u_t||^2 \quad \text{for } t \ge t_4.
\end{aligned} \tag{3.49}
$$

Summing up (3.43)–(3.49), we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla u_t||^2 + ||\nabla b_t||^2) \le C(||\nabla u_t||^2 + ||\nabla b_t||^2). \tag{3.50}
$$

By the uniform Gronwall's lemma, there exists a t_5 such that for every $t \ge t_5$,

$$
||\nabla u_t||^2 + ||\nabla b_t||^2 \le C. \tag{3.51}
$$

4 Global Attractors

In this section, we will show the existence of a global attractor for system (2.2) in $H^2 \times H^2$. Inspired by [13,14], we introduce the following the main lemmas.

Lemma 4.1 *$\{S(t)\}_{t\geq 0}$ is Lipschitz continuous in $V \times V$.*

Proof Let (u_1, b_1) and (u_2, b_2) be two solutions of system (2.2) with initial values (u_{01}, b_{01}) and (u_{02}, b_{02}). We set $\bar{u} = u_1 - u_2$ and $\bar{b} = b_1 - b_2$. We multiply the inner product with $A\bar{u}$ and $A\bar{b}$, respectively. Then we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla\bar{u}||^2 + ||\nabla\bar{b}||^2) + ||A\bar{u}||^2 + ||A\bar{b}||^2 \\
&\leq \alpha\int_D ||u_1|^{\beta-1}u_1 - |u_2|^{\beta-1}u_2||A\bar{u}|\mathrm{d}x + \int_D |\bar{u}\nabla u_1 A\bar{u}|\mathrm{d}x + \int_D |u_2\nabla\bar{u}A\bar{u}|\mathrm{d}x \\
&\quad + \int_D |\bar{b}\nabla b_1 A\bar{u}|\mathrm{d}x + \int_D |b_2\nabla\bar{b}A\bar{u}|\mathrm{d}x + \int_D |u_1\nabla\bar{b}A\bar{b}|\mathrm{d}x + \int_D |\bar{u}\nabla b_2 A\bar{b}|\mathrm{d}x \\
&\quad + \int_D |\bar{b}\nabla u_1 A\bar{b}|\mathrm{d}x + \int_D |b_2\nabla\bar{u}A\bar{b}|\mathrm{d}x = \sum_{i=1}^{9} L_i.
\end{aligned} \tag{4.1}
$$

Inspired by [13,14], since $\int_0^t(||u_1||_{3(\beta-1)}^{2(\beta-1)} + ||\nabla u_2||^2(||u_1||_{6(\beta-2)}^{2(\beta-2)} + ||u_2||_{6(\beta-2)}^{2(\beta-2)}))\mathrm{d}s < C$ for $4 \leq \beta < 5$, then we get

$$
L_1 \leq \frac{1}{8}||A\bar{u}||^2 + C(||u_1||_{3(\beta-1)}^{2(\beta-1)} + ||\nabla u_2||^2(||u_1||_{6(\beta-2)}^{2(\beta-2)} + ||u_2||_{6(\beta-2)}^{2(\beta-2)}))||\nabla\bar{u}||^2. \tag{4.2}
$$

For L_2 and L_3, applying the Sobolev inequality, we have

$$
\begin{aligned}
L_2 &\leq C||\nabla\bar{u}||||\nabla u_1||^{\frac{1}{2}}||Au_1||^{\frac{1}{2}}||A\bar{u}|| \\
&\leq \frac{1}{16}||A\bar{u}||^2 + C||\nabla u_1||||Au_1||||\nabla\bar{u}||^2,
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
L_3 &\leq C||\nabla u_2||||\nabla\bar{u}||^{\frac{1}{2}}||A\bar{u}||^{\frac{3}{2}} \\
&\leq \frac{1}{16}||A\bar{u}||^2 + C||\nabla u_2||^4||\nabla\bar{u}||^2.
\end{aligned} \tag{4.4}
$$

Similarly, for the rest of terms $L_4 - L_9$, we get

$$
\begin{aligned}
L_4 + L_5 &\leq C||\nabla\bar{b}||||\nabla b_1||^{\frac{1}{2}}||Ab_1||^{\frac{1}{2}}||A\bar{u}|| + C||\nabla b_2||||\nabla\bar{b}||^{\frac{1}{2}}||A\bar{b}||^{\frac{1}{2}}||A\bar{u}|| \\
&\leq \frac{1}{8}||A\bar{u}||^2 + \frac{1}{8}||A\bar{b}||^2 + C(||\nabla b_1||||Ab_1|| + ||\nabla b_2||^4)||\nabla\bar{b}||^2,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
L_6+L_7 &\le C||\nabla u_1||||\nabla \bar{b}||^{\frac{1}{2}}||A\bar{b}||^{\frac{3}{2}}+C||\nabla \bar{u}||||\nabla b_2||^{\frac{1}{2}}||Ab_2||^{\frac{1}{2}}||A\bar{b}||\\
&\le \frac{1}{8}||A\bar{b}||^2+C(||\nabla u_1||^4+||\nabla b_2||||Ab_2||)(||\nabla \bar{u}||^2+||\nabla \bar{b}||^2), \qquad (4.6)
\end{aligned}
$$

and

$$
\begin{aligned}
L_8+L_9 &\le C||\nabla \bar{b}||||\nabla u_1||^{\frac{1}{2}}||Au_1||^{\frac{1}{2}}||A\bar{b}||+C||\nabla b_2||||\nabla \bar{u}||^{\frac{1}{2}}||A\bar{u}||^{\frac{1}{2}}||A\bar{b}||\\
&\le \frac{1}{8}||A\bar{u}||^2+\frac{1}{8}||A\bar{b}||^2+C(||\nabla u_1||||Au_1||+||\nabla b_2||^4)(||\nabla \bar{u}||^2+||\nabla \bar{b}||^2). \qquad (4.7)
\end{aligned}
$$

Adding up (4.1)–(4.7), it is easy to get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla \bar{u}||^2+||\nabla \bar{b}||^2) \le\; & C[||\nabla u_1||^2+||\nabla u_1||^4+||u_1||_{3(\beta-1)}^{2(\beta-1)}+||\nabla b_1||||Ab_1||\\
&+||\nabla u_2||^2(||u_1||_{6(\beta-2)}^{2(\beta-2)}+||u_2||_{6(\beta-2)}^{2(\beta-2)})+||\nabla u_2||^4\\
&+||Au_1||^2+||\nabla b_2||^4+||\nabla b_2||^2+||Ab_2||^2](||\nabla \bar{u}||^2+||\nabla \bar{b}||^2). \qquad (4.8)
\end{aligned}
$$

Applying the Gronwall inequality and Lemma 3.1–Lemma 3.6, this completes the proof of Lemma 4.1.

Lemma 4.2 *Assume that $\mathcal{A}$ is a $(V\times V, V\times V)-$global attractor for $\{S(t)\}_{t\ge 0}$. $\mathcal{A}$ is a $(V\times V, H^2\times H^2)-$global attractor if and only if*

(i) *$\{S(t)\}_{t\ge 0}$ is a bounded $(V\times V, H^2\times H^2)-$absorbing set.*
(ii) *$\{S(t)\}_{t\ge 0}$ is $(V\times V, H^2\times H^2)-$asymptotically compact.*

Firstly, we will prove the operator $\{S(t)\}_{t\ge 0}$ has a $(V\times V, V\times V)-$global attractor, then by using above Lemma 4.2, we get the attractor is a $(V\times V, H^2\times H^2)-$global attractor. Let

$$
B_1=\{u,b\in V: ||\nabla u||^2+||\nabla b||^2\le C\}
$$

and

$$
B_2=\{u,b\in D(A): ||Au||^2+||Ab||^2\le C\}.
$$

By above Lemma 3.2, we deduce that B_1 is bounded absorbing set of $\{S(t)\}_{t\ge 0}$ in the space $(V\times V, V\times V)$. By above Lemma 3.5, we get that B_2 is bounded absorbing set of $\{S(t)\}_{t\ge 0}$ in the space $(V\times V, H^2\times H^2)$. By Lemma 3.5, the $\{S(t)\}_{t\ge 0}$ is $(V\times V, V\times V)$-asymptotically compact. Inspired by [13–15], we get a $(V\times V, V\times V)$-global attractor $\mathcal{A}$. Finally, we will show $\{S(t)\}_{t\ge 0}$ is $(V\times V, H^2\times H^2)$-asymptotically compact. We need the following lemma.

Lemma 4.3 *Let $(u_0,b_0)\in V\times V$ and $f_1,f_2\in H$ for $4\le\beta<5$ with any $\alpha>0$. The dynamical system $\{S(t)\}_{t\ge 0}$ is $(V\times V, H^2\times H^2)$-asymptotically compact.*

Proof Assume that (u_{0n}, b_{0n}) is a bounded in $V \times V$ and $t_n \to \infty$. We will show $\{S(t_n)(u_{0n}, b_{0n})\}$ has a convergent subsequence in $H^2 \times H^2$. Let

$$(u_n(t), b_n(t)) = S(t)(u_{0n}, b_{0n}), \quad (\bar{u}_n(t_n), \bar{b}_n(t_n)) = (\frac{\partial u_n}{\partial t}|_{t=t_n}, \frac{\partial b_n}{\partial t}|_{t=t_n}).$$

For the first equation and the second equation of (2.2), we get

$$\begin{aligned} Au_n(t_n) &= f_1 - \bar{u}_n(t_n) - B(u_n(t_n), u_n(t_n)) + B(b_n(t_n), b_n(t_n)) - g(u_n(t_n)), \\ Ab_n(t_n) &= f_2 - \bar{b}_n(t_n) - B(u_n(t_n), b_n(t_n)) + B(b_n(t_n), u_n(t_n)). \end{aligned}$$

By Lemma 3.5 and Lemma 3.6, then there exists a positive constant $T > 0$ such that for every $t \geq T$,

$$||\nabla \frac{\partial u_n}{\partial t}(t)|| + ||\nabla \frac{\partial b_n}{\partial t}(t)|| \leq C, \quad ||Au_n(t)|| + ||Ab_n(t)|| \leq C. \tag{4.9}$$

When $t_n \to \infty$, there exists a $N > 0$ such that $t_n \geq T$ for every $n \geq N$. Applying (4.9), we deduce for $n \geq N$,

$$||\nabla \bar{u}_n(t_n)|| + ||\nabla \bar{b}_n(t_n)|| \leq C, \quad ||Au_n(t_n)|| + ||Ab_n(t_n)|| \leq C. \tag{4.10}$$

Applying the compactness of embedding $V \hookrightarrow H$ and $D(A) \hookrightarrow V$ and (4.10), then there exist $(\bar{u}, \bar{b}) \in V \times V$ and $(\hat{u}, \hat{b}) \in D(A) \times D(A)$ such that

$$u_n(t_n) \to \hat{u} \ \text{ strongly in } \ V, \tag{4.11}$$

$$b_n(t_n) \to \hat{b} \ \text{ strongly in } \ V, \tag{4.12}$$

$$\bar{u}_n(t_n) \to \bar{u} \ \text{ strongly in } \ H, \tag{4.13}$$

$$\bar{b}_n(t_n) \to \bar{b} \ \text{ strongly in } \ H. \tag{4.14}$$

By (4.10) and $H^2 \hookrightarrow L^\infty$, we get

$$||u_n(t_n)||_\infty + ||b_n(t_n)||_\infty \leq C, \quad \forall n \geq N. \tag{4.15}$$

Inspired by [13,14], applying (4.11), we get

$$||F(u_n(t_n)) - F(\hat{u})||^2 \leq C||u_n(t_n) - \hat{u}||^2 \to 0, \text{as}\, n \to \infty.$$

Hence,

$$g(u_n(t_n)) \to g(\hat{u}) \ \text{ strongly in } \ H. \tag{4.16}$$

Then, by Sobolev inequality, we have

$$\begin{aligned}
&||B(u_n(t_n), u_n(t_n)) - B(\hat{u}, \hat{u})||^2 \\
&\quad \le C(||(u_n(t_n)\cdot\nabla)(u_n(t_n)-\hat{u})||^2 + ||(u_n(t_n)-\hat{u})\cdot\nabla\hat{u}||^2) \\
&\quad \le C(||\nabla u_n(t_n)||^2||\nabla(u_n(t_n)-\hat{u})||||A(u_n(t_n)-\hat{u})|| + ||\nabla(u_n(t_n)-\hat{u})||^2||\nabla\hat{u}||||A\hat{u}||) \\
&\quad \to 0, \quad \text{as } n\to\infty. \qquad (4.17)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&||B(b_n(t_n), b_n(t_n)) - B(\hat{b}, \hat{b})||^2 \\
&\quad \le C(||(b_n(t_n)\cdot\nabla)(b_n(t_n)-\hat{b})||^2 + ||(b_n(t_n)-\hat{b})\cdot\nabla\hat{b}||^2) \\
&\quad \le C(||\nabla b_n(t_n)||^2||\nabla(b_n(t_n)-\hat{b})||||A(b_n(t_n)-\hat{b})|| + ||\nabla(b_n(t_n)-\hat{b})||^2||\nabla\hat{b}||||A\hat{b}||) \\
&\quad \to 0, \quad \text{as } n\to\infty, \\
&||B(u_n(t_n), b_n(t_n)) - B(\hat{u}, \hat{b})||^2 \qquad (4.18) \\
&\quad \le C(||(u_n(t_n)\cdot\nabla)(b_n(t_n)-\hat{b})||^2 + ||(u_n(t_n)-\hat{u})\cdot\nabla\hat{b}||^2) \\
&\quad \le C(||\nabla u_n(t_n)||^2||\nabla(b_n(t_n)-\hat{b})||||A(b_n(t_n)-\hat{b})|| + ||\nabla(u_n(t_n)-\hat{u})||^2||\nabla\hat{b}||||A\hat{b}||) \\
&\quad \to 0, \quad \text{as } n\to\infty, \qquad (4.19)
\end{aligned}$$

and

$$\begin{aligned}
&||B(b_n(t_n), u_n(t_n)) - B(\hat{b}, \hat{u})||^2 \\
&\quad \le C(||(b_n(t_n)\cdot\nabla)(u_n(t_n)-\hat{u})||^2 + ||(b_n(t_n)-\hat{b})\cdot\nabla\hat{u}||^2) \\
&\quad \le C(||\nabla b_n(t_n)||^2||\nabla(u_n(t_n)-\hat{u})||||A(u_n(t_n)-\hat{u})|| + ||\nabla(b_n(t_n)-\hat{b})||^2||\nabla\hat{u}||||A\hat{u}||) \\
&\quad \to 0, \quad \text{as } n\to\infty. \qquad (4.20)
\end{aligned}$$

(4.17)–(4.20) imply that

$$-B(u_n(t_n), u_n(t_n)) + B(b_n(t_n), b_n(t_n)) \to -B(\hat{u}, \hat{u}) + B(\hat{b}, \hat{b}) \text{ strongly in } H, \qquad (4.21)$$

$$-B(u_n(t_n), b_n(t_n)) + B(b_n(t_n), u_n(t_n)) \to -B(\hat{u}, \hat{b}) + B(\hat{b}, \hat{u}) \text{ strongly in } H. \qquad (4.22)$$

Applying (4.13), (4.14), (4.16), (4.21) and (4.22), then we get

$$Au_n(t_n) \to f_1 - \bar{u} - B(\hat{u}, \hat{u}) + B(\hat{b}, \hat{b}) - g(\hat{u}) \text{ strongly in } H, \qquad (4.23)$$

$$Ab_n(t_n) \to f_2 - \bar{b} - B(\hat{u}, \hat{b}) + B(\hat{b}, \hat{u}) \text{ strongly in } H, \qquad (4.24)$$

as $n \to \infty$. We get $\{S(t)\}_{t\ge 0}$ is $(V\times V, H^2\times H^2)$-asymptotically compact.

Proof of Theorem 2.1 Applying Lemma 3.5, we get $B_2 = \{u, b \in D(A) : ||Au||^2 + ||Ab||^2 \le C\}$ denotes a bounded $(V\times V, H^2\times H^2)-$absorbing set. Next, applying

Lemma 4.3, we obtain the $\{S(t)\}_{t\geq 0}$ is $(V \times V, H^2 \times H^2)$-asymptotically compact. Finally, by Lemma 4.2, $\mathcal{A}$ is a $(V \times V, H^2 \times H^2)-$global attractor.

References

1. Anh, C.T., Son, D.T.: Pullback attractors for non-autonomous 2D MHD equations on some unbounded domains. Ann. Polon. Math. **113**, 129–154 (2015)
2. Cai, X.J., Jiu, Q.S.: Weak and strong solutions for the incompressible Navier–Stokes equations with damping. J. Math. Anal. Appl. **343**, 799–809 (2008)
3. Catania, D.: Finite dimensional global attractor for 3D MHD-α models: a comparison. J. Math. Fluid Mech. **14**, 95–115 (2012)
4. Gao, H.J., Liu, H.: Well-posedness and invariant measures for a class of stochastic 3D Navier-Stokes equations with damping driven by jump noise. J. Differ. Equ. **267**, 5938–5975 (2019)
5. Hajduk, K.W., Robinson, J.C.: Energy equality for the 3D critical convective Brinkman–Forchheimer equations. J. Differ. Equ. **263**, 7141–7161 (2017)
6. Li, B., Zhu, H.J., Zhao, C.D.: Time decay rate of solutions to the hyperbolic MHD equations in $\mathbb{R}^3$, Acta Math. Sci. Ser. B (Engl. Ed.) **36**, 1369–1382 (2016)
7. Liu, H., Gao, H.J.: Decay of solutions for the 3D Navier–Stokes equations with damping. Appl. Math. Lett. **68**, 48–54 (2017)
8. Liu, H., Gao, H.J.: Ergodicity and dynamics for the stochastic 3D Navier–Stokes equations with damping. Commun. Math. Sci. **16**, 97–122 (2018)
9. Liu, H., Lin, L., Sun, C.F., Xiao, Q.K.: The exponential behavior and stabilizability of the stochastic 3D Navier–Stokes equations with damping. Rev. Math. Phys. **31**, 1950023 (2019)
10. Liu, H., Sun, C.F., Meng, F.W.: Global well-posedness of the 3D magneto–micropolar equations with damping. Appl. Math. Lett. **94**, 38–43 (2019)
11. Markowich, P.A., Titi, E.S., Trabelsi, S.: Continuous data assimilation for the three-dimensional Brinkman–Forchheimer-extended Darcy model. Nonlinearity **29**, 1292–1328 (2016)
12. Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. Comm. Pure Appl. Math. **36**, 635–664 (1983)
13. Song, X.L., Hou, Y.R.: Attractors for the three-dimensional incompressible Navier–Stokes equations with damping. Discrete Contin. Dyn. Syst. **31**, 239–252 (2011)
14. Song, X.L., Hou, Y.R.: Uniform attractors for three-dimensional Navier–Stokes equations with non-linear damping. J. Math. Anal. Appl. **422**, 337–351 (2015)
15. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, vol. 68. Springer, New York (1997)
16. Titi, E.S., Trabelsi, S.: Global well-posedness of a 3D MHD model in porous media. J. Geom. Mech. **11**, 621–637 (2019)
17. Zhao, C.D., Caraballo, T.: Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier–Stokes equations. J. Differ. Equ. **266**, 7205–7229 (2019)
18. Zhao, C.D., Li, B.: Time decay rate of weak solutions to the generalized MHD equations in $\mathbb{R}^2$. Appl. Math. Comput. **292**, 1–8 (2017)
19. Zhao, C.D., Li, B.: Analyticity of the global attractor for the 3D regularized MHD equations. Electron. J. Differ. Equ. **179**, 20 (2016)
20. Zhou, Y.: Regularity and uniqueness for the 3D incompressible Navier–Stokes equations with damping. Appl. Math. Lett. **25**, 1822–1825 (2012)
21. Zhu, Z.Q., Zhao, C.D.: Pullback attractor and invariant measures for the three-dimensional regularized MHD equations. Discrete Contin. Dyn. Syst. **38**, 1461–1477 (2018)