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Further Study on Z-Eigenvalue Localization Set and Positive Definiteness of Fourth-Order Tensors

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Abstract

Fourth-order tensors play a fundamental role in signal processing, wireless communication systems, image processing, data analysis and higher-order statistics. In this paper, we introduce a Z-identity tensor and establish two Z-eigenvalue inclusion sets for fourth-order tensors, which are sharper than some existing results. Numerical examples are proposed to verify the efficiency of the obtained results. As applications, we provide some checkable sufficient conditions for the positive definiteness of fourth-order symmetric tensors. Further, we propose upper bounds on the Z-spectral radius of fourth-order nonnegative tensors and estimate the convergence rate of the greedy rank-one algorithms under suitable conditions.

Keywords Fourth-order tensors · Positive definiteness · Z-eigenvalue inclusion sets · Z-identity tensor · Best rank-one approximation

Mathematics Subject Classification 15A18 · 15A42

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1 Introduction

Consider the following homogeneous polynomials with spherical constraint:

$$\begin{aligned} \min f_{\mathcal{A}}(x) = \mathcal{A}x^m &= \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m} \\ \text{s.t.} \quad x^\top x &= 1, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n, m, n \geq 2,\ f_{\mathcal{A}}(x)$ is a homogeneous polynomial of degree m with n variables and $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an mth-order n-dimensional real tensor with entries [13, 15]:

$$a_{i_1 \ldots i_m} \in \mathbb{R},\ i_j \in N = \{1, \ldots, n\},\ j = 1, \ldots, m.$$

The positive definiteness of homogeneous polynomials plays an important role in medical resonance [1,2], automatic control [5,6,14] and imaging spectral hypergraph theory [12,17]. For instance, in diffusion weighted magnetic resonance image processing, to analyze the connectivity of the tissues being scanned, we need to compute the diffusivity at each image lattice point and approximate it by a positive definite tensor [16]. Clearly, the critical points of (1.1) satisfy the following equations of for some $\lambda \in \mathbb{R}$:

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^\top x = 1, \tag{1.2}$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$. The real number λ and the real vector x satisfying with (1.2) are called Z-eigenvalue and Z-eigenvector, respectively.

Note that $f_{\mathcal{A}}(x)$ is positive definite if and only if tensor $\mathcal{A}$ is positive definite, and that an even-order real symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive [15]. Some effective algorithms for finding Z-eigenvalue and the corresponding eigenvector have been implemented [3,10,12,17,24–26,28]. When m and n are very large, it is difficult to compute all the Z-eigenvalues or the smallest Z-eigenvalue. Thus, many researchers turned to investigating the inclusion sets of Z-eigenvalues [3,7,11,19–23]. Unfortunately, the mentioned inclusion sets always include zero and could not be used to identify the positive definiteness of $\mathcal{A}$ and homogeneous polynomials. Recently, there have been breakthroughs in judging the positive definiteness of fourth-order tensors [8,29]. Based on special structure of fourth-order tensor, Zhao [29] proposed a Geršgorin-type E-eigenvalue inclusion set, which can identify the positive definiteness.

Lemma 1.1 (Corollary 1 of [29]) *Let* $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,n]}$ *with* $a_{iiii} = a_{iijj} + a_{ijij} + a_{ijji},\ i, j \in N, i \neq j$. *If for each* $i \in N$

$$a_{iiii} > R_i(\mathcal{A}) - r_i^{\widetilde{\Delta}_i}(\mathcal{A}),$$

then $\mathcal{A}$ *is positive definite, where* $R_i(\mathcal{A}) = \sum_{i_2,i_3,i_4 \in N} |a_{i i_2 i_3 i_4}|,\ r_i^{\widetilde{\Delta}_i}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \widetilde{\Delta}_i} |a_{i i_2 i_3 i_4}|$.

It is noted that the localization sets given in Lemma 1.1 need to satisfy the condition $a_{iiii} = a_{iijj} + a_{ijij} + a_{ijji}$, which is a relatively strict condition. To overcome the drawback, we want to introduce Z-identity tensor and establish new Z-eigenvalue inclusion sets to test the positive definiteness of $\mathcal{A}$. Meanwhile, He et al. [8] proposed some Z-eigenvalue inclusion sets based on classification of indicator sets, which can identify the positive definiteness of fourth-order tensors. However, some information of a given eigenvector $x = (x_1, \ldots, x_n)^\top$ on fourth-order tensors is not fully mined, such as $\max_{i \neq j \in N} |x_i||x_j| \leq \frac{1}{2}$. Inspired by these results, we want to explore properties of eigenvectors of fourth-order tensors and establish sharp Z-eigenvalue inclusion sets by a Z-identity tensor, which show that new Z-eigenvalue inclusion sets are tighter than existing results. Further, we propose some sufficient conditions for testing the positive definiteness of fourth-order tensors and estimate the convergence rate of the greedy rank-one algorithms under suitable conditions.

The remainder of the paper is organized as follows. In Sect. 2, some definitions and preliminary results are recalled. In Sect. 3, two sharp Z-eigenvalue inclusion sets with n parameters are established. In Sect. 4, some sufficient conditions are proposed for identifying positive definiteness of fourth-order tensors. Further, the upper bounds on Z-spectral radius of weakly symmetric nonnegative tensors are given and the convergence rate of the greedy rank-one algorithms is estimated.

2 Preliminaries

In this section, we firstly introduce important definitions of tensors [3,10,15].

Definition 2.1 Let $\mathcal{A}$ and $\mathcal{I}_Z$ be mth-order n-dimensional tensors.

(i) We define $\sigma_Z(\mathcal{A})$ as the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma_Z(\mathcal{A}) \neq \emptyset$. Then the Z-spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

(ii) We say that $\mathcal{A}$ is symmetric if

$$a_{i_1 \ldots i_m} = a_{i_{\pi(1)} \ldots i_{\pi(m)}}, \quad \forall \pi \in \Gamma_m,$$

where Γ_m is the permutation group of m indices.

(iii) We say that $\mathcal{A}$ is weakly symmetric if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

Obviously, if tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ weakly symmetric. However, the converse result may not hold.

Definition 2.2 We call $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ a fourth-order Z-identity tensor if

$$\mathcal{I}_Z x^3 = x \quad \text{with} \quad x^\top x = 1.$$

Note that the fourth-order Z-identity tensor is not unique in general. For instance, each fourth-order tensor in the following is a Z-identity tensor:
Case 1

$$(\mathcal{I}_Z)_{ijkl} = \begin{cases} 1, & \text{if } i = j \textit{ and } k = l \\ 0, & \text{otherwise.} \end{cases}$$

Case 2 (Property 2.4 of [10]):

$$(\mathcal{I}_Z)_{ijkl} = \begin{cases} 1, & \text{if } i = j = k = l, i, j, k, l \in N \\ \frac{1}{3}, & \text{if } i = j \neq k = l, i, j, k, l \in N \\ \frac{1}{3}, & \text{if } i = k \neq j = l, i, j, k, l \in N \\ \frac{1}{3}, & \text{if } i = l \neq k = j, i, j, k, l \in N \\ 0, & \text{otherwise.} \end{cases}$$

Next, we partition the index sets and recall some properties on Z-eigenvalues of a tensor [8,29]. For $k, i \in N$, define

$\Delta^{ki} = \{(i_2, i_3, i_4)$: at least two of the indices i_2, i_3, i_4 are equal to k and $i \in \{i_2, i_3, i_4\}\}$;

$\Delta^{k\bar{i}} = \{(i_2, i_3, i_4)$: at least two of the indices i_2, i_3, i_4 are equal to k and $i \notin \{i_2, i_3, i_4\}\}$;

$\Delta^i = \Delta^{1i} \cup \Delta^{2i} \cup \ldots \cup \Delta^{ni}$; $\Delta = \Delta^1 \cup \Delta^2 \cup \ldots \cup \Delta^n$;

$\Delta^{\bar{i}} = \Delta \backslash \Delta^i$; $\bar{\Delta} = \{(i_2, i_3, i_4) : i_2, i_3, i_4 \in N\} \backslash \Delta$;

$\Delta^{\bar{i},\bar{j}} = \{(i_2, i_3, i_4) \in \Delta : (i_2, i_3, i_4) \notin \Delta^i$ and $(i_2, i_3, i_4) \notin \Delta^j\}$.

Lemma 2.1 (Theorem 5 of [29]) *Let* $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,n]}$ *with* $a_{iiii} = a_{iijj} + a_{ijij} + a_{ijji}$, $i, j \in N, i \neq j$. *Then*

$$\sigma_E(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{iiii}| \leq R_i(\mathcal{A}) - r_i^{\Delta^i}(\mathcal{A})\}$ *and* $r_i^{\Delta^i}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^i} |a_{i i_2 i_3 i_4}|$.

Lemma 2.2 (Theorem 5 of [8]) *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[4,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \Big(\bigcup_{i,j \in N, i \neq j} \Upsilon_{ij}(\mathcal{A}) \Big) \bigcup \Big(\bigcup_{i \in N} \mathcal{M}_i(\mathcal{A}) \Big),$$

where $\Upsilon_{ij}(\mathcal{A}) = \{z \in \mathbb{R} : (|z| - \beta_i^{\Delta^i}(\mathcal{A}) - r_i^{\bar{\Delta}^i}(\mathcal{A}))(|z| - \beta_j^{\Delta^j}(\mathcal{A}) - r_j^{\bar{\Delta}^j}(\mathcal{A})) \leq (\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}))\}$, $\mathcal{M}_i(\mathcal{A}) = \{(z \in \mathbb{R} : |z| - \beta_i^{\Delta^i}(\mathcal{A}) - r_i^{\bar{\Delta}^i}(\mathcal{A})) \leq 0\}$ *and*

$$\beta_i^{\Delta^i}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{ki}} |a_{ii_2i_3i_4}|\right\}, r_i^{\bar{\Delta}^i}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^i} |a_{ii_2i_3i_4}|,$$

$$\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{k\bar{i}}} |a_{ii_2i_3i_4}|\right\}, r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{i}}} |a_{ii_2i_3i_4}|,$$

$$\beta_j^{\Delta^j}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{kj}} |a_{ji_2i_3i_4}|\right\}, r_j^{\bar{\Delta}^j}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^j} |a_{ji_2i_3i_4}|,$$

$$\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{k\bar{j}}} |a_{ji_2i_3i_4}|\right\}, r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{j}}} |a_{ji_2i_3i_4}|.$$

3 Sharp *Z*-eigenvalue Inclusion Sets with Parameters

In this section, we establish two sharp Z-eigenvalue inclusion theorems for fourth-order tensors. We begin our work by collecting the information of $\sum_{i\neq j\in N} |x_i||x_j|$.

Lemma 3.1 *For any $x \in \mathbb{R}^n$, if*

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1,$$

then

$$\max_{i\neq j\in N} x_i^2 x_j^2 \leq \frac{1}{4}.$$

Further, $\max_{i\neq j\in N} |x_i||x_j| \leq \frac{1}{2}$.

Proof Define Lagrange function

$$f(x_1, \ldots, x_n) = x_i^2 x_j^2 - \lambda(x_1^2 + x_1^2 + \ldots + x_n^2 - 1),$$

where λ denotes Lagrange multiplier. For all $i \neq j$, deriving the above equation x_i and x_j, respectively, we get

$$\begin{cases} 2x_j^2 x_i = 2\lambda x_i, \\ 2x_i^2 x_j = 2\lambda x_j. \end{cases}$$

Hence, we obtain $x_i^2 = x_j^2$, $\lambda = \frac{1}{2}$. Particularly, set

$$x_i = \pm x_j = \pm\frac{\sqrt{2}}{2}, x_n = 0, n \neq i, j$$

with $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. So,

$$\max_{i \neq j \in N} x_i^2 x_j^2 \leq \frac{1}{4}.$$

Further,

$$\max_{i \neq j \in N} |x_i||x_j| \leq \frac{1}{2}.$$

□

Theorem 3.1 *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,n]}$ and $\mathcal{I}_Z$ be an Z-identity tensor given in Definition 2.2. For any $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then,*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}, \alpha) = \bigcup_{i \in N} \Omega_i(\mathcal{A}, \alpha),$$

where $\Omega_i(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : |z - \alpha_i| \leq \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}}(\mathcal{A})\}$, $r_i^{\bar{\Delta}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \bar{\Delta}} |a_{i i_2 i_3 i_4}|$, and $\gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) = \max_{k \in N}\{\sum_{(i_2,i_3,i_4) \in \Delta^{ki}} |a_{i i_2 i_3 i_4} - \alpha_i (\mathcal{I}_Z)_{i i_2 i_3 i_4}|\}$, $\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) = \max_{k \in N}\{\sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{i}}} |a_{i i_2 i_3 i_4}|\}$. Further, $\sigma(\mathcal{A}) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \Omega(\mathcal{A}, \alpha)$.

Proof Let (λ, x) be a Z-eigenpair of $\mathcal{A}$ and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor, i.e.,

$$\mathcal{A}x^3 = \lambda x = \lambda \mathcal{I}_Z x^3, \quad x^\top x = 1. \tag{3.1}$$

Assume $|x_t| = \max_{i \in N} |x_i|$, then $0 < |x_t| \leq 1$. From the tth equality of (3.1), we have

$$\sum_{i_2,i_3,i_4 \in N} \lambda (\mathcal{I}_Z)_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4} = \sum_{i_2,i_3,i_4 \in N} a_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4}. \tag{3.2}$$

Hence, for any real number α_t, it follows that

$$\begin{aligned}
(\lambda - \alpha_t) x_t &= \sum_{i_2,i_3,i_4 \in N} (\lambda - \alpha_t)(\mathcal{I}_Z)_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4} \\
&= \sum_{i_2,i_3,i_4 \in N} \big(a_{t i_2 i_3 i_4} - \alpha_t (\mathcal{I}_Z)_{t i_2 i_3 i_4}\big) x_{i_2} x_{i_3} x_{i_4} \\
&= \sum_{(i_2,i_3,i_4) \in \Delta} \big(a_{t i_2 i_3 i_4} - \alpha_t (\mathcal{I}_Z)_{t i_2 i_3 i_4}\big) x_{i_2} x_{i_3} x_{i_4} + \sum_{(i_2,i_3,i_4) \in \bar{\Delta}} a_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4} \\
&= \sum_{(i_2,i_3,i_4) \in \Delta^t} \big(a_{t i_2 i_3 i_4} - \alpha_t (\mathcal{I}_Z)_{t i_2 i_3 i_4}\big) x_{i_2} x_{i_3} x_{i_4} + \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{t}}} a_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4} \\
&\quad + \sum_{(i_2,i_3,i_4) \in \bar{\Delta}} a_{t i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4}.
\end{aligned} \tag{3.3}$$

Taking modulus in (3.3) and using the triangle inequality and for any $k \in N$, we get

$$\begin{aligned}
|\lambda-\alpha_t||x_t| &= |\lambda-\alpha_t|\Big|\sum_{i_2,i_3,i_4\in N}(\mathcal{I}_Z)_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\Big| \\
&\le \sum_{(i_2,i_3,i_4)\in\Delta^t}\big|\big(a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}\big)\big||x_{i_2}||x_{i_3}||x_{i_4}| \\
&\quad+\sum_{(i_2,i_3,i_4)\in\Delta^{\bar t}}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}|+\sum_{(i_2,i_3,i_4)\in\bar\Delta}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| \\
&\le \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{kt}}|a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}|\right\}|x_t| \\
&\quad+\max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{k\bar t}}|a_{ti_2i_3i_4}|\right\}|x_t|+\frac{1}{2}\sum_{(i_2,i_3,i_4)\in\bar\Delta}|a_{ti_2i_3i_4}||x_t| \\
&= \gamma_t^{\Delta^t}(\mathcal{A},\alpha_t)|x_t|+\beta_t^{\Delta^{\bar t}}(\mathcal{A})|x_t|+\frac{1}{2}r_t^{\bar\Delta}(\mathcal{A})|x_t|,
\end{aligned} \tag{3.4}$$

i.e., $|\lambda-\alpha_t|\le\gamma_t^{\Delta^t}(\mathcal{A},\alpha_t)+\beta_t^{\Delta^{\bar t}}(\mathcal{A})+\frac{1}{2}r_t^{\bar\Delta}(\mathcal{A})$, which implies that $\lambda\in\Omega_t(\mathcal{A},\alpha)$. It follows from the arbitrariness of α that $\sigma(\mathcal{A})\subseteq\bigcap_{\alpha\in\mathbb{R}^n}\Omega(\mathcal{A},\alpha)$. □

Remark 3.1 (i) From Theorem 5 of [29], for $a_{iiii}=a_{iijj}+a_{ijij}+a_{ijji}$, we observe

$$R_i(\mathcal{A})-r_i^{\Delta^i}(\mathcal{A})=\sum_{(i_2,i_3,i_4)\notin\Delta^i}|a_{ii_2i_3i_4}|=\sum_{(i_2,i_3,i_4)\in\Delta^{\bar i}}|a_{ii_2i_3i_4}|+\sum_{(i_2,i_3,i_4)\in\bar\Delta}|a_{ii_2i_3i_4}|.$$

Setting $\alpha_i=a_{iiii}$, from Theorem 3.1, we obtain

$$\Omega_i(\mathcal{A},a_{iiii})=\left\{z\in\mathbb{R}:|z-a_{iiii}|\le\beta_i^{\Delta^{\bar i}}(\mathcal{A})+\frac{1}{2}r_i^{\bar\Delta}(\mathcal{A})\right\}.$$

It is easy to verify that

$$\beta_i^{\Delta^{\bar i}}(\mathcal{A})=\max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{k\bar i}}|a_{ii_2i_3i_4}|\right\}\le\sum_{(i_2,i_3,i_4)\in\Delta^{\bar i}}|a_{ii_2i_3i_4}|$$

and

$$\frac{1}{2}r_i^{\bar\Delta}(\mathcal{A})=\frac{1}{2}\sum_{(i_2,i_3,i_4)\in\bar\Delta}|a_{ii_2i_3i_4}|\le\sum_{(i_2,i_3,i_4)\in\bar\Delta}|a_{ii_2i_3i_4}|,\quad\forall i\in N,$$

which show $\beta_i^{\Delta^{\bar{i}}}(\mathcal{A})+\frac{1}{2}r_i^{\bar{\Delta}}(\mathcal{A}) \le R_i(\mathcal{A})-r_i^{\Delta^i}(\mathcal{A})$. Thus, $\Omega_i(\mathcal{A}, a_{iiii}) \subseteq \Gamma_i(\mathcal{A})$ for all $i \in N$. Further, $\bigcap_{\alpha\in\mathbb{R}^n} \Omega(\mathcal{A}, \alpha) \subseteq \Gamma(\mathcal{A})$. Hence, the result of Theorem 3.1 is always sharper than the result given in Theorem 5 of [29].

(ii) When $a_{iiii} \neq a_{iijj} + a_{ijij} + a_{ijji}$, we cannot determine the distribution of Z-eigenvalues by Theorem 5 of [29]. However, Theorem 3.1 still works.

(iii) Similar to the above characterizations, setting $\alpha = 0$, we prove that Theorem 3.1 can determine Z-eigenvalues more accurately than that of Theorem 4 of [8].

Theorem 3.2 *Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,n]}$ and $\mathcal{I}_Z$ be an Z-identity tensor given in Definition 2.2. For any $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then,*

$$\sigma(\mathcal{A}) \subseteq \Phi(\mathcal{A}, \alpha) = \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \Phi_{i,j}(\mathcal{A}, \alpha),$$

where $\Phi_{i,j}(\mathcal{A}, \alpha) = \big(\mathcal{P}_{i,j}(\mathcal{A}, \alpha) \cup \mathcal{M}_{i,j}(\mathcal{A}, \alpha)\big)$,

$$\begin{aligned}\mathcal{P}_{i,j}(\mathcal{A}, \alpha) = \Bigg\{z \in \mathbb{R} : &\left(|z-\alpha_i| - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2}r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)\\ &\times \left(|z-\alpha_j| - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2}r_j^{\bar{\Delta}^j}(\mathcal{A})\right)\\ &\le \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2}r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2}r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)\Bigg\},\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{i,j}(\mathcal{A}, \alpha) = \{&z \in \mathbb{R} : |z-\alpha_i| - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A})\\ &- \frac{1}{2}r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) < 0 \text{ and } |z-\alpha_j| - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2}r_j^{\bar{\Delta}^j}(\mathcal{A}) < 0\},\end{aligned}$$

$$\begin{aligned}\gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) &= \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{ki}} |a_{ii_2i_3i_4} - \alpha_i(\mathcal{I}_Z)_{ii_2i_3i_4}|\right\}, r_j^{\bar{\Delta}^j}(\mathcal{A})\\ &= \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^j} |a_{ji_2i_3i_4}|,\end{aligned}$$

$$\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{k\bar{j}}} |a_{ji_2i_3i_4}|\right\}, r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{j}}} |a_{ji_2i_3i_4}|,$$

$$\beta_i^{\Delta^j}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{kj}} |a_{ii_2i_3i_4}|\right\}, r_i^{\bar{\Delta}^j}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^j} |a_{ii_2i_3i_4}|,$$

$$\beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) = \max_{k\in N}\left\{\sum_{\substack{(i_2,i_3,i_4)\in\Delta\\(i_2,i_3,i_4)\notin\Delta^{ki}\\(i_2,i_3,i_4)\notin\Delta^{kj}}}|a_{ii_2i_3i_4}|\right\}, r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{j}}}|a_{ii_2i_3i_4}|.$$

Proof Let (λ, x) be a Z-eigenpair of $\mathcal{A}$ and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor. Setting $|x_t| = \max_{i\in N}|x_i|$, then one has $0 < |x_t| \le 1$. For any $s \ne t \in N$ and any real number α_t, from the tth equality of (3.1), we have

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{i_2,i_3,i_4\in N}(\lambda-\alpha_t)(\mathcal{I}_Z)_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\
&= \sum_{i_2,i_3,i_4\in N}\big(a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4}\\
&= \sum_{(i_2,i_3,i_4)\in\Delta}\big(a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4} + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}}a_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\
&= \sum_{(i_2,i_3,i_4)\in\Delta^t}\big(a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4} + \sum_{(i_2,i_3,i_4)\in\Delta^s}a_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\
&\quad + \sum_{\substack{(i_2,i_3,i_4)\in\Delta\\(i_2,i_3,i_4)\notin\Delta^t\\(i_2,i_3,i_4)\notin\Delta^s}}a_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4} + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s}a_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\
&\quad + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}}a_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}.\end{aligned}\tag{3.5}$$

Taking modulus in (3.5) and using the triangle inequality and for any $k \in N$, we obtain

$$\begin{aligned}|\lambda-\alpha_t||x_t| &= |\lambda-\alpha_t||\sum_{i_2,i_3,i_4\in N}(\mathcal{I}_Z)_{ti_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}|\\
&\le \sum_{(i_2,i_3,i_4)\in\Delta^t}|\big(a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}\big)||x_{i_2}||x_{i_3}||x_{i_4}|\\
&\quad + \sum_{(i_2,i_3,i_4)\in\Delta^s}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| + \sum_{\substack{(i_2,i_3,i_4)\in\Delta\\(i_2,i_3,i_4)\notin\Delta^t\\(i_2,i_3,i_4)\notin\Delta^s}}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}|\\
&\quad + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}}|a_{ti_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}|\\
&\le \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{kt}}|a_{ti_2i_3i_4}-\alpha_t(\mathcal{I}_Z)_{ti_2i_3i_4}|\right\}|x_t| + \max_{k\in N}\left\{\sum_{(i_2,i_3,i_4)\in\Delta^{ks}}|a_{ti_2i_3i_4}|\right\}|x_s|\end{aligned}$$

$$+\max_{k\in N}\left\{\sum_{\substack{(i_2,i_3,i_4)\in\Delta\\(i_2,i_3,i_4)\notin\Delta^{kt}\\(i_2,i_3,i_4)\notin\Delta^{ks}}}|a_{ti_2i_3i_4}|\right\}|x_t|+\frac{1}{2}\sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s}|a_{ti_2i_3i_4}||x_s|$$
$$+\frac{1}{2}\sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}}|a_{ti_2i_3i_4}||x_t|$$
$$=\gamma_t^{\Delta^t}(\mathcal{A},\alpha_t)|x_t|+\beta_t^{\Delta^s}(\mathcal{A})|x_s|+\beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A})|x_t|+\frac{1}{2}r_t^{\bar{\Delta}^s}(\mathcal{A})|x_s|+\frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})|x_t|. \tag{3.6}$$

Therefore,

$$(|\lambda-\alpha_t|-\gamma_t^{\Delta^t}(\mathcal{A},\alpha_t)-\beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A})-\frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}))|x_t|\leq\big(\beta_t^{\Delta^s}(\mathcal{A})+\frac{1}{2}r_t^{\bar{\Delta}^s}(\mathcal{A})\big)|x_s|. \tag{3.7}$$

If $|x_s|=0$, by (3.7), we deduce $|\lambda-\alpha_t|-\gamma_t^{\Delta^t}(\mathcal{A},\alpha_t)-\beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A})-\frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\leq 0$. Further, we obtain $\lambda\in\mathcal{P}_{t,s}(\mathcal{A},\alpha)$ or $\lambda\in\mathcal{M}_{t,s}(\mathcal{A},\alpha)$.

Otherwise, $|x_s|>0$. For any $s\neq t\in N$, from the sth equality of (3.1), it holds that

$$\begin{aligned}(\lambda-\alpha_s)x_s&=\sum_{i_2,i_3,i_4\in N}(\lambda-\alpha_s)(\mathcal{I}_Z)_{si_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\&=\sum_{i_2,i_3,i_4\in N}\big(a_{si_2i_3i_4}-\alpha_s(\mathcal{I}_Z)_{si_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4}\\&=\sum_{(i_2,i_3,i_4)\in\Delta}\big(a_{si_2i_3i_4}-\alpha_s(\mathcal{I}_Z)_{si_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4}+\sum_{(i_2,i_3,i_4)\in\bar{\Delta}}a_{si_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\&=\sum_{(i_2,i_3,i_4)\in\Delta^s}\big(a_{si_2i_3i_4}-\alpha_s(\mathcal{I}_Z)_{si_2i_3i_4}\big)x_{i_2}x_{i_3}x_{i_4}\\&\quad+\sum_{(i_2,i_3,i_4)\in\Delta^{\bar{s}}}a_{si_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}\\&\quad+\sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s}a_{si_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}+\sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}}a_{si_2i_3i_4}x_{i_2}x_{i_3}x_{i_4}.\end{aligned} \tag{3.8}$$

Taking modulus in (3.8) and using the triangle inequality and for any $k\in N$, we obtain

$$
\begin{aligned}
|\lambda-\alpha_s||x_s| &= |\lambda-\alpha_s|| \sum_{i_2,i_3,i_4\in N} (\mathcal{I}_Z)_{si_2i_3i_4} x_{i_2}x_{i_3}x_{i_4}| \\
&\le \sum_{(i_2,i_3,i_4)\in\Delta^s} |(a_{si_2i_3i_4}-\alpha_s(\mathcal{I}_Z)_{si_2i_3i_4})||x_{i_2}||x_{i_3}||x_{i_4}| + \sum_{(i_2,i_3,i_4)\in\Delta^{\bar{s}}} |a_{si_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| \\
&\quad + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s} |a_{si_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}} |a_{si_2i_3i_4}||x_{i_2}||x_{i_3}||x_{i_4}| \\
&\le \max_{k\in N}\left\{ \sum_{(i_2,i_3,i_4)\in\Delta^{ks}} |a_{si_2i_3i_4}-\alpha_s(\mathcal{I}_Z)_{si_2i_3i_4}| \right\}|x_s| + \max_{k\in N}\left\{ \sum_{(i_2,i_3,i_4)\in\Delta^{k\bar{s}}} |a_{si_2i_3i_4} \right\}|x_t| \\
&\quad + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^s} |a_{si_2i_3i_4}||x_{i_2}||x_{i_3}||x_s| + \sum_{(i_2,i_3,i_4)\in\bar{\Delta}^{\bar{s}}} |a_{si_2i_3i_4}||x_{i_2}||x_{i_3}||x_t| \\
&\le \gamma_s^{\Delta^s}(\mathcal{A},\alpha_s)|x_s| + \beta_s^{\Delta^{\bar{s}}}(\mathcal{A})|x_t| + \frac{1}{2}r_s^{\bar{\Delta}^s}(\mathcal{A})|x_s| + \frac{1}{2}r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})|x_t|.
\end{aligned} \tag{3.9}
$$

Thus,

$$
\left(|\lambda-\alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A},\alpha_s) - \frac{1}{2}r_s^{\bar{\Delta}^s}(\mathcal{A})\right)|x_s| \le \left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2}r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)|x_t|. \tag{3.10}
$$

When $|\lambda-\alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A},\alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) \ge 0$ or $|\lambda-\alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A},\alpha_s) - \frac{1}{2}r_s^{\bar{\Delta}^s}(\mathcal{A}) \ge 0$, multiplying inequalities (3.7) with (3.10), we have

$$
\begin{aligned}
&(|\lambda-\alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A},\alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}))(|\lambda-\alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A},\alpha_s) - \frac{1}{2}r_s^{\bar{\Delta}^s}(\mathcal{A})) \\
&\le (\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2}r_t^{\bar{\Delta}^s}(\mathcal{A}))(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2}r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})),
\end{aligned}
$$

which implies that $\lambda \in \mathcal{P}_{t,s}(\mathcal{A},\alpha)$.

When $|\lambda-\alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A},\alpha_t) - \beta_t^{\Delta_s^{\bar{t}}}(\mathcal{A}) - \frac{1}{2}r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) < 0$ and $|\lambda-\alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A},\alpha_s) - \frac{1}{2}r_s^{\bar{\Delta}^s}(\mathcal{A}) < 0$, then $\lambda \in \mathcal{M}_{t,s}(\mathcal{A},\alpha)$. Thus, the desired result holds. From the arbitrariness of s, we have $\lambda \in \bigcap_{s\in N, s\neq t}\Phi_{t,s}(\mathcal{A},\alpha)$. Further, $\lambda \in \bigcup_{i\in N}\bigcap_{j\in N, j\neq i}\Phi_{i,j}(\mathcal{A},\alpha)$. It follows from the arbitrariness of α that $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^n}\Phi(\mathcal{A},\alpha)$. □

Now, we give a proof to show $\Phi(\mathcal{A},\alpha) \subseteq \Omega(\mathcal{A},\alpha)$.

Corollary 3.1 *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ and $I_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor given in Definition 2.2. For any real vector $\alpha = (\alpha_1,\ldots,\alpha_n)^\top \in \mathbb{R}^n$, then*

$$
\sigma_Z(\mathcal{A}) \subseteq \Phi(\mathcal{A},\alpha) \subseteq \Omega(\mathcal{A},\alpha).
$$

Proof For any $\lambda \in \Phi(\mathcal{A},\alpha)$, we now break up the argument into two cases.

Case 1 Without loss of generality, there exists $t \in N$, for any $s \in N, s \neq t$ such that $\lambda \in \mathcal{P}_{t,s}(\mathcal{A}, \alpha)$, that is,

$$\begin{aligned}
&\left(|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right) \\
&\times \left(|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right) \\
&\leq \left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right).
\end{aligned}$$

If $(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A}))(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})) = 0$, then, $|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) \leq 0$, or $|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}) \leq 0$, which implies that $\lambda \in \Omega_t(\mathcal{A}, \alpha) \bigcup \Omega_s(\mathcal{A}, \alpha) \subseteq \Omega(\mathcal{A}, \alpha)$.

If $(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A}))(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})) > 0$, we have

$$\frac{\left(|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)}{\left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)} \frac{\left(|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right)}{\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)} \leq 1.$$

Therefore,

$$\frac{\left(|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)}{\left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)} \leq 1$$

or

$$\frac{(|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}))}{(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}))} \leq 1,$$

which implies that $\lambda \in \Omega_t(\mathcal{A}, \alpha) \bigcup \Omega_s(\mathcal{A}, \alpha) \subseteq \Omega(\mathcal{A}, \alpha)$.

Case 2 There exists $t \in N$, for any $s \in N, s \neq t$ such that $\lambda \in \mathcal{M}_{t,s}(\mathcal{A}, \alpha)$, that is,

$$\begin{aligned}
&|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) < 0 \textit{ and} \\
&|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}) < 0.
\end{aligned}$$

Obviously, $\lambda \in \Omega_t(\mathcal{A}, \alpha) \bigcap \Omega_s(\mathcal{A}, \alpha) \subseteq \Omega(\mathcal{A}, \alpha)$. Thus, we obtain the desired results. □

Table 1 Comparisons among He's method [8], Zhao's method [29] and our method

References	Inclusion sets
Theorem 4 of [8]	$\Theta(\mathcal{A}) = \{z \in \mathbb{R} : \lvert z\rvert \leq 18.0000\}$
Theorem 5 of [8]	$\Upsilon(\mathcal{A}) = \{z \in \mathbb{R} : \lvert z\rvert \leq \sqrt{54} + 9 \approx 16.3485\}$
Theorem 5 of [29]	$\Gamma(\mathcal{A}) = \{z \in \mathbb{C} : \lvert z - 9\rvert \leq 11.0000\}$
Theorem 7 of [29]	$\Theta(\mathcal{A}) = \{z \in \mathbb{C} : \lvert z - 9\rvert \leq \frac{15+\sqrt{33}}{2} \approx 10.3723\}$
Theorem 3.1	$\Omega(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : \lvert z - 9\rvert \leq 9.0000\}$
Theorem 3.2	$\Phi(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : \lvert z - 9\rvert \leq \sqrt{54} \approx 7.3485\}$.

Table 2 Comparisons between He's method [8] and our method

References	Inclusion sets
Theorem 4 of [8]	$\Theta(\mathcal{A}) = \{z \in \mathbb{R} : \lvert z\rvert \leq 26.0000\}$
Theorem 5 of [8]	$\Upsilon(\mathcal{A}) = \{z \in \mathbb{R} : \lvert z\rvert \leq 30 + \sqrt{240} \approx 22.7459\}$
Theorem 3.1	$\Omega(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : -9 \leq z \leq 25.0000\}$
Theorem 3.2	$\Phi(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : -5.3739 \leq z \leq 20.9373\}$

Now, we introduce Example 1 of [29] to show the improvement of the obtained results.

Example 3.1 Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor with elements defined as follows:

$$a_{1111} = 9, a_{2222} = 9, a_{1122} = 3, a_{1222} = 2, a_{2111} = 3.$$

By computations, all different E-eigenvalues are $3.9045, 9.000 - 5.7686i, 9.000 + 5.7686i, 14.0955$. Setting $\alpha = (9, 9)^\top$ and Z-identity tensor as Case 2 in Definition 2.2, we obtain that different estimations given in the literature are shown (Table 1).

Since $a_{iiii} \neq a_{iijj} + a_{ijij} + a_{ijji}$, Theorems 5-7 of [29] are invalid in the following example. Hence, our results are compared with Theorem 4 and Theorem 5 of [8]. The following example reveals that the results given in Theorems 3.1–3.2 are sharper than some existing results.

Example 3.2 Consider symmetric the tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ with entries defined as follows:

$$a_{ijkl} = \begin{cases} a_{1111} = 7, a_{2222} = 8, a_{3333} = 9, a_{1122} = 2, a_{1133} = 2, a_{2233} = 3; \\ a_{1222} = a_{3111} = a_{3222} = 2, a_{1333} = a_{2111} = a_{2333} = 0; \\ a_{1123} = a_{2123} = a_{3123} = 1. \end{cases}$$

Setting $\alpha = (6.5, 8, 7.5)^\top$ and Z-identity tensor as Case 2 in Definition 2.2, we obtain that different estimations given in the literature are shown (Table 2).

4 Applications

In this section, based on the inclusion sets $\Omega(\mathcal{A}, \alpha)$ and $\Phi(\mathcal{A}, \alpha)$ in Theorems 3.1 and 3.2, we propose some sufficient conditions for the positive definiteness of fourth-order symmetric tensors, as well as the estimations of Z-spectral radius.

4.1 Positive Definiteness of Fourth-Order Symmetric Tensors

We establish some sufficient conditions to check the positive definiteness of fourth-order symmetric tensor $\mathcal{A}$ without the condition $a_{iiii} = a_{iijj} + a_{ijij} + a_{ijji}$. Before proceeding further, we introduce the results of [8].

Lemma 4.1 (Theorem 16 of [8]) *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ be a symmetric tensor with $\beta_i^{1i} > 0, \ldots, \beta_i^{ni} > 0$ and*

$$\min_{i\in N}\{\beta_i^{1i}, \ldots, \beta_i^{ni}\} = C_i.$$

If for all $i, j \in N$, $j \neq i$,

$$C_i > r_i^{\bar{\Delta}^i}(\mathcal{A})$$

and

$$(C_i - r_i^{\bar{\Delta}^i}(\mathcal{A}))(C_j - r_j^{\bar{\Delta}^j}(\mathcal{A})) > (\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})),$$

where $\beta_i^{1i} = \sum_{(i_2,i_3,i_4)\in\Delta^{1i}} |a_{ii_2i_3i_4}|, \ldots, \beta_i^{ni} = \sum_{(i_2,i_3,i_4)\in\Delta^{ni}} |a_{ii_2i_3i_4}|$, then $\mathcal{A}$ is positive definite.

Theorem 4.1 *Let λ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,n]}$ and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor given in Definition 2.2. For $i \in N$, if there exists positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top$ such that*

$$\alpha_i > \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}}(\mathcal{A}), \tag{4.1}$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1.1) is positive definite.

Proof Suppose on the contrary that $\lambda \leq 0$. From Theorem 3.1, there exists $t \in N$ with $\lambda \in \Omega_t(\mathcal{A}, \alpha)$, i.e.,

$$|\lambda - \alpha_t| \leq \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}}(\mathcal{A}).$$

Further, it follows from $\alpha_t > 0$ and $\lambda \leq 0$ that

$$\alpha_t \leq |\lambda - \alpha_t| \leq \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}}(\mathcal{A}),$$

which contradicts (4.1). Thus, $\lambda > 0$. When $\mathcal{A}$ is a symmetric tensor and all Z-eigenvalues are positive, $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1.1) is positive definite. □

Theorem 4.2 *Let λ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,n]}$ and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor given in Definition 2.2. For $i \neq j \in N$, if there exists positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top$ such that the following two statements hold:*

$$\left(\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)\left(\alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\right)$$
$$> \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right), \tag{4.2}$$

$$\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) \geq 0 \text{ or } \alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) \geq 0, \tag{4.3}$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1.1) is positive definite.

Proof Suppose on the contrary that $\lambda \leq 0$. It follows from Theorem 3.2 that $\lambda \in \Phi(\mathcal{A}, \alpha)$. Now, we divide the following argument into two cases.
Case 1 There exists $t \in N$ with $\lambda \in \mathcal{P}_{t,s}(\mathcal{A}, \alpha)$ such that

$$\begin{aligned}
&\left(|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right) \\
&\times \left(|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right) \\
&\leq \left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right), \quad \forall s \neq t \in N.
\end{aligned}$$

Further, it follows from $\alpha_t > 0$ and $\lambda \leq 0$ that

$$\begin{aligned}
&\left(\alpha_t - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)\left(\alpha_s - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right) \\
&\leq \left(|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right) \\
&\times \left(|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right) \\
&\leq \left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right),
\end{aligned}$$

which contradicts (4.2). Thus, $\lambda > 0$.
Case 2 There exists $t \in N$ with $\lambda \in \mathcal{M}_{t,s}(\mathcal{A}, \alpha)$ such that

$$|\lambda - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) < 0, \quad \forall s \neq t \in N$$

Table 3 Testing the positive definiteness of tensor $\mathcal{A}$ by Theorem 16 of [8]

	$(C_i - r_i^{\bar{\Delta}^i}(\mathcal{A}))(C_j - r_j^{\bar{\Delta}^j}(\mathcal{A}))$	$(\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}))$
$i = 1, j = 2$	22.5	18
$i = 1, j = 3$	22.5	18
$i = 2, j = 1$	22.5	15
$i = 2, j = 3$	22.5	15
$i = 3, j = 1$	20.25	7.5
$i = 3, j = 2$	20.25	7.5

and

$$|\lambda - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}) < 0, \quad \forall s \neq t \in N.$$

Further, it follows from $\alpha_t > 0, t \in N$ and $\lambda \leq 0$ that

$$\alpha_t \leq |\lambda - \alpha_t| < \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})$$

and

$$\alpha_s \leq |\lambda - \alpha_s| < \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) + \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}),$$

which contradict (4.3). Thus, $\lambda > 0$.

From the above two cases, when $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1.1) is positive definite. □

The following example shows the validity of Theorems 4.1–4.2.

Example 4.1 Consider the symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ with entries defined as follows:

$$a_{ijkl} = \begin{cases} a_{1111} = 7.5, a_{2222} = 8, a_{3333} = 7.5, a_{1122} = 3, a_{1133} = 2.5, a_{2233} = 3; \\ a_{1222} = a_{3111} = a_{3222} = 1, a_{1333} = a_{2111} = a_{2333} = 0; \\ a_{1123} = a_{2123} = a_{3123} = 0.5. \end{cases}$$

By computations, we obtain that the minimum Z-eigenvalue and corresponding with the Z-eigenvector are $(\bar{\lambda}, \bar{x}) = (6.1312, (0.8682, 0.1374, 0.4769))$. Hence, $\mathcal{A}$ is positive definite. From Lemma 4.1, it is easy to get that $C_1 = 7.5$, $C_2 = 8$, $C_3 = 7.5$, we can calculate the following corresponding values (Table 3).

From Table 3, we verify

$$(C_i - r_i^{\bar{\Delta}^i}(\mathcal{A}))(C_j - r_j^{\bar{\Delta}^j}(\mathcal{A})) > (\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})), \quad \forall\, i \neq j \in N$$

Table 4 Testing the positive definiteness of tensor $\mathcal{A}$ by Theorem 4.1

	$\gamma_i^{\Delta^i}(\mathcal{A},\alpha_i)+\beta_i^{\Delta^{\bar{i}}}(\mathcal{A})+\frac{1}{2}r_i^{\bar{\Delta}}(\mathcal{A})$
$i=1$	5.5
$i=2$	8
$i=3$	5

Table 5 Testing the positive definiteness of tensor $\mathcal{A}$ by Theorem 4.2

	A_1	A_2
$i=1, j=2$	29.25	18.00
$i=1, j=3$	36.00	11.25
$i=2, j=1$	30.00	13.50
$i=2, j=3$	30.00	11.25
$i=3, j=1$	36.00	29.00
$i=3, j=2$	39.00	18.00

and

$$C_i > r_i^{\bar{\Delta}^i}(\mathcal{A}), \quad \forall\, i \in N,$$

which imply that $\mathcal{A}$ is positive definite.

Taking $\alpha = (8.5, 8.5, 8.5)^\top$ and the Z-identity tensor as Case 2 in Definition 2.2, from Theorem 4.1, we obtain the following corresponding values (Table 4).

It follows from Table 4 that

$$\alpha_i > \gamma_i^{\Delta^i}(\mathcal{A},\alpha_i)+\beta_i^{\Delta^{\bar{i}}}(\mathcal{A})+\frac{1}{2}r_i^{\bar{\Delta}}(\mathcal{A}), \quad \forall\, i \in N,$$

which implies that $\mathcal{A}$ is positive definite.

By Theorem 4.2, we define

$$A_1 = \left(\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A},\alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2}r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)\left(\alpha_j - \gamma_j^{\Delta^j}(\mathcal{A},\alpha_j) - \frac{1}{2}r_j^{\bar{\Delta}^j}(\mathcal{A})\right),$$
$$A_2 = \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2}r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2}r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)$$

and compute the following corresponding values (Table 5).

From Table 5, we verify

$$\left(\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A},\alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2}r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)\left(\alpha_j - \gamma_j^{\Delta^j}(\mathcal{A},\alpha_j) - \frac{1}{2}r_j^{\bar{\Delta}^j}(\mathcal{A})\right)$$
$$> \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2}r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2}r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right), \ \forall\, i \neq j \in N$$

Table 6 Testing the positive definiteness of tensor $\mathcal{A}$ by Theorem 16 of [8]

	$(C_i - r_i^{\bar{\Delta}^i}(\mathcal{A}))(C_j - r_j^{\bar{\Delta}^j}(\mathcal{A}))$	$(\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}))$
$i = 1, j = 2$	0.84	0.72
$i = 1, j = 3$	0.56	0.30
$i = 2, j = 1$	0.84	0.72
$i = 2, j = 3$	0.32	0.60
$i = 3, j = 1$	0.56	0.30
$i = 3, j = 2$	0.32	0.60

and

$$\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) \geq 0 \; or \; \alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) \geq 0,$$
$$\forall\, i \neq j \in N,$$

which show that $\mathcal{A}$ is positive definite.

The following example reveals that Theorem 4.2 can judge the positive definiteness of fourth-order symmetric tensors more accurately than Theorem 4.1 and Lemma 4.1.

Example 4.2 Consider the symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ with entries defined as follows:

$$a_{ijkl} = \begin{cases} a_{1111} = 2, a_{2222} = 1.4, a_{3333} = 1, a_{1122} = 1.5, a_{1133} = 2, a_{2233} = 2; \\ a_{1222} = a_{3111} = a_{3222} = 0.2, a_{1333} = a_{2111} = a_{2333} = 0; \\ a_{1123} = a_{2123} = a_{3123} = 0.1. \end{cases}$$

By computations, we obtain that the minimum Z-eigenvalue and corresponding with the Z-eigenvector are $(\bar{\lambda}, \bar{x}) = (1.9799, (0.9987, 0.0057, 0.0501))$. Hence, $\mathcal{A}$ is positive definite.

By Lemma 4.1, it is easy to get that $C_1 = 2$, $C_2 = 1.4$, $C_3 = 1$, and we can calculate the following corresponding values (Table 6).

Hence, $\mathcal{A}$ does not satisfy

$$(C_i - r_i^{\bar{\Delta}^i}(\mathcal{A}))(C_j - r_j^{\bar{\Delta}^j}(\mathcal{A})) > (\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A})$$
$$+ r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})), \quad \forall\, i \neq j \in N,$$

which shows that Lemma 4.1 is not suitable to check the positive definiteness of $\mathcal{A}$.

By Theorem 4.1, for any positive real number α_2, we have

$$\gamma_2^{\Delta^2}(\mathcal{A}, \alpha_2) + \beta_2^{\Delta^{\bar{2}}}(\mathcal{A})$$
$$+ \frac{1}{2} r_2^{\bar{\Delta}}(\mathcal{A}) = \max\{|4.5 - \alpha_2|, |1.4 - \alpha_2|, |6 - \alpha_2|\} + 1.2 + 0.3 > \alpha_2,$$

Table 7 Testing the positive definiteness of tensor $\mathcal{A}$ by Theorem 4.2

	A_1	A_2
$i=1, j=2$	1.54	0.72
$i=1, j=3$	1.19	0.45
$i=2, j=1$	1.36	0.54
$i=2, j=3$	0.56	0.45
$i=3, j=1$	1.19	0.36
$i=3, j=2$	0.77	0.72

which implies that Theorem 4.1 is not suitable to check the positive definiteness of $\mathcal{A}$.

However, taking $\alpha = (4, 4, 3.5)^\top$ and the Z-identity tensor as Case 2 in Definition 2.2, by Theorem 4.2, we define

$$A_1 = \left(\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\right),$$

$$A_2 = \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right)$$

and compute the following corresponding values (Table 7).

From Table 7, it holds that

$$\begin{aligned}
&\left(\alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\right)\\
&\quad > \left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right),\ \forall\, i \neq j \in N,\ \alpha_i - \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i)\\
&\quad - \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) - \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A}) \geq 0 \ or\ \alpha_j - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) \geq 0,\ \forall\, i \neq j \in N,
\end{aligned}$$

which imply that $\mathcal{A}$ is positive definite.

4.2 Estimations of Z-spectral Radius and Convergence Rate on the Greedy Rank-One Algorithms

The best rank-one approximation of $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is to find a rank-one tensor $\kappa x^m = (\kappa x_{i_1} x_{i_2} \ldots x_{i_m})$ such that

$$\min_{\kappa \in \mathbb{R}, x} \{||\mathcal{A} - \kappa x^m||_F : x^T x = 1\},$$

where $||\mathcal{A}||_F := \sqrt{\sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \ldots i_m}^2}$. When $\mathcal{A}$ is nonnegative and weakly symmetric, $\rho(\mathcal{A}) x_0^m$ is a best rank-one approximation of $\mathcal{A}$, i.e.,

$$\min_{\kappa \in \mathbb{R}, x^T x = 1} ||\mathcal{A} - \kappa x^m||_F = ||\mathcal{A} - \rho(\mathcal{A}) x_0^m||_F = \sqrt{||\mathcal{A}||_F^2 - \rho(\mathcal{A})^2}.$$

Further, the quotient on the residual of the best rank-one approximation of $\mathcal{A}$ is:

$$\omega = \frac{||\mathcal{A} - \rho(\mathcal{A})x_0^m||_F}{||\mathcal{A}||_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{||\mathcal{A}||_F^2}},$$

which can estimate the convergence rate of the greedy rank-one algorithm [4,9,18,19,27]. Thus, we shall propose upper bounds of the Z-spectral radius and estimate the convergence rate of greedy rank-one algorithms. We start this subsection with some fundamental results of nonnegative tensors [3,8,29].

Lemma 4.2 (Theorem 3.11 of [3]) *Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then, $\rho(\mathcal{A}) = \lambda^*$, where λ^* denotes the largest Z-eigenvalue.*

Lemma 4.3 (Corollary 4.10 of [3]) *Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then,*

$$\rho(\mathcal{A}) \geq \max_{i \in N} a_{i \ldots i}.$$

Lemma 4.4 (Theorem 11 of [29]) *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,n]}$ be a weakly symmetric nonnegative tensor with $a_{iiii} = a_{iijj} + a_{ijij} + a_{ijji}$, $i \neq j \in N$. Then*

$$\rho(\mathcal{A}) \leq \max_{i \in N}\{R_i(\mathcal{A}) - r_i^{\widehat{\Delta}_i}(\mathcal{A})\},$$

where $\widehat{\Delta}_i = \Delta^i \backslash \{i, i, i\}$, $r_i^{\widehat{\Delta}_i}(\mathcal{A}) = \sum_{(i_2, i_3, i_4) \in \widehat{\Delta}_i} |a_{i i_2 i_3 i_4}|$.

Lemma 4.5 (Theorem 8 of [8]) *Suppose $\mathcal{A} \in \mathbb{R}^{[4,n]}$ is weakly symmetric, nonnegative and irreducible. Then*

$$\rho(\mathcal{A}) \leq \max_{i,j \in N, i \neq j}\{\nu, \beta_i^{\Delta^i}(\mathcal{A}) + r_i^{\bar{\Delta}^i}(\mathcal{A})\},$$

where

$$\nu = \frac{1}{2}(\beta_i^{\Delta^i}(\mathcal{A}) + r_i^{\bar{\Delta}^i}(\mathcal{A}) + \beta_j^{\Delta^j}(\mathcal{A}) + r_j^{\bar{\Delta}^j}(\mathcal{A}) + \Lambda_{ij}^{\frac{1}{2}}(\mathcal{A}))$$

and

$$\Lambda_{ij}(\mathcal{A}) = (\beta_i^{\Delta^i}(\mathcal{A}) + r_i^{\bar{\Delta}^i}(\mathcal{A}) - \beta_j^{\Delta^j}(\mathcal{A}) - r_j^{\bar{\Delta}^j}(\mathcal{A}))^2 + 4(\beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + r_i^{\bar{\Delta}^{\bar{i}}}(\mathcal{A}))(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})).$$

In the following, we shall devote to finding sharp upper bounds of the Z-spectral radius of weakly symmetric nonnegative tensors.

Theorem 4.3 *Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,n]}$ be a weakly symmetric nonnegative tensor and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor given in Definition 2.2. For real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$ with $\alpha_i \leq \max_{i\in N} a_{iiii}$, then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} \min_{\alpha\in\mathbb{R}^n} \{\alpha_i + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}}(\mathcal{A})\}. \tag{4.4}$$

Proof From Lemma 4.2, we assume that $\rho(\mathcal{A}) = \lambda^*$ is the largest Z-eigenvalue. It follows from Theorem 3.1 that there exists $t \in N$ such that

$$|\rho(\mathcal{A}) - \alpha_t| \leq \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}}(\mathcal{A}).$$

Then, we get

$$\rho(\mathcal{A}) \leq \alpha_t + \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}}(\mathcal{A}).$$

Further,

$$\rho(\mathcal{A}) \leq \max_{i\in N} \left\{\alpha_i + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}}(\mathcal{A})\right\}.$$

Since α is chosen arbitrarily, it holds

$$\rho(\mathcal{A}) \leq \max_{i\in N} \min_{\alpha\in\mathbb{R}^n} \left\{\alpha_i + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}}(\mathcal{A})\right\}.$$

Thus, (4.4) holds. □

Theorem 4.4 *Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,n]}$ be a weakly symmetric nonnegative tensor and $\mathcal{I}_Z \in \mathbb{R}^{[4,n]}$ be a Z-identity tensor given in Definition 2.2. For real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$ with $\alpha_i \leq \max_{i\in N} a_{iiii}$, then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} \min_{j\in N, i\neq j, \alpha\in\mathbb{R}^n} \{\nu, \alpha_i + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A})$$
$$+ \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}), \alpha_j + \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) + \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\},$$

where

$$\nu = \frac{1}{2}\left(\alpha_i + \alpha_j + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) + \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) + \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\right),$$

$$\Lambda_{i,j}(\mathcal{A}) = \left(\alpha_i - \alpha_j + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) - \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) - \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\right)^2$$
$$+ 4\left(\beta_i^{\Delta^j}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^j}(\mathcal{A})\right)\left(\beta_j^{\Delta^{\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_j^{\bar{\Delta}^{\bar{j}}}(\mathcal{A})\right).$$

Proof From Lemma 4.2, we assume that $\rho(\mathcal{A}) = \lambda^*$ is the largest Z-eigenvalue. It follows from Theorem 3.2 that $\rho(\mathcal{A}) \in \Phi_{t,s}(\mathcal{A}, \alpha)$. Now, we break the proof into two parts.
Case 1 There exists $t \in N$ with $\rho(\mathcal{A}) \in \mathcal{P}_{t,s}(\mathcal{A}, \alpha)$ such that

$$\left(|\rho(\mathcal{A}) - \alpha_t| - \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) - \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) - \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right)$$
$$\times \left(|\rho(\mathcal{A}) - \alpha_s| - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right)$$
$$\leq \left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right), \quad \forall s \neq t \in N.$$

Then, solving for $\rho(\mathcal{A})$, we get

$$\rho(\mathcal{A}) \leq \frac{1}{2}(\alpha_t + \alpha_s + \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A})$$
$$+ \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) + \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) + \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}) + \Lambda_{t,s}^{\frac{1}{2}}(\mathcal{A})),$$

where

$$\Lambda_{t,s}(\mathcal{A}) = \left(\alpha_t - \alpha_s + \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}) - \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) - \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A})\right)^2$$
$$+ 4\left(\beta_t^{\Delta^s}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^s}(\mathcal{A})\right)\left(\beta_s^{\Delta^{\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_s^{\bar{\Delta}^{\bar{s}}}(\mathcal{A})\right).$$

Further, $\rho(\mathcal{A}) \leq \max_{i \in N} \frac{1}{2}(\alpha_i + \alpha_j + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i) + \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) + \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) + \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}))$. Since α and j are chosen arbitrarily, it holds

$$\rho(\mathcal{A}) \leq \max_{i \in N} \min_{j \in N, i \neq j, \alpha \in \mathbb{R}^n} \frac{1}{2}(\alpha_i + \alpha_j + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i)$$
$$+ \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}) + \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) + \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})).$$

Case 2 There exists $t \in N$ with $\rho(\mathcal{A}) \in \mathcal{M}_{t,s}(\mathcal{A}, \alpha)$ such that

$$\rho(\mathcal{A}) < \alpha_t + \gamma_t^{\Delta^t}(\mathcal{A}, \alpha_t) + \beta_t^{\Delta^{\bar{t},\bar{s}}}(\mathcal{A}) + \frac{1}{2} r_t^{\bar{\Delta}^{\bar{s}}}(\mathcal{A}),$$
$$\rho(\mathcal{A}) < \alpha_s + \gamma_s^{\Delta^s}(\mathcal{A}, \alpha_s) + \frac{1}{2} r_s^{\bar{\Delta}^s}(\mathcal{A}), \quad \forall s \neq t \in N.$$

Since α and j are chosen arbitrarily, one has

$$\rho(\mathcal{A}) \leq \max_{i \in N} \min_{j \in N, i \neq j, \alpha \in \mathbb{R}^n} \{\alpha_i + \gamma_i^{\Delta^i}(\mathcal{A}, \alpha_i)$$

Table 8 Upper bounds of $\rho(\mathcal{A})$ with different methods

References	Upper bound	Parameter α
Theorem 8 of [8]	$\rho(\mathcal{A}) \leq 9.0000$	No
Theorem 11 of [29]	$\rho(\mathcal{A}) \leq 11.1000$	No
Theorem 4.3	$\rho(\mathcal{A}) \leq 7.5000$	$\alpha = (3, 3, 3)^\top$
Theorem 4.4	$\rho(\mathcal{A}) \leq 7.0100$	$\alpha = (3, 3, 3)^\top$

$$+ \beta_i^{\Delta^{\bar{i},\bar{j}}}(\mathcal{A}) + \frac{1}{2} r_i^{\bar{\Delta}^{\bar{j}}}(\mathcal{A}), \alpha_j + \gamma_j^{\Delta^j}(\mathcal{A}, \alpha_j) + \frac{1}{2} r_j^{\bar{\Delta}^j}(\mathcal{A})\}.$$

From the above two cases, the conclusion holds. □

The following numerical experiment shows the validity of Theorems 4.3 and 4.4, and gives an estimation for the convergence rate of the greedy rank-one algorithms.

Example 4.3 Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a symmetric tensor with elements defined as follows:

$$a_{ijkl} = \begin{cases} a_{1111} = a_{2222} = a_{3333} = 3, a_{1122} = a_{1133} = a_{2233} = 1; \\ a_{1222} = a_{1333} = a_{2111} = a_{2333} = 0, a_{3111} = a_{3222} = 1; \\ a_{1123} = a_{2123} = 0.5, a_{3123} = 0.2. \end{cases}$$

By simple computation, we obtain $(\rho(\mathcal{A}), x) = (5.5283, (0.6027, 0.6027, 0.5230))$ and $||\mathcal{A}||_F = 7.7123$. The bounds via different estimations given in the literature are shown in Table 8.

Using Table 8 and Theorem 4.4, we give the best rank-one approximation of $\mathcal{A}$

$$\min_{\kappa \in \mathbb{R}, \kappa \in \mathbb{R}^n, x^T x = 1} ||\mathcal{A} - \kappa x^m||_F = \sqrt{||\mathcal{A}||_F^2 - \rho(\mathcal{A})^2} \geq 3.2156$$

and the quotient on the residual of the best rank-one approximation of $\mathcal{A}$

$$\omega = \frac{||\mathcal{A} - \rho(\mathcal{A}) x_0^m||_F}{||\mathcal{A}||_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{||\mathcal{A}||_F^2}} \geq 0.4169.$$

Since $a_{iiii} \neq a_{iijj} + a_{ijij} + a_{ijji}$, Theorem 11 of [29] is invalid in the following example. Hence, our results are compared with Theorem 8 of [8].

Example 4.4 Consider symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 1.5, a_{2222} = 2.5, a_{3333} = 1.5, a_{1122} = 0.5, a_{1133} = 1, a_{2233} = 0.5; \\ a_{1222} = a_{1333} = a_{2333} = a_{3111} = a_{3222} = 0.1, a_{2111} = 0; \\ a_{1123} = a_{2123} = 0.1, a_{3123} = 0.2. \end{cases}$$

By simple computation, we obtain $(\rho(\mathcal{A}), x) = (2.5388, (0.0592, 0.9906, 0.0980))$ and $||\mathcal{A}||_F = 4.5464$. The bounds via different estimations given in the literature are shown in Table 9.

Table 9 Upper bounds of $\rho(\mathcal{A})$ with different methods

References	Upper bound	Parameter α
Theorem 8 of [8]	$\rho(\mathcal{A}) \leq 4.6141$	No
Theorem 4.3	$\rho(\mathcal{A}) \leq 4.2000$	$\alpha = (2.25, 2, 2.25)^\top$
Theorem 4.4	$\rho(\mathcal{A}) \leq 3.8822$	$\alpha = (2.25, 2, 2.25)^\top$

From Table 9, by Theorem 4.4, we obtain the best rank-one approximation of $\mathcal{A}$

$$\min_{\kappa\in\mathbb{R},\kappa\in\mathbb{R}^n,x^T x=1} ||\mathcal{A} - \kappa x^m||_F = \sqrt{||\mathcal{A}||_F^2 - \rho(\mathcal{A})^2} \geq 2.3661$$

and the quotient on the residual of the best rank-one approximation of $\mathcal{A}$

$$\omega = \frac{||\mathcal{A} - \rho(\mathcal{A})x_0^m||_F}{||\mathcal{A}||_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{||\mathcal{A}||_F^2}} \geq 0.5204.$$

5 Conclusions

In this paper, without the condition $a_{iiii} \neq a_{iijj} + a_{ijij} + a_{ijji}$, we established new Z-eigenvalue inclusion sets for fourth-order tensors by Z-identity tensor and proposed some sufficient conditions for the positive definiteness based on exploring the information of eigenvectors. Further, we gave upper bounds for the Z-spectral radius and estimated the convergence rate of the greedy rank-one algorithms for fourth-order nonnegative tensors. Note that the suitable parameter α has a great influence on the numerical effects and positive definiteness of fourth-order tensors. Therefore, how to select the suitable parameter α is our further research.

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