



Carathéodory Density of the Hurwitz Metric on Plane Domains

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Abstract

It is well known that the Carathéodory metric is a natural generalization of the Poincaré metric, namely, the hyperbolic metric of the unit disk. In 2016, the Hurwitz metric was introduced by D. Minda in arbitrary proper subdomains of the complex plane and he proved that this metric coincides with the hyperbolic metric when the domains are simply connected. In this paper, we define a new metric which generalizes the Hurwitz metric in the sense of Carathéodory. Our main focus is to study its various basic properties in connection with the Hurwitz metric.

Keywords Hyperbolic density · Hurwitz density · Kobayashi density · Carathéodory density · Conformal mapping · Covering mapping

Mathematics Subject Classification Primary 30F45; Secondary 30C20 · 30C80

1 Introduction

Studying families of holomorphic functions associated with the hyperbolic metric always remains the hot topic in geometric function theory. Several researchers (for instance see [2,5,6]) have introduced new metrics which are closely related to the hyperbolic metric and established their comparison properties in possible situations. In particular, the generalized Kobayashi metric is one such metric which is always greater than or equal to the hyperbolic metric (see [3, Proposition 1]). The generalized

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Kobayashi metric in a domain is defined by the smallest push forward of the hyperbolic metric from a hyperbolic domain to a plane domain by holomorphic functions. In [3], it is further proved that the generalized Kobayashi metric agrees with the hyperbolic metric on simply connected domains (see also [4]). Coincidence of the hyperbolic and the generalized Kobayashi densities on other plane domains are studied in [7]. Moreover, the Kobayashi density satisfies the generalized Schwarz lemma for holomorphic function between two domains.

In 2016, Minda introduced a new metric, namely, the Hurwitz metric that also exceeds the hyperbolic metric in hyperbolic domains (see [6]) and investigated several basic properties such as distance decreasing property, conformal invariance property, domain monotonicity property, bilipschitz equivalent properties with the hyperbolic and the quasihyperbolic metrics. In our recent work (see [1]), we studied a new metric that generalizes the Hurwitz metric in the sense of Kobayashi. This new work focuses on some basic properties of this generalized metric.

On the other hand, the classical Carathéodory metric is another generalized metric which is always less than or equal to the hyperbolic metric. The Carathéodory metric in a domain is the largest pull back of the hyperbolic metric. Similar to the case of the Kobayashi metric, the Carathéodory metric also agrees with the hyperbolic metric on simply connected domains. Furthermore, it satisfies the generalized Schwarz lemma for holomorphic function between two domains. Analogous to the Carathéodory metric, in this paper, we generalize the Hurwitz metric and study its basic properties.

Rest of this document is organized as follows: Section 2 contains preliminary information including terminology, definitions and well known results. We define the generalized Hurwitz metric in the sense of Carathéodory in Sect. 3 and study its various properties including distance decreasing property for special class of holomorphic function between two domains. Finally, Sect. 4 is devoted to the distance between two points induced by the generalized Hurwitz metric.

2 preliminaries

Throughout the paper, unless it is specified, we assume that Ω is an arbitrary domain and Y is a proper subdomain in \mathbb{C} , the complex plane. Symbolically, we write $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$. We denote $\mathcal{H}(\Omega, Y)$ by the set of all holomorphic functions from Ω into Y . For a fixed $w \in \Omega$, we define the following notation:

$$\mathcal{H}_w^s(\Omega, Y) = \{h \in \mathcal{H}(\Omega, Y), h(w) = s, h(z) \neq s \text{ for all } z \in \Omega \setminus \{w\}\}.$$

The open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by \mathbb{D} . The family $\{h \in \mathcal{H}_0^w(\mathbb{D}, Y) : h'(0) > 0\}$ is known as the *Hurwitz family*. More about the Hurwitz family and several other classes of holomorphic functions analogous to the Hurwitz family are discussed in [6]. By setting

$$F'(0) = r_Y(w) = \max\{h'(0) : h \in \mathcal{H}_0^w(\mathbb{D}, Y), h'(0) > 0\},$$

the *Hurwitz density* is defined as

$$\eta_Y(w) = \frac{2}{F'(0)} = \frac{2}{r_Y(w)}.$$

An equivalent definition of the Hurwitz density can be found in [1]. A domain Y is said to be hyperbolic provided its complement $\mathbb{C} \setminus Y$ contains at least two points. The supremum of $\{|h'(0)| : h \in \mathcal{H}(\mathbb{D}, Y)\}$ leads to the definition of the hyperbolic density. Indeed, for any point $w \in Y$ the *hyperbolic density* λ_Y is defined as

$$\lambda_Y(w) = \frac{2}{|g'(0)|},$$

where g is a universal covering mapping from D onto Y . Note that existence of such g is guaranteed by the uniformization theorem. Analogous to the hyperbolic density, we now describe the maximizer for the Hurwitz family. For $s \in Y$, there exists a holomorphic covering map $F : \mathbb{D} \setminus \{0\} \rightarrow Y \setminus \{s\}$ which extends to \mathbb{D} holomorphically in such a way that $F(0) = s$ and $F'(0) > 0$. This is determined by the subgroup generated by the curve, namely the circle centered at s and radius ρ , for some $\rho > 0$, in $Y \setminus \{s\}$ of the fundamental group of $Y \setminus \{s\}$. The function F is the unique extremal function for the Hurwitz-extremal problem $\max\{h'(0) : h \in \mathcal{H}_0^w(\mathbb{D}, Y), h'(0) > 0\}$ and is defined as the *Hurwitz covering map* (see [6]).

3 Carathéodory Density of the Hurwitz Metric

In [1], by adopting the idea of the Kobayashi metric, we generalized the Hurwitz density for a domain $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$ as follows:

$$\eta_\Omega^Y(w) = \inf \frac{\eta_Y(s)}{|h'(s)|},$$

where η_Y is the Hurwitz density on Y and the infimum is taken over all $h \in \mathcal{H}(Y, \Omega)$ satisfying $h(s) = w, h(t) \neq w$ for all $t \in Y \setminus \{s\}$, and $h'(s) \neq 0$. We name the quantity η_Ω^Y by the *Kobayashi density of the Hurwitz metric of Ω relative to Y* .

As stated at the end of Sect. 1, this section is devoted to the introduction of a new density that generalizes the Hurwitz density in the sense of Carathéodory. This is defined as follows:

Definition 3.1 Let $w \in \Omega \subsetneq \mathbb{C}$. For an element $s \in \mathbb{D}$, we define a new quantity

$$\mathcal{C}_\Omega^{\mathbb{D},s}(w) = \sup \eta_{\mathbb{D}}(h(w))|h'(w)|, \tag{3.1}$$

where the supremum is taken over all $h \in \mathcal{H}(\Omega, \mathbb{D})$ such that $h(w) = s, h(z) \neq s$ for all $z \in \Omega \setminus \{w\}$, i.e. for all $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$. We call this quantity by the *Carathéodory density of the Hurwitz metric of Ω relative to \mathbb{D}* . Setting $\mathcal{C}_\Omega^{\mathbb{D}} := \mathcal{C}_\Omega^{\mathbb{D},0}$.

- Remark 3.2** (1) Note that on simply connected domains the Hurwitz density agrees with the hyperbolic density, so one can replace $\eta_{\mathbb{D}}$ by the hyperbolic density $\lambda_{\mathbb{D}}$ in Definition 3.1.
- (2) If $\Omega = \mathbb{C}$, then by Liouville’s theorem, the only holomorphic function from Ω into \mathbb{D} is a constant function, which does not belong to the class $\mathcal{H}_w^s(\Omega, \mathbb{D})$. Hence, it can be defined that $\mathcal{C}_{\mathbb{C}}^{\mathbb{D},s}(w) = 0$ when the set $\mathcal{H}_w^s(\Omega, \mathbb{D})$ becomes empty. It suggests us to assume that $\mathcal{H}_w^s(\Omega, Y) \neq \emptyset$ throughout the paper for an arbitrary base domain $Y \subsetneq \mathbb{C}$.

The first basic property of the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D},s}$ is that the supremum is attained by some holomorphic function $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$ in (3.1).

Proposition 3.3 *Let $\Omega \subsetneq \mathbb{C}$ be a domain and $\mathcal{H}_w^0(\Omega, \mathbb{D}) \neq \emptyset$. Then, the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D}}$ can be computed by the formula:*

$$\mathcal{C}_{\Omega}^{\mathbb{D}}(w) = 2 \max\{|h'(w)| : h \in \mathcal{H}_w^0(\Omega, \mathbb{D})\}.$$

Proof Since the members of the family $\mathcal{H}_w^0(\Omega, \mathbb{D})$ are uniformly bounded by 1, by Montel’s theorem, $\mathcal{H}_w^0(\Omega, \mathbb{D})$ is a normal family. By Definition 3.1, there exists a sequence of holomorphic functions $h_n \in \mathcal{H}_w^0(\Omega, \mathbb{D})$ such that $2|h'_n(w)| \rightarrow \mathcal{C}_{\Omega}^{\mathbb{D}}(w)$, since $h_n(w) = 0$ and $\eta_{\mathbb{D}}(0) = \lambda_{\mathbb{D}}(0) = 2$. Furthermore, by the open mapping theorem, there exists a subsequence h_{n_k} of h_n which converges to either an open map h or a constant map. Since $h_n \in \mathcal{H}(\Omega, \mathbb{D})$, it follows that $|h(z)| \leq 1$ for all $z \in \Omega$. Note that, if $h(z)$ attains 1 for some $z \in \Omega$, then by the maximum modulus principle, $|h| = 1$, contradicting to the fact that $h(w) = 0$. Moreover, by Hurwitz theorem, there exists an $N \in \mathbb{N}$ such that h_{n_k} and h have the same number of zeros for all $n_k \geq N$ in some neighborhood of w . Since $h(z) \neq 0$ for all $z \in \Omega \setminus \{w\}$, we conclude by the uniqueness of limit that $2|h'(w)| = \mathcal{C}_{\Omega}^{\mathbb{D}}(w)$, which completes the proof. \square

Remark 3.4 By a suitable composition of the disk automorphism with the function obtained in Proposition 3.3, we can prove the existence of the holomorphic function h in Definition 3.1 when $s \neq 0$.

Alike to the case of coinciding of the hyperbolic and Carathéodory density on simply connected domains, we now prove that the Hurwitz density η_{Ω} and the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D},s}$ too agree on simply connected domains Ω .

Proposition 3.5 *If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D},s}$ coincides with the Hurwitz density η_{Ω} as well as with the Kobayashi density of the Hurwitz metric $\eta_{\Omega}^{\mathbb{D}}$. That is, we have $\mathcal{C}_{\Omega}^{\mathbb{D},s} \equiv \eta_{\Omega} \equiv \eta_{\Omega}^{\mathbb{D}}$.*

Proof By the distance decreasing property of the Hurwitz density (see [6, Theorem 6.1]), for a point $w \in \Omega$ and for any $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$ we have $\eta_{\mathbb{D}}(h(w))|h'(w)| \leq \eta_{\Omega}(w)$. By taking supremum over all $h \in \mathcal{H}_w^s(\Omega, \mathbb{D})$, in one hand, we obtain $\mathcal{C}_{\Omega}^{\mathbb{D},s}(w) \leq \eta_{\Omega}(w)$. On the other hand, to prove the reverse inequality, we consider the conformal homeomorphism $f : \Omega \rightarrow \mathbb{D}$ which is guaranteed by Riemann mapping theorem. By [6, Corollary 6.2], it follows that

$$\eta_{\Omega}(w) = \eta_{\mathbb{D}}(h(w))|h'(w)| \leq \mathcal{C}_{\Omega}^{\mathbb{D},s}(w),$$

where the inequality holds by Definition 3.1. Thus, we have the identity $\mathcal{C}_{\Omega}^{\mathbb{D},s} \equiv \eta_{\Omega}$.

The second required identity follows from [1, Corollary 3.10], completing the proof. \square

Due to [1, Corollary 3.10], the Kobayashi density of the Hurwitz metric $\eta_{\Omega}^{\mathbb{D}}$ and the Hurwitz density η_{Ω} both agree on any domain Ω , whereas in the following result we show that on non-simply connected domains the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D},s}$ is strictly less than the Hurwitz density η_{Ω} .

Proposition 3.6 *Let $\Omega \subsetneq \mathbb{C}$ be a non-simply connected domain and $\mathcal{C}_{\Omega}^{\mathbb{D}} > 0$. Then for an element $w \in \Omega$ we have the strict inequality: $\mathcal{C}_{\Omega}^{\mathbb{D}}(w) < \eta_{\Omega}(w)$.*

Proof Let $w \in \Omega$. Since $\Omega \subsetneq \mathbb{C}$, there exists a Hurwitz covering map $g : \mathbb{D} \rightarrow \Omega$ with $g(0) = w$. By Proposition 3.3, there exists a function $h \in \mathcal{H}_w^0(\Omega, \mathbb{D})$ such that

$$\mathcal{C}_{\Omega}^{\mathbb{D}}(w) = 2|h'(w)| = \eta_{\mathbb{D}}(h(w))|h'(w)| \tag{3.2}$$

holds, since $h(w) = 0$ and $\eta_{\mathbb{D}}(0) = \lambda_{\mathbb{D}}(0) = 2$. Thus, we observe that the composition $h \circ g$ is a holomorphic function from \mathbb{D} to \mathbb{D} that fixes the origin. Since Ω is non-simply connected, the covering map g cannot be one–one and hence the composition $h \circ g$ can never be conformal. Thus, by the classical Schwarz lemma we conclude the strict inequality

$$\lambda_{\mathbb{D}}((h \circ g)(0))|(h \circ g)'(0)| < \lambda_{\mathbb{D}}(0).$$

Note that the hyperbolic density coincides with the Hurwitz density on simply connected hyperbolic domains (see [6, p. 15]). Therefore, it follows that

$$\eta_{\mathbb{D}}((h \circ g)(0))|(h \circ g)'(0)| < \eta_{\mathbb{D}}(0). \tag{3.3}$$

Since g is a Hurwitz covering map, by [6, Theorem 6.1], we have the equality

$$\eta_{\Omega}(g(0))|g'(0)| = \eta_{\mathbb{D}}(0). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain from (3.2) that

$$\mathcal{C}_{\Omega}^{\mathbb{D}}(w) = \eta_{\mathbb{D}}(h(w))|h'(w)| = \eta_{\mathbb{D}}((h \circ g)(0)) \frac{|(h \circ g)'(0)|}{|g'(0)|} < \frac{\eta_{\mathbb{D}}(0)}{|g'(0)|} = \eta_{\Omega}(w),$$

where the second equality follows by the chain rule. \square

Since the Hurwitz density can be defined on a proper subdomain of the complex plane, a natural way of further generalizing the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{\mathbb{D},s}$ by changing the base domain from the unit disk to a proper subdomain Y of \mathbb{C} . The definition is as follows:

Definition 3.7 Let $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be domains. For $w \in \Omega$ and $s \in Y$, the Carathéodory density of the Hurwitz metric of Ω relative to the base domain Y is defined as

$$\mathcal{C}_\Omega^{Y,s}(w) = \sup \eta_Y(h(w))|h'(w)|,$$

where the supremum is taken over all $h \in \mathcal{H}(\Omega, Y)$ such that $h(w) = s$, $h(z) \neq s$ for all $z \in \Omega \setminus \{w\}$, i.e. for all $h \in \mathcal{H}_w^s(\Omega, Y)$.

In [1], we have noticed that the Kobayashi density of the Hurwitz metric η_Ω^Y exceeds over the Hurwitz density η_Ω , whereas in the case of the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$, we prove that it lacks the Hurwitz density on proper subdomains of \mathbb{C} .

Proposition 3.8 Let Ω and Y be proper subdomains of the complex plane \mathbb{C} . If for an element $s \in Y$, we assume $\mathcal{C}_\Omega^{Y,s} > 0$ then

$$\eta_\Omega(w) \geq \mathcal{C}_\Omega^{Y,s}(w)$$

holds for every $w \in \Omega$.

Proof By the distance decreasing property of the Hurwitz density, for $w \in \Omega$, $s \in Y$ and for any $h \in \mathcal{H}_w^s(\Omega, Y)$ we have

$$\eta_Y(h(w))|h'(w)| \leq \eta_\Omega(w).$$

Taking the supremum over all $h \in \mathcal{H}_w^s(\Omega, Y)$ on both sides, we obtain

$$\mathcal{C}_\Omega^{Y,s}(w) \leq \eta_\Omega(w).$$

Since $w \in \Omega$ was arbitrary, we conclude the proof as desired. □

Recall that the Hurwitz density and the hyperbolic density agree on simply connected domains. Analogous to this, we now prove that upon some specific conditions the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$ and the Hurwitz density η_Ω coincide and in a more special situation, they also coincide with the Kobayashi density of the Hurwitz metric η_Ω^Y .

Proposition 3.9 Let $\Omega, Y \subsetneq \mathbb{C}$ be domains. Suppose that for every $s \in Y$ there exists a point $w \in \Omega$ and a holomorphic covering map $g_s : \Omega \setminus \{w\} \rightarrow Y \setminus \{s\}$ which extends to a holomorphic function $g : \Omega \rightarrow Y$ with $g(w) = s$ and $g'(w) \neq 0$. If $\mathcal{C}_\Omega^{Y,s} > 0$, then

$$\mathcal{C}_\Omega^{Y,s} \equiv \eta_\Omega.$$

In particular, when $Y = \Omega$, we have

$$\mathcal{C}_\Omega^{\Omega,w} \equiv \eta_\Omega \equiv \zeta_\Omega^\Omega.$$

Proof By the distance decreasing property of the Hurwitz density (see the second part of [6, Theorem 6.1]), we have

$$\eta_Y(g(w))|g'(w)| = \eta_\Omega(w).$$

Now, plugging the holomorphic covering map g_s into Definition 3.7, on the one hand we obtain

$$\mathcal{C}_\Omega^{Y,s}(w) \geq \eta_Y(g(w))|g'(w)| = \eta_\Omega(w).$$

On the other hand, the reverse inequality follows from Proposition 3.8. Since w is arbitrary, the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$ and the Hurwitz density η_Ω both agree over Ω .

The proof of the second part is a combination of the above identity that we just proved and the identity proved in [1, Proposition 3.9]. □

An instant corollary to Proposition 3.9 is that on simply connected domains both the Hurwitz density η_Ω and the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$ agree.

Corollary 3.10 *If $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $Y \subsetneq \mathbb{C}$ be an arbitrary domain, then*

$$\mathcal{C}_\Omega^{Y,s} \equiv \eta_\Omega,$$

where $s \in Y$.

Proof Since $Y \subsetneq \mathbb{C}$, there exists a Hurwitz covering map $g : \mathbb{D} \rightarrow Y$. Now, $\Omega \subsetneq \mathbb{C}$ being a simply connected domain, by Riemann mapping theorem, we would get a conformal mapping $h : \Omega \rightarrow \mathbb{D}$ with $h(w) = 0$ and $h'(w) > 0$ for some $w \in \Omega$. Then the composition $g \circ h$ is a holomorphic covering map from $\Omega \setminus \{w\}$ onto $Y \setminus \{s\}$ for some $s \in Y$ that can be extended from Ω onto Y by taking w to s . The proof now follows by Proposition 3.9. □

Recall that the hyperbolic density λ_Ω , the Hurwitz density η_Ω and the Kobayashi density of the Hurwitz metric η_Ω^Y satisfy the distance decreasing property. Note that, in the case of the hyperbolic metric the distance decreasing property is also known as the generalized Schwarz-Pick lemma. Alike to these properties we here show that the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$ too satisfies the distance decreasing property.

Theorem 3.11 (Distance decreasing property) *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$ be domains. If there exists a holomorphic function f from Ω_1 into Ω_2 with $f(a) = b$, $f(s) \neq b$ for all $s \in \Omega_1 \setminus \{a\}$, then*

$$\mathcal{C}_{\Omega_2}^{Y,c}(f(a))|f'(a)| \leq \mathcal{C}_{\Omega_1}^{Y,c}(a),$$

where $c \in Y$.

Proof If $\mathcal{H}_b^c(\Omega_2, Y) = \emptyset$, then $\mathcal{C}_{\Omega_2}^{Y,c} = 0$ and hence there is nothing to prove. Therefore, without loss of generality we assume that $\mathcal{H}_b^c(\Omega_2, Y) \neq \emptyset$.

By the definition of $\mathcal{C}_{\Omega_2}^{Y,c}(b)$, for every $\epsilon > 0$ there exists a holomorphic function h from Ω_2 into Y with $h(b) = c$, $h(s) \neq c$ for all $s \in \Omega_2 \setminus \{b\}$ for some $c \in Y$, such that

$$\mathcal{C}_{\Omega_2}^{Y,c}(b) - \epsilon \leq \eta_Y(h(b))|h'(b)|. \tag{3.5}$$

Suppose that f is a holomorphic function from Ω_1 into Ω_2 with $f(a) = b$, $f(s) \neq b$ for all $s \in \Omega_1 \setminus \{a\}$. Now the composition function $h \circ f \in \mathcal{H}(\Omega_1, Y)$ satisfies $(h \circ f)(a) = c$. Furthermore, $(h \circ f)(t) \neq c$ for all $t \in \Omega_1 \setminus \{a\}$ as $b \notin f(\Omega_1) \setminus \{a\}$ and $c \notin h(\Omega_2) \setminus \{b\}$. Now, by plugging the map $h \circ f$ into the definition of $\mathcal{C}_{\Omega_1}^{Y,c}(a)$, it follows that

$$\mathcal{C}_{\Omega_1}^{Y,c}(a) \geq \eta_Y((h \circ f)(a))|(h \circ f)'(a)| = \eta_Y(h(b))|h'(b)||f'(a)|. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\mathcal{C}_{\Omega_1}^{Y,c}(a) \geq (\mathcal{C}_{\Omega_2}^{Y,c}(b) - \epsilon)|f'(a)|$$

which holds for every $\epsilon > 0$. Letting $\epsilon \rightarrow 0$, we have the desired inequality. □

As a direct consequence of Theorem 3.11, we obtain the conformal invariance property and monotonicity property of the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{Y,s}$ as follows:

Corollary 3.12 (Conformal invariance property) *If f is a conformal mapping from a domain $\Omega_1 \subset \mathbb{C}$ onto another domain $\Omega_2 \subset \mathbb{C}$, then for a base domain $Y \subsetneq \mathbb{C}$ we have*

$$\mathcal{C}_{\Omega_2}^{Y,s}(f(w))|f'(w)| = \mathcal{C}_{\Omega_1}^{Y,s}(w),$$

for all $w \in \Omega_1$ and $s \in Y$.

Corollary 3.13 (Domain monotonicity property) *If $\Omega_1 \subsetneq \Omega_2$ and Y are domains as in Theorem 3.11, then $\mathcal{C}_{\Omega_2}^{Y,s}(w) \leq \mathcal{C}_{\Omega_1}^{Y,s}(w)$ for all $w \in \Omega_1$ and $s \in Y$.*

Until now, we studied the properties of the Carathéodory density of the Hurwitz metric $\mathcal{C}_{\Omega}^{Y,s}$ by fixing the base domain Y . For two different base domains, the comparison result is given below.

Theorem 3.14 *Let $Y_1, Y_2 \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be subdomains. If for every point $b \in Y_2$, there exist a point $a \in Y_1$ and a holomorphic covering map $g_b : Y_1 \setminus \{a\} \rightarrow Y_2 \setminus \{b\}$ which extends to the holomorphic function with $g_b(a) = b$ and $g_b'(a) \neq 0$, then*

$$\mathcal{C}_{\Omega}^{Y_1,a}(w) \leq \mathcal{C}_{\Omega}^{Y_2,b}(w)$$

for all $w \in \Omega$.

Proof By the distance decreasing property for Hurwitz density, it follows that

$$\eta_{Y_2}(g_b(a))|g'_b(a)| = \eta_{Y_1}(a) \tag{3.7}$$

since g_b is the extended holomorphic covering map from Y_1 onto Y_2 . Let $\epsilon > 0$ be arbitrary.

If $\mathcal{H}_w^a(\Omega, Y_1) = \emptyset$, then $\mathcal{C}_\Omega^{Y_1,a} = 0$ and hence there is nothing to prove. Therefore, without loss of generality we assume that $\mathcal{H}_w^a(\Omega, Y_1) \neq \emptyset$.

By the definition of $\mathcal{C}_\Omega^{Y_1,a}$, for $a \in Y_1$ and $w \in \Omega$, there exists a function $h \in \mathcal{H}_w^a(\Omega, Y_1)$ such that

$$\mathcal{C}_\Omega^{Y_1,a}(w) \leq \eta_{Y_1}(h(w))|h'(w)| + \epsilon. \tag{3.8}$$

Now we notice that the composed function $g_b \circ h \in \mathcal{H}(\Omega, Y_2)$ satisfies $(g_b \circ h)(w) = b$, $(g_b \circ h)(z) \neq b$ for all $z \in \Omega \setminus \{w\}$. Hence, $g_b \circ h \in \mathcal{H}_w^b(\Omega, Y_2)$. Applying $g_b \circ h$ in the definition of $\mathcal{C}_\Omega^{Y_2,b}(w)$, we conclude that

$$\mathcal{C}_\Omega^{Y_2,b}(w) \geq \eta_{Y_2}((g_b \circ h)(w))|(g_b \circ h)'(w)| = \eta_{Y_2}(g_b(a))|g'_b(a)||h'(w)| \tag{3.9}$$

for all $w \in \Omega$. Combining (3.7), (3.8), (3.9) and applying the chain rule, we obtain

$$\mathcal{C}_\Omega^{Y_1,a}(w) \leq \eta_{Y_1}(a)|h'(w)| + \epsilon = \eta_{Y_2}(g_b(a))|g'_b(a)||h'(w)| + \epsilon \leq \mathcal{C}_\Omega^{Y_2,b}(w) + \epsilon$$

for all $w \in \Omega$. Since ϵ is arbitrary, we can let it approach to zero to obtain the desired inequality. □

Corollary 3.15 *If Y_1 and Y_2 are conformally equivalent proper subdomains of \mathbb{C} and Ω is an arbitrary subdomain of \mathbb{C} , then*

$$\mathcal{C}_\Omega^{Y_1,a}(w) = \mathcal{C}_\Omega^{Y_2,b}(w)$$

holds for every $w \in \Omega$ and for some $a \in Y_1, b \in Y_2$.

Proof We consider the inverse image of the conformal mapping in Theorem 3.14 to obtain the reverse inequality $\mathcal{C}_\Omega^{Y_1,a}(w) \geq \mathcal{C}_\Omega^{Y_2,b}(w)$. □

4 A Distance Function

In this section, we consider the usual distance function associated with the Carathéodory density of the Hurwitz metric $\mathcal{C}_\Omega^{Y,s}$ for the domains $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

Definition 4.1 Let $Y \subsetneq \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be domains. For $w_1, w_2 \in \Omega$ and $s \in Y$, define

$$\mathcal{C}_\Omega^Y(w_1, w_2) = \inf \int_\gamma \mathcal{C}_\Omega^{Y,s}(w) |dw|,$$

where the infimum is taken over all rectifiable paths $\gamma \subset \Omega$ joining w_1 to w_2 . If \mathcal{C}_Ω^Y defines a metric, then we say $(\Omega, \mathcal{C}_\Omega^Y)$ a metric space.

It is easy to see from Definition 4.1 that $\mathcal{C}_\Omega^Y(w_1, w_1) = 0$ and $\mathcal{C}_\Omega^Y(w_1, w_2) = \mathcal{C}_\Omega^Y(w_2, w_1)$ for any $w_1, w_2 \in \Omega$. Further, it can also be verified that \mathcal{C}_Ω^Y satisfies the triangle inequality. Hence, at least we can say that \mathcal{C}_Ω^Y is a pseudo-metric. At present we do not know whether \mathcal{C}_Ω^Y defines a metric or not. However, we have a partial solution to this whenever $\Omega \subset Y$.

Theorem 4.2 If $\Omega \subset Y \subsetneq \mathbb{C}$ are domains, then $(\mathcal{C}_\Omega^Y, \Omega)$ becomes a metric space.

Proof Since \mathcal{C}_Ω^Y is a pseudo-metric on Ω , it is enough to show that $\mathcal{C}_\Omega^Y(w_1, w_2) > 0$ for two distinct points $w_1, w_2 \in \Omega$. Let γ be an arbitrary rectifiable curve joining w_1 to w_2 in Ω . Since $\Omega \subset Y$, plugging the inclusion mapping $i \in \mathcal{H}_w^w(\Omega, Y)$ into the definition of $\mathcal{C}_\Omega^{Y,w}(w)$, we conclude that

$$\int_\gamma \mathcal{C}_\Omega^{Y,w}(w) |dw| \geq \int_\gamma \eta_Y(i(w)) |i'(w)| |dw| = \int_\gamma \eta_Y(w) |dw|$$

By the definition of Hurwitz distance (see [1]) between two points, it follows that

$$\int_\gamma \mathcal{C}_\Omega^{Y,w}(w) |dw| > \eta_Y(w_1, w_2).$$

Now, taking infimum over γ , we obtain

$$\mathcal{C}_\Omega^Y(w_1, w_2) \geq \eta_Y(w_1, w_2) > 0,$$

where the last inequality follows from [1, Theorem 2.3]. Hence $(\Omega, \mathcal{C}_\Omega^Y)$ defines a metric space, completing the proof. \square

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