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On the Irregularity of π-Permutation Graphs, Fibonacci Cubes, and Trees

Yaser Alizadeh[1] · **Emeric Deutsch**[2] · **Sandi Klavžar**[3,4,5]

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Abstract

The irregularity of a graph G is the sum of $|\deg(u) - \deg(v)|$ over all edges uv of G. In this paper, this invariant is considered on π-permutation graphs, Fibonacci cubes, and trees. An upper bound on the irregularity of π-permutations graphs is given, and π-permutation graphs that attain the equality are characterized. The concept of the irregularity is extended to arbitrary edge subsets and applied to permutation edges of π-permutation graphs. An exact formula for the irregularity of Fibonacci cubes is proved. An upper bound on the irregularity of trees in terms of the diameter is given, and trees that attain the equality are characterized.

Keywords Irregularity of graph · π-Permutation graph · Fibonacci cube · Tree

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✉ Sandi Klavžar
sandi.klavzar@fmf.uni-lj.si

Yaser Alizadeh
y.alizadeh@hsu.ac.ir

Emeric Deutsch
EmericDeutsch@msn.com

1 Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran

2 Polytechnic Institute of New York University, Brooklyn, USA

3 Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia

4 Faculty of Natural Sciences and Mathematics, University of Maribor, Maribor, Slovenia

5 Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

1 Introduction

The degree of a vertex v of a graph $G = (V(G), E(G))$ is denoted by $\deg_G(v)$. Graphs in which all vertices have the same degree, that is, regular graphs, are in the center of interest of the graph theory community. If G is not regular, then it is called *irregular*, cf. [5], and one is interested in how irregular it is. For this sake, let the *imbalance* $\mathrm{imb}_G(e)$ of an edge $e = uv \in E(G)$ be defined by

$$\mathrm{imb}_G(e) = |\deg_G(u) - \deg_G(v)|\,.$$

The imbalance of an edge is thus a local measure of non-regularity of a given graph, cf. [7], where Ramsey problems with repeated degrees were investigated. To measure graph's global non-regularity, different approaches have been proposed; they are nicely presented in two recent papers [4,31]. One of the most natural such measures is the *irregularity* $\mathrm{irr}(G)$ of G (lately also called the *Albertson index*) defined as [6]:

$$\mathrm{irr}(G) = \sum_{uv \in E(G)} |\deg_G(u) - \deg_G(v)| = \sum_{e \in E(G)} \mathrm{imb}_G(e)\,.$$

Let us explicitly mention some of the papers in which the irregularity has been studied. In [29] related extremal problems are proved; the paper [40] reports several bounds on irregularity; the paper [15] gives a spectral bound for graph irregularity that improves a bound from [40]; in [20] bipartite graphs having maximum possible irregularity are determined; the irregularity of some graph families that are important in chemistry is reported in [3]; for the irregularity of cacti see [25]; see also [30] for the role of irregularity indices used as molecular descriptors. We also point out that graphs in which $\mathrm{imb}_G(e) = 1$ holds for all edges have been recently investigated in [18] and named stepwise irregular graphs.

In this paper we focus on the irregularity of three classes of graphs: π-permutation graphs, Fibonacci cubes, and trees. In the next section we give an upper bound on the irregularity of π-permutation graphs which is, roughly speaking, stronger by a factor of 4 than the corresponding bound for general graphs. We also characterize the π-permutation graphs that attain the equality. In Sect. 3 we extend the concept of the irregularity to arbitrary edge subsets and prove a related upper bound for the irregularity of permutation edges in π-permutation graphs. In Sect. 4 we prove an exact formula for the irregularity of Fibonacci cubes. In the final section we prove an upper bound for the irregularity of trees in terms of the diameter and characterize the graphs that attain equality. We also give bounds for the irregularity of π-permutation graphs over trees.

In the rest of the introduction we give some further, basic definitions used in this paper. All graphs in this paper are simple and connected. The order (= number of vertices) and the size (= number of edges) of a graph $G = (V(G), E(G))$ are denoted with $n(G)$ and $m(G)$, respectively. If $W \subseteq V(G)$, then $\langle W \rangle$ denotes the subgraph of G induced by W. The minimum and the maximum degrees of vertices from G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex $v \in V(G)$ with $\deg_G(v) = n(G) - 1$ is a *universal vertex* of G. The distance $d_G(u, v)$ between vertices u and v of a graph G

is the number of edges on a u, v-geodesic. The *diameter* $\mathrm{diam}(G)$ of G is the length of a longest geodesic in G. For a positive integer n we will denote the set $\{1, \ldots, n\}$ with $[n]$.

2 π-Permutation Graphs

Let G' and G'' be disjoint copies of a graph G, and let $\pi : V(G') \to V(G'')$ be a bijection; in other words, π is a permutation on $V(G)$. The π-*permutation graph* G^π of G has the vertex set $V(G^\pi) = V(G') \cup V(G'')$ and the edge set $E(G^\pi) = E(G') \cup E(G'') \cup E_G^\pi$, where

$$E_G^\pi = \{uv : u \in V(G'), v \in V(G''), v = \pi(u)\}.$$

Hence, a π-permutation graph is obtained from two disjoint copies of a given graph by adding a matching between them. This concept was introduced half a century ago in [11] and further investigated in a series of papers including [10,14,17,19,39]. We point out that the term "permutation graph" is also frequently used for intersection graphs of the lines representing a permutation; see, for example, [13]. In this paper we are only interested in the former interpretation which will be emphasized by speaking of π-permutation graphs and by the notation G^π.

Let G^π be a π-permutation and G' and G'' be two isomorphic copies of G in G^π. If $u \in V(G)$, then the vertices corresponding to u in G' and G'' will be denoted, respectively, by u' and u''. We begin with the following simple result.

Proposition 2.1 *If G is a graph and π is a permutation on $V(G)$, then*

$$\mathrm{irr}(G^\pi) = 2 \cdot \mathrm{irr}(G) + \sum_{u \in V(G)} |\deg_G(u) - \deg_G(\pi(u))|.$$

In particular, if π induces an automorphism of G, then $\mathrm{irr}(G^\pi) = 2 \cdot \mathrm{irr}(G)$.

Proof Let G' and G'' be the isomorphic copies of G in G^π. If $u \in V(G)$, then $\deg_{G^\pi}(u') = \deg_{G^\pi}(u'') = \deg_G(u) + 1$, and, consequently, $E(G')$ and $E(G'')$ each contribute $\mathrm{irr}(G)$ to $\mathrm{irr}(G^\pi)$. For the same reason, each matching edge $u'\pi(u')$ contributes $|\deg_G(u) - \deg_G(\pi(u))|$ to $\mathrm{irr}(G^\pi)$, and, consequently, the first assertion follows. The second assertion then follows because automorphisms preserve degrees. □

In [2,36] it was proved in two different ways that if G is a graph of order $n = n(G)$, then

$$\mathrm{irr}(G) \le \left\lfloor \frac{n}{3} \right\rfloor \left\lceil \frac{2n}{3} \right\rceil \left(\left\lceil \frac{2n}{3} \right\rceil - 1 \right) = \left\lfloor \frac{n}{3} \right\rfloor \left(n - \left\lfloor \frac{n}{3} \right\rfloor \right) \left(n - \left\lfloor \frac{n}{3} \right\rfloor - 1 \right). \quad (1)$$

Moreover, let $KS_{p,q}$, $p, q \ge 1$, be the *clique-star graph*, that is, the join of a complete graph K_p and an edge-less graph $\bar{K}_q$. (The *join* of graphs G and H is the graph obtained from the disjoint union of G and H by adding all possible edges between vertices of G and vertices of H.) Then, the equality in (1) is attained if and only if G is $KS_{\lfloor \frac{n}{3} \rfloor, \lceil \frac{2n}{3} \rceil}$ or, if $n \equiv 2 \bmod 3$, G is $KS_{\lceil \frac{n}{3} \rceil, \lfloor \frac{2n}{3} \rfloor}$, see [2, Theorem 2.2]. This immediately implies the following lemma which we need in the proof of our below result analogous to (1) for π-permutation graphs.

Lemma 2.2 *If G is a graph of maximum irregularity among all graphs of order n, then G has at most $\lceil \frac{n}{3} \rceil$ universal vertices.*

Theorem 2.3 *If G is a graph of order $n = n(G)$ and π is a permutation on $V(G)$, then*

$$\mathrm{irr}(G^{\pi}) \leq 2\left\lceil \frac{n}{3} \right\rceil \left(\left\lfloor \frac{2n}{3} \right\rfloor^{2} - 1 \right).$$

Moreover, the equality holds if and only if $G = KS_{\lceil \frac{n}{3} \rceil, \lfloor \frac{2n}{3} \rfloor}$.

Proof Let a graph G and a permutation π of $V(G)$ be selected such that the graph G^{π} has the maximum irregularity among all permutation graphs over graphs of order n. Let $V(G) = U \cup W$, where $U = \{u_1, \ldots, u_s\}$ is the set of universal vertices of G and $W = V(G) \setminus U = \{w_1, \ldots, w_{n-s}\}$. Without loss of generality, we may assume that $\deg_G(w_1) \leq \cdots \leq \deg_G(w_{n-s})$. Under these assumptions and with Lemma 2.2 in mind, we may assume without loss of generality that

$$\pi(u_i') = w_i'', \pi(w_i') = u_i'', i \in [s], \text{ and } \pi(w_{s+i}') = w_{n-i+1}'', i \in [n-s]. \quad (2)$$

Let G' be the spanning subgraph of G obtained from G by removing all the edges of $\langle W \rangle$. If $e = uw \in E(G)$, where $u \in U$ and $w \in W$, then

$$\mathrm{imb}_{G'}(e) = \mathrm{imb}_{G}(e) + \deg_{\langle W \rangle}(w). \quad (3)$$

If $e \in E(\langle W \rangle)$, then $\mathrm{imb}_G(e) \leq n - s - 3$ and consequently

$$\sum_{e \in E(\langle W \rangle)} \mathrm{imb}_G(e) \leq \frac{1}{2} \sum_{w \in W} \deg_{\langle W \rangle}(w)(n - s - 3). \quad (4)$$

Setting

$$X = \sum_{e \in E_G^{\pi}} \mathrm{imb}_{(G')^{\pi}}(e) - \mathrm{imb}_{G^{\pi}}(e),$$

and having in mind that $E_G^\pi = E_{G'}^\pi$, we can do the following estimations:

$$
\begin{aligned}
X &= 2\sum_{i=1}^{s}\left(\mathrm{imb}_{(G')^\pi}(u_i'w_i'') - \mathrm{imb}_{G^\pi}(u_i'w_i'')\right) \\
&\quad + 2\sum_{i=1}^{n-s}|\mathrm{imb}_{(G')^\pi}(w_{s+i}'w_{n-i+1}'') - \mathrm{imb}_{G^\pi}(w_{s+i}'w_{n-i+1}'')| \\
&\geq 2\sum_{i=1}^{s}\deg_{\langle W\rangle}(w_i) - 2\sum_{i=1}^{n-s}|\deg_{\langle W\rangle}(w_{s+i}) - \deg_{\langle W\rangle}(w_{n-i+1})| \\
&\geq 2\left(\sum_{i=1}^{s}\deg_{\langle W\rangle}(w_i) - \sum_{i=s+1}^{n-s}\deg_{\langle W\rangle}(w_i)\,.\right) \qquad (5)
\end{aligned}
$$

From (3)–(5) we get

$$
\begin{aligned}
\mathrm{irr}((G')^\pi) - \mathrm{irr}(G^\pi) &\geq 2\sum_{i=1}^{n-s}\deg_{\langle W\rangle}(w_i)s - \sum_{i=1}^{n-s}\deg_{\langle W\rangle}(w_i)(n-s-3) \\
&\quad + 2\left(\sum_{i=1}^{s}\deg_{\langle W\rangle}(w_i) - \sum_{i=s+1}^{n-s}\deg_{\langle W\rangle}(w_i)\right) \\
&\geq 2\left(s - \frac{1}{2}(n-s-3) - 1\right)\sum_{i=1}^{n-s}\deg_{\langle W\rangle}(w_i) \\
&= (3s-n+1)\sum_{i=1}^{n-s}\deg_{\langle W\rangle}(w_i)\,.
\end{aligned}
$$

Since G^π has maximum irregularity and the expression $3s-n+1$ is positive for $s \geq \lceil \frac{n}{3} \rceil$, we infer that $\sum_{i=1}^{n-s}\deg_{\langle W\rangle}(w_i) = 0$. Hence, G is a clique-star graph $KS_{s,n-s}$. For a fixed value of s we have

$$
\max\{\mathrm{irr}(KS_{s,n-s}^\pi) :\ \pi \text{ is a permutation}\} = 2s((n-s)^2-1)\,.
$$

If $f(s) = 2s((n-s)^2-1)$, then $f(s)$ is maximized at $s = \lceil \frac{n}{3} \rceil$. Therefore, we conclude that

$$
\mathrm{irr}(G^\pi) \leq 2\left\lceil \frac{n}{3} \right\rceil \left(\left\lfloor \frac{2n}{3} \right\rfloor^2 - 1\right).
$$

□

Note that if $n = n(G)$, then $n(G^\pi) = 2n$. Hence, the upper bound of Theorem 2.3 bounds $\mathrm{irr}(G^\pi)$ from the above with, roughly, $\frac{1}{27}n(G^\pi)^3$. On the other hand, the general bound (1) yields, roughly, $\frac{4}{27}n(G^\pi)^3$. Hence, Theorem 2.3 improves the general upper bound in the case of π-permutation graphs by, roughly speaking, again a factor of 4.

3 Irregularity of Edge Subsets and π-Permutation Graphs

As a variant of the irregularity measure, Abdo et al. [1] suggested to consider the imbalance over all pairs of vertices, introducing in this way the *total irregularity* $\mathrm{irr}_t(G)$ of a graph G with

$$\mathrm{irr}_t(G) = \sum_{\{u,v\}\subseteq V(G)} |\deg_G(u) - \deg_G(v)|\,.$$

The total irregularity has been compared with the irregularity in [12] see also [9,35,38]. However, for our purposes, it is useful to extend the concept of irregularity from the sum of the imbalances of all the edges of a graph to arbitrary edge subsets. More precisely, if $F \subseteq E(G)$, then let

$$\mathrm{irr}_G(F) = \sum_{f\in F} \mathrm{imb}(f)\,.$$

Note that with this notation $\mathrm{irr}(G) = \mathrm{irr}_G(E(G))$ and that Proposition 2.1 reads as:

$$\mathrm{irr}(G^\pi) = 2\cdot \mathrm{irr}(G) + \mathrm{irr}_{G^\pi}(E_G^\pi)\,.$$

Hence, $\mathrm{irr}_{G^\pi}(E_G^\pi)$ is of special interest, and in our next result we give a sharp upper bound for it.

Theorem 3.1 *If G is a graph of order n and π a permutation on $V(G)$, then*

$$\mathrm{irr}_{G^\pi}(E_G^\pi) \le 2\left\lfloor \frac{n-1}{2}\right\rfloor \left\lceil \frac{n-1}{2}\right\rceil\,.$$

Moreover, equality holds if and only if $G = KS_{\lfloor \frac{n-1}{2}\rfloor,\lceil \frac{n+1}{2}\rceil}$ or $G = KS_{\lceil \frac{n-1}{2}\rceil,\lfloor \frac{n+1}{2}\rfloor}$.

Proof Let G^π be a permutation graph that has maximum $\mathrm{irr}_{G^\pi}(E_G^\pi)$ among all G of order n and all permutations on $V(G)$. Moreover, assume that among such graphs, G has maximum number of universal vertices. Let $U = \{u_1,\ldots,u_s\}$ and $W = \{w_1,\ldots,w_{n-s}\}$ be the sets of its universal and its non-universal vertices, respectively, defined just as in the proof of Theorem 2.3. Then, (2) applies also to π.

We claim that $\deg_W(w_i) = 0$ for $i \in [n-s]$. On the contrary, suppose that $\deg_W(w_{n-s}) \ge 1$. Let w_p and w_q be two adjacent vertices in W, where w_{n-s} is adjacent to w_p and non-adjacent to w_q. Consider the following transformation: remove the edge w_pw_q and then add an edge between w_{n-s} and w_q. Let H be the newly obtained graph. Then, we have $\deg_H(w_{n-s}) = \deg_G(w_{n-s}) + 1$ and $\deg_H(w_p) = \deg_G(w_p) - 1$. Moreover, the degrees of the other vertices remain the same. If $\pi(w'_{n-s}) = w''_q$, then $\mathrm{imb}_H(w'_{n-s}\pi(w'_{n-s})) = \mathrm{imb}_G(w'_{n-s}\pi(w'_{n-s})) + 2$; otherwise, $\mathrm{imb}_H(w'_{n-s}\pi(w'_{n-s})) = \mathrm{imb}_G(w'_{n-s}\pi(w'_{n-s})) + 1$.

Also for w_q we have $\mathrm{imb}_H(w'_q\pi(w'_q)) \geq \mathrm{imb}_G(w'_q\pi(w'_q)) - 1$. Therefore, $\mathrm{irr}_H(E^\pi_H) \geq \mathrm{irr}_G(E^\pi_G))$. Then we apply the above transformation until w_{n-s} is adjacent to all vertices of W. As we have assumed that G has the largest possible number of universal vertices, we have a contradiction.

Hence, $\deg_W(w_{n-s}) = 0$ and consequently $\deg_W(w_i) = 0$, $i \in [n-s]$). This implies that $\mathrm{irr}_{G^\pi}(E^\pi_G) = \sum_{i=1}^{s} \mathrm{imb}(u_i w_i) = 2(n-1-s)s$. By Lemma 2.2, $s \leq \lfloor \frac{n}{2} \rfloor$. Hence, $\mathrm{irr}_{G^\pi}(E^\pi_G)$ is maximized when $s = \lfloor \frac{n-1}{2} \rfloor$ or $s = \lceil \frac{n-1}{2} \rceil$ and then $G = KS_{p,q}$ where $p = \lfloor \frac{n-1}{2} \rfloor$ or $p = \lceil \frac{n-1}{2} \rceil$ and $q = n - p$. □

We close this section with a certain sub-additivity result on $\mathrm{irr}_{G^\pi}(E^\pi_G)$, where $\pi\alpha$ denotes the composition of the permutations π and α.

Theorem 3.2 *If π and α are permutations on $V(G)$, then*

$$\mathrm{irr}_{G^{\pi\alpha}}(E^{\pi\alpha}_G) \leq \mathrm{irr}_{G^\pi}(E^\pi_G) + \mathrm{irr}_{G^\alpha}(E^\alpha_G)\,.$$

Proof Set $\beta = \pi\alpha$. Then, having in mind that the degree of each vertex of G^β (as well as of G^π and G^α) is larger by 1 than the degree of its corresponding vertex in G, we can estimate as follows:

$$\begin{aligned}
\mathrm{irr}_{G^\beta}(E^\beta_G) &= \sum_{v\in V(G)} \mathrm{imb}_{G^\beta}(v'\beta(v')) = \sum_{v\in V(G)} |\deg_{G^\beta}(\beta(v')) - \deg_{G^\beta}(v')| \\
&= \sum_{v\in V(G)} |\deg_G(\beta(v)) - \deg_G(v)| \\
&= \sum_{v\in V(G)} |\deg_G(\beta(v)) - \deg_G(\alpha(v)) + \deg_G(\alpha(v)) - \deg_G(v)| \\
&\leq \sum_{v\in V(G)} \big(|\deg_G(\beta(v)) - \deg_G(\alpha(v))| + |\deg_G(\alpha(v)) - \deg_G(v)|\big) \\
&= \sum_{v\in V(G)} |\deg_{G^\pi}(\pi(v)) - \deg_{G^\pi}(v)| \\
&\quad + \sum_{v\in V(G)} |\deg_{G^\alpha}(\alpha(v)) - \deg_{G^\alpha}(v)| \\
&= \mathrm{irr}_{G^\pi}(E^\pi_G) + \mathrm{irr}_{G^\alpha}(E^\alpha_G)
\end{aligned}$$

and we are done. □

4 Fibonacci Cubes

Fibonacci cubes were introduced by Hsu [21] as an interconnection network model. Afterward, they have been studied from different perspectives; the developments until

2013 are summarized in the survey article [23]. Among the subsequent developments on Fibonacci cubes we point out the studies of the structure of their induced hypercubes [16,27,33] and the investigations of their domination invariants [8,22,32]. Moreover, Fibonacci cubes can be recognized in linear time [37]. In this section we add to the literature on the Fibonacci cubes their irregularity.

A *Fibonacci string* of length n is a binary string $b_1 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \le i < n$, that is, a binary string that contains no consecutive 1s. The *Fibonacci cube* Γ_n, $n \ge 1$, is the graph whose vertices are all Fibonacci strings of length n, two vertices being adjacent if they differ in a single coordinate. It is well known that $|V(\Gamma_n)| = F_{n+2}$, where $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, $n \ge 0$, are the Fibonacci numbers.

Theorem 4.1 *If $n \ge 1$, then*

$$\mathrm{irr}(\Gamma_n) = \frac{2}{5}\Big((n-1)F_n + 2nF_{n-1}\Big).$$

Proof We proceed by induction on n. By a direct computation we see that $\mathrm{irr}(\Gamma_1) = 0$, $\mathrm{irr}(\Gamma_2) = 2$, $\mathrm{irr}(\Gamma_3) = 4$, $\mathrm{irr}(\Gamma_4) = 10$, and $\mathrm{irr}(\Gamma_5) = 20$; hence, the stated formula holds for $n \le 5$. From now on assume that $n \ge 6$.

Define the following subsets of vertices of Γ_n:

$$\begin{aligned} A_n &= \{00b_3 \ldots b_n : \ b_3, \ldots, b_n \in \{0,1\}\}, \\ B_n &= \{10b_3 \ldots b_n : \ b_3, \ldots, b_n \in \{0,1\}\}, \\ C_n &= \{010b_4 \ldots b_n : \ b_4, \ldots, b_n \in \{0,1\}\}. \end{aligned}$$

The sets A_n, B_n, and C_n are disjoint and $V(\Gamma_n) = A_n \cup B_n \cup C_n$. In addition, the subgraphs $\langle A_n \rangle$, $\langle B_n \rangle$, and $\langle C_n \rangle$ are isomorphic to Γ_{n-2}, Γ_{n-2}, and Γ_{n-3}, respectively. Since each vertex from B_n has exactly one neighbor outside B_n (more precisely in A_n), we see that

- $\mathrm{irr}_{\Gamma_n}(E(\langle B_n \rangle)) = \mathrm{irr}(\Gamma_{n-2})$.

Similarly, since each vertex from C_n has exactly one neighbor outside C_n (more precisely in A_n), we get

- $\mathrm{irr}_{\Gamma_n}(E(\langle C_n \rangle)) = \mathrm{irr}(\Gamma_{n-3})$.

Consider now the edges uv from $G[A_n]$, where $u = 00u_3 \ldots u_n$ and $v = 00v_3 \ldots v_n$. If $u_3 = v_3$, then u and v have the same number of neighbors outside A_n. Hence, the irregularity of uv in Γ_n is equal to the irregularity of the corresponding edge in $G[A_n] = \Gamma_{n-2}$. Suppose now that $u = 000u_4 \ldots u_n$ and $v = 001v_4 \ldots v_n$. In this case $v_4 = 0$ and hence, also $u_4 = 0$, so that $u = 0000u_5 \ldots u_n$ and $v = 0010v_5 \ldots v_n$. The irregularity of such an edge in Γ_n is by 1 larger than the irregularity of the corresponding edge in Γ_{n-2}. Since there are precisely F_{n-2} such edges, we get

- $\mathrm{irr}_{\Gamma_n}(E(\langle A_n \rangle)) = \mathrm{irr}(\Gamma_{n-2}) + F_{n-2}$.

We still need to consider the edges between A_n and B_n and between A_n and C_n. (Note that there are no edges between B_n and C_n.)

Consider first the edges uv between A_n and B_n, in which case $u = 00u_3 \dots u_n$ and $v = 10u_3 \dots u_n$. Among them, the edges where $u = 0000u_5 \dots u_n$ contribute F_{n-2}, the edges where $u = 0010u_5 \dots u_n$ contribute nothing, and the edges where $u = 00010u_6 \dots u_n$ contribute F_{n-3}. Hence,

- the contribution of the edges between A_n and B_n is $F_{n-2} + F_{n-3}$.

Consider next the edges uv between A_n and C_n, in which case $u = 000u_4 \dots u_n$ and $v = 010u_3 \dots u_n$. Among them, the edges where $u = 0000u_5 \dots u_n$ contribute $2F_{n-2}$ and the edges where $u = 00010u_6 \dots u_n$ contribute F_{n-3}. Hence,

- the contribution of the edges between A_n and C_n is $2F_{n-2} + F_{n-3}$.

We have thus considered all the edges of Γ_n. Putting together the above itemized contributions, we infer that

$$\begin{aligned}\operatorname{irr}(\Gamma_n) &= 2 \cdot \operatorname{irr}(\Gamma_{n-2}) + \operatorname{irr}(\Gamma_{n-3}) + 4F_{n-2} + 2F_{n-3} \\ &= 2 \cdot \operatorname{irr}(\Gamma_{n-2}) + \operatorname{irr}(\Gamma_{n-3}) + 2F_n \,.\end{aligned}$$

Using the induction assumption we thus get

$$\begin{aligned}\operatorname{irr}(\Gamma_n) &= \frac{4}{5}\Big((n-3)F_{n-2} + 2(n-2)F_{n-3}\Big) \\ &\quad + \frac{2}{5}\Big((n-4)F_{n-3} + 2(n-3)F_{n-4}\Big) + 2F_n \\ &= \frac{2}{5}\Big((n-1)F_n + 2nF_{n-1}\Big),\end{aligned}$$

where the last equality follows by a lengthy but straightforward computation using the definition of the Fibonacci numbers. □

In [28] it is proved that $m(\Gamma_n) = (nF_{n+1} + 2(n+1)F_n)/5$. Hence, Theorem 4.1 has the following interesting consequence.

Corollary 4.2 *If $n \geq 2$, then*

$$\operatorname{irr}(\Gamma_n) = 2 \cdot m(\Gamma_{n-1}) \,.$$

To conclude the section we remark that in [34] the so-called boundary enumerator polynomials D_k of hypercubes in Fibonacci cubes are considered and that the proof of D_1 has similarities with our proof of Theorem 4.1. Namely, D_1 is related to $\sum_{uv \in E(\Gamma_n)}(\deg(u) + \deg(v) - 2)$ and the proof in [34] uses the same decomposition as we do here. Both proofs were produced independently, though. (We are adding this remark while preparing a revised version.)

5 Trees

The irregularity of trees has been already investigated. In [26] the irregularity of trees (and of unicyclic graphs) with given matching number was studied, while in [24] trees with minimum/maximum irregularity among the trees with given degree sequence and among the trees with given branching number were investigated. In the main result of this section we add to these studies the irregularity of trees of a given diameter and characterize the trees that attain the equality. Connecting the present section with Sects. 2 and 3 we also find lower and upper bounds for the irregularity of the π-permutation graph of an arbitrary tree.

Let $T_i(n,d)$, $2 \le i \le d$, $n \ge 2$, $d \le n-1$, denote the tree obtained from the path P_{d+1} by attaching $n-d-1$ leaves to the i^{th} vertex of P_{d+1}. Note that $n(T_i(n,d)) = n$ and that $\operatorname{diam}(T_i(n,d)) = d$. Observe also that if $d = n-1$, then for every i the graph $T_i(n,d) = T_i(n,n-1)$ is the path on n vertices. Recall also that a tree is called a *caterpillar* if, after its leaves are removed, a path graph remains. In other words, a caterpillar is obtained from a path graph by attaching some leaves to its vertices. Thus, the trees $T_i(n,d)$ belong to the class of caterpillars.

Theorem 5.1 *If T is a tree of order $n \ge 2$ and with* $\operatorname{diam}(T) = d$*, then*

$$\operatorname{irr}(T) \le (n-d)(n-d+1),$$

where the equality holds if and only if $T \in \{T_2(n,d), \ldots, T_d(n,d)\}$.

Proof Let T be a tree of order $n \ge 2$ and diameter d such that $\operatorname{irr}(T)$ is the largest possible.

We claim first that T is a caterpillar and assume on the contrary that it is not. Let P be a diametrical path of T. Since T is not a caterpillar and the path P is diametrical, P contains an inner vertex y such that the rooted tree T_y, defined as the maximal subtree of T with $V(T_y) \cap V(P) = \{y\}$, is of depth $r \ge 2$. Clearly, $n(T_y) \ge 3$.

Consider the following transformation. Let $u \in V(T_y)$ be a vertex with $d_{T_y}(y,u) = r-2$. (If $r = 2$, then $u = y$.) Let $A = \{w_1, \ldots, w_k\}$ be the set of down-neighbors of u in the rooted tree T_y and $B = N_T(u) - A$. Let $S_i = N_T(w_i) - \{u\}$, $i \in [k]$, and $S = \bigcup_{i=1}^k S_i$. Set $s = |S|$ and note that $s = \sum_{i=1}^k (\deg_T(w_i) - 1)$. Let now T' be the tree obtained from T by removing the edges between w_i and the vertices of S_i, $i \in [k]$, and then adding edges between vertices of S_i and u.

In T' the vertices w_i, $i \in [k]$, as well as all the vertices from S are leaves. The contribution to the irregularity in T and T' differs only for the edges incident with u and w_i. Therefore, setting $D = \operatorname{irr}(T') - \operatorname{irr}(T)$, we have

$$\begin{aligned} D &= \sum_{i=1}^{k} (\operatorname{imb}_{T'}(uw_i) - \operatorname{imb}_T(uw_i)) + \sum_{i=1}^{k} \sum_{x \in S_i} (\operatorname{imb}_{T'}(ux) - \operatorname{imb}_T(w_i x)) \\ &\quad + \sum_{z \in B} (\operatorname{imb}_{T'}(uz) - \operatorname{imb}_T(uz)) \\ &= F_1 + F_2 + F_3\,, \end{aligned}$$

where

$$F_1 = \sum_{i=1}^{k} \left[((\deg_T(u) + s) - 1) - |\deg_T(u) - \deg_T(w_i)|\right],$$
$$F_2 = \sum_{i=1}^{k} \sum_{x \in S_i} \left[(\deg_T(u) + s - 1) - (\deg_T(w_i) - 1)\right],$$
$$F_3 = \sum_{z \in B} \left[(\deg_T(u) + s - 1) - |\deg_T(u) - \deg_T(z)|\right].$$

Now we have

$$\begin{aligned} F_1 &\geq \sum_{i=1}^{k} 2 \cdot \min\{\deg_T(u) - 1, \deg_T(w_i) - 1\} > 0, \\ F_2 &= \sum_{i=1}^{k} \sum_{x \in S_i} \left[(\deg_T(u) + s - 1) - (\deg_T(w_i) - 1)\right] \\ &= s \cdot \deg_T(u) + s^2 - \sum_{i=1}^{k} \deg_T(w_i)(\deg_T(w_i) - 1) \\ &= s \cdot \deg_T(u) + s^2 - \sum_{i=1}^{k} (\deg_T(w_i) - 1)^2 - s \\ &= s \cdot (\deg_T(u) - 1) + s^2 - \sum_{i=1}^{k} (\deg_T(w_i) - 1)^2, \\ F_3 &= \sum_{z \in B} \left[(\deg_T(u) + s - 1) - |\deg_T(u) - \deg_T(z)| \geq -|B| \cdot s\right]. \end{aligned}$$

Note that $|B| \in [2]$. If $|B| = 1$, then

$$F_2 + F_3 \geq (s + s^2 - \sum_{i=1}^{k} (\deg_T(w_i) - 1)^2) - s \geq 0.$$

If $|B| = 2$, then $u = y$ and $\deg_T(u) \geq 3$. Therefore,

$$F_2 + F_3 \geq (2s + s^2 - \sum_{i=1}^{k} (\deg(w_i) - 1)^2 - 2s) \geq 0.$$

Hence, if $S \neq \emptyset$, then $D > 0$. Applying iteratively the above transformation as many times as required, we arrive at a caterpillar.

Let now T be a caterpillar of diameter d, and let P be its diametrical path. Suppose that $T \notin \{T_2(n, d), \ldots, T_d(n, d)\}$. Let v_1 and v_2 be vertices with the first and the second largest degree among the vertices of P, respectively. Note that $\deg(v_1) \geq \deg(v_2) \geq 3$. Let $S_i = N(v_i) - V(P)$, $i \in [2]$. Remove the edges between v_2 and S_2, add edges between the vertices of S_2 and v_1, and denote the obtained tree with T'. The contribution of the edges incident by vertices v_1 and v_2 is changed in T' compared with T, while the contribution of every other edge to the irregularity is the same. Therefore, setting $D = \mathrm{irr}(T') - \mathrm{irr}(T)$, we have

$$
\begin{aligned}
D = & \sum_{w \in S_1} (\mathrm{imb}_{T'}(v_1 w) - \mathrm{imb}_T(v_1 w)) + \sum_{w \in S_2} (\mathrm{imb}_{T'}(v_1 w) - \mathrm{imb}_T(v_2 w)) \\
& + \sum_{w \in N(v_1) \cap V(P)} (\mathrm{imb}_{T'}(v_1 w) - \mathrm{imb}_T(v_1 w)) \\
& + \sum_{w \in N(v_2) \cap V(P)} (\mathrm{imb}_{T'}(v_2 w) - \mathrm{imb}_T(v_2 w)) \\
= & \; |S_1|(\deg_T(v_2) - 2) + \big(|S_2|((\deg_T(v_1) - 2) - (\deg_T(v_2) - 1)\big) \\
& + 2(\deg_T(v_2) - 2) + \sum_{w \in N(v_2) \cap V(P)} (\mathrm{imb}_{T'}(v_2 w) - \mathrm{imb}_T(v_2 w)) \\
\geq & \; |S_1| \big(\deg_T(v_2) - 2\big) + |S_2| \big(\deg_T(v_1) - \deg_T(v_2) - 1\big) \\
& + 2(\deg_T(v_2) - 2) - 2(\deg_T(v_2) - 2) \\
= & \; (\deg_T(v_2) - 2)(2\deg_T(v_1) + \deg_T(v_2) - 3) > 0\,.
\end{aligned}
$$

This proves the theorem. □

To conclude the paper we find bounds for the irregularity of the π-permutation graphs of tree.

Theorem 5.2 *If T is a tree of order $n \geq 3$ and π a permutation on $V(T)$, then*

$$4 \leq \mathrm{irr}(T^\pi) \leq 2n(n-2)\,.$$

Moreover, the left equality holds if and only if $T = P_n$ and $\pi = \mathrm{id}$, and the right equality holds if and only if $T = S_n$ and π is a permutation that maps the center of S_n into a leaf.

Proof From the fact that path P_n has the minimum possible irregularity among the trees of order $n \geq 3$, that is, irregularity 2, and by Proposition 2.1, we infer that $\mathrm{irr}(T^\pi) \geq \mathrm{irr}(P_n^\pi)$ with equality holding if and only if $T = P_n$ and $\pi = \mathrm{id}$.

Let π be an arbitrary permutation on $V(T)$. Then, we have

$$\begin{aligned}\operatorname{irr}(T^{\pi}) &= 2\cdot\operatorname{irr}(T) + \sum_{v\in V(T)} |\deg_{T^{\pi}}(v') - \deg_{T^{\pi}}(\pi(v'))| \\ &\le 2\cdot\operatorname{irr}(T) + 2\sum_{v\in V(T)} |\deg_{T}(v) - \delta(T)| \\ &= 2\cdot\operatorname{irr}(T) + 2(2m(T) - n(T)\delta(T)) \\ &\le 2\cdot\operatorname{irr}(S_n) + 2(n-2) \\ &= \operatorname{irr}(S_n^{\pi'}).\end{aligned}$$

In the last equality π' is a permutation on $V(S_n)$ that maps the center of S_n to a leaf. Since S_n is the only tree with maximum irregularity, the right equality holds if and only if $T = S_n$ and π is a permutation as just described. □

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