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Markov's Inequality and C^∞ Functions on Certain Algebraic Hypersurfaces

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Abstract
It is known that if E is a C^∞ determining set, then E is a Markov set if and only if it has Bernstein's property. This article provides the equivalent of this result for compact subsets of some algebraic varieties.

Keywords Markov inequality · C^∞ functions · Algebraic sets

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1 Introduction

Jackson's famous estimate of the error of the best polynomial approximation for a fixed function is one of the main theorems in constructive function theory. According to a multivariate version of the classical Jackson theorem (see, e.g., [10]), if I is a compact cube in $\mathbb{R}^N$ and $f : I \to \mathbb{R}$ is a $\mathcal{C}^{k+1}$ function on I, then

$$n^k \mathrm{dist}_I(f, \mathcal{P}_n) \leq C_k \sum_{j=1}^{N} \sup_{x \in I} \left| \frac{\partial^{k+1} f}{\partial x_j^{k+1}}(x) \right|,$$

where the constant C_k depends only on N, I and k. As usual, $\mathrm{dist}_I(f, \mathcal{P}_n) = \inf\{\|f - p\|_I : p \in \mathcal{P}_n\}$, $\mathcal{P}_n$ is the space of all algebraic polynomials of degree at most n and $\|\cdot\|_I$ is the sup norm on I.

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As an application of Jackson's theorem, one can prove classical results like the well-known Bernstein theorem (see, e.g., [5,6]) which allows to obtain a characterization of C^∞ functions:
A function f defined on I can be extended to a C^∞function on $\mathbb{R}^N$ if and only if

$$\lim_{n\to\infty} n^k \operatorname{dist}_I(f,\mathcal{P}_n) = 0 \quad for\ all\ positive\ integer\ numbers\ k.$$

A natural question arises: For which compact subsets E of $\mathbb{R}^N$ the following Bernstein property holds?
For every function $f : E \to \mathbf{R}$ if the sequence $\{\operatorname{dist}_E(f,\mathcal{P}_n)\}_n$ is rapidly decreasing (i.e. $\lim_{n\to\infty} n^k \operatorname{dist}_E(f,\mathcal{P}_n) = 0$ for all $k > 0$), then there exists a C^∞ function $F : \mathbb{R}^N \to \mathbb{R}$ such that $F = f$ on E.

It turns out that these matters were considered by Pleśniak in 1990 (see [8,9] for previous results). He proved that the Markov inequality

$$\|D^\alpha P\|_E \le M(\deg P)^{r|\alpha|}\|P\|_E, \quad \alpha \in \mathbb{Z}_+^N,$$

and Bernstein's property are equivalent for C^∞ determining sets. Our goal is to find a generalization of this fact for sets which are not C^∞ determining.

2 Markov Inequality

Our intention in this section is to study an extension of the Markov inequality to compact subsets of algebraic set. We will consider nonempty sets of the form

$$V = \left\{(x_1,\ldots,x_N) \in \mathbb{R}^N : x_k^d = Q_0(y) + Q_1(y)x_k + \cdots + Q_{d-1}(y)x_k^{d-1}\right\}, \quad (1)$$

where Q_i are polynomials for every $0 \le i \le d-1$ and the variable $y = (x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_N) \in \mathbb{R}^{N-1}$. One can verify that every polynomial P from the space $\mathcal{P}(x_1,\ldots,x_N)$, on V, coincides with some polynomial from $\mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$ (see [3]). Here $\mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$ denotes the subspace of $\mathcal{P}(x_1,\ldots,x_N)$ formed of all polynomials of the form $\sum_{i=0}^{d-1} G_i(y)x_k^i$ with $G_i \in \mathcal{P}(y)$. Hence

$$\mathcal{P}(V) := \left\{P_{|V}, P \in \mathcal{P}(x_1,\ldots,x_N)\right\} = \left\{P_{|V}, P \in \mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)\right\}. \quad (2)$$

Considerations in [2,3] suggest the following definition:
(Markov set and Markov inequality on $\mathbf{F}$) *Let $\mathbf{F}$ be an infinite-dimensional subspace of $\mathcal{P}(x_1,\ldots,x_N)$such that $P \in \mathbf{F}$ implies $D^\alpha P \in \mathbf{F}$* for all $\alpha \in \mathbb{Z}_+^N$. *A compact set $\emptyset \ne E \subset \mathbb{R}^N$is said to be a $\mathbf{F}$-Markov set if there exist $M, m > 0$ such that*

$$\|D^\alpha P\|_E \le M^{|\alpha|}(\deg P)^{m|\alpha|}\|P\|_E, \quad P \in \mathbf{F}, \quad \alpha \in \mathbb{Z}_+^N. \quad (3)$$

This inequality is called a $\mathbf{F}$-Markov inequality for E.

Note that, similarly as in the classical case, it is enough to check the property for $|\alpha| = 1$.

It is clear that if $P \in \mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$, then $D^\alpha P \in \mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$ for all $\alpha \in \mathbb{Z}_+^N$. Now we give an example to demonstrate that the above definition makes sense.

Example 1 Let $V = \{y^3 = (1-x^2)y\} \subset \mathbb{R}^2$. The compact set $E = \{(x, y) \in V : x \in [-1, 1]\}$ is a $\mathcal{P}(x) \otimes \mathcal{P}_2(y)$-Markov.

Proof We recall first the three classical inequalities Markov's inequality: For any polynomial P

$$\|P'\|_{[-1,1]} \leq (\deg P)^2 \|P\|_{[-1,1]}. \tag{4}$$

Bernstein's inequality: If T_n is a trigonometric polynomial of degree at most n, then

$$\|T'\| \leq n\|T\|, \tag{5}$$

where $\|\cdot\|$ denotes the supremum norm. If P_n is an algebraic polynomial of degree at most n, then $T_n(t) = P_n(\cos t)$ is a trigonometric polynomial of degree at most n, and (5) yields

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|P\|_{[-1,1]}, \quad x \in (-1, 1), \tag{6}$$

which is also known as Bernstein inequality. The classical inequality of Schur states that

$$\|P\|_{[-1,1]} \leq (\deg P + 1) \left\| P(x)\sqrt{1-x^2} \right\|_{[-1,1]} \tag{7}$$

holds for every polynomial P. This can be generalized to weights $(1-x^2)^\alpha$ with $\alpha \geq 1/2$ (see [1], Lemma 2.4, p. 73):

$$\|P\|_{[-1,1]} \leq n^{2\alpha} \left\| P(x)(1-x^2)^\alpha \right\|_{[-1,1]} \quad P \in \mathcal{P}_{n-1}. \tag{8}$$

Combining the above inequality and Markov's inequality (4), we obtain

$$\left\| P'(x)(1-x^2) \right\|_{[-1,1]} \leq 3(n+2)^2 \left\| P(x)(1-x^2) \right\|_{[-1,1]} \quad P \in \mathcal{P}_n. \tag{9}$$

Let $P \in \mathcal{P}(x) \otimes \mathcal{P}_2(y)$. Then $P(x, y) = G_0(x) + G_1(x)y + G_2(x)y^2$ for some $G_i \in \mathcal{P}(x)$ $(i = 0, 1, 2)$. Now

$$\begin{aligned} \left\| D^{(1,0)} P(x, y) \right\|_E &\leq \left\| G_0'(x) \right\|_E + \left\| G_1'(x)y + G_2'(x)y^2 \right\|_E \\ &= \left\| G_0'(x) \right\|_{[-1,1]} + \left\| G_1'(x)y + G_2'(x)y^2 \right\|_{E'}, \end{aligned}$$

where $E' = \{(x, y) \in \mathbb{R}^2 : y^2 = 1 - x^2\}$. Since $(x, y) \in E' \iff (x, -y) \in E'$, we have

$$\left\| D^{(1,0)} P(x, y) \right\|_E \leq \left\| G_0'(x) \right\|_{[-1,1]} + \left\| G_1'(x)\sqrt{1 - x^2} \right\|_{[-1,1]} + \left\| G_2'(x)(1 - x^2) \right\|_{[-1,1]}.$$

By (4), (5), and (9), respectively, we get

$$\left\| D^{(1,0)} P(x, y) \right\|_E \leq (\deg G_0)^2 \left\| G_0(x) \right\|_{[-1,1]} + \deg G_1 \left\| G_1(x) \right\|_{[-1,1]} + 3(2 + \deg G_2)^2 \left\| G_2(x)(1 - x^2) \right\|_{[-1,1]}.$$

The inequality (7) yields the following

$$\begin{aligned}\left\| D^{(1,0)} P(x, y) \right\|_E \leq\ & (\deg G_0)^2 \left\| G_0(x) \right\|_{[-1,1]} \\ & + (\deg G_1 + 1)^2 \left\| G_1(x)\sqrt{1 - x^2} \right\|_{[-1,1]} \\ & + 3(2 + \deg G_2)^2 \left\| G_2(x)(1 - x^2) \right\|_{[-1,1]}.\end{aligned}$$

Using again the fact that $(x, y) \in E' \iff (x, -y) \in E'$, we obtain

$$\left\| D^{(1,0)} P(x, y) \right\|_E \leq 5(\deg P)^2 \left(\left\| G_0(x) \right\|_{[-1,1]} + \left\| G_1(x)y + G_2(x)y^2 \right\|_{E'} \right).$$

Now if $-1 \leq \xi \leq 1$, then $(\xi, 0) \in E$ and $G_0(\xi) = P(\xi, 0)$. Hence

$$\left\| G_0(x) \right\|_{[-1,1]} \leq \left\| P \right\|_E.$$

This together with the triangle inequality, implies

$$\left\| D^{(1,0)} P(x, y) \right\|_E \leq 15(\deg P)^2 \left\| P \right\|_E.$$

Next, we consider the case of $D^{(0,1)}$. It is clear that

$$\left\| D^{(0,1)} P(x, y) \right\|_E \leq \left\| G_1(x) \right\|_E + 2 \left\| G_2(x)y \right\|_E \leq \left\| G_1(x) \right\|_E + 2 \left\| G_2(x) \right\|_E.$$

Then, using (7) and (8), we have

$$\left\| D^{(0,1)} P(x, y) \right\|_E \leq (\deg G_1 + 1) \left\| G_1(x)\sqrt{1 - x^2} \right\|_{[-1,1]} + 2(1 + \deg G_2)^2 \left\| G_2(x)(1 - x^2) \right\|_{[-1,1]}.$$

Now a similar proof to that of the previous case gives the following

$$\left\| D^{(0,1)} P(x, y) \right\|_E \leq 6(\deg P)^2 \|P\|_E .$$

That is what we wished to prove. □

Next example shows that **F**-Markov inequality depends not only on the set but also on the family **F**.

Example 2 Consider set $V = \{y^3 = 1-x^2\} \subset \mathbb{R}^2$. The compact set $E = \{(x, y) \in V : x \in [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]\}$ is a $\mathcal{P}(y) \otimes \mathcal{P}_1(x)$-Markov, but it is not $\mathcal{P}(x) \otimes \mathcal{P}_2(y)$-Markov.

Proof The fact that $E = \{(x, y) \in V : x \in [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]\}$ is a $\mathcal{P}(y) \otimes \mathcal{P}_1(x)$-Markov follows from [2,4]. So we need only show that E is not $\mathcal{P}(x) \otimes \mathcal{P}_2(y)$-Markov. Seeking a contradiction, we consider the sequence of polynomials

$$P_n(x, y) = y - \sum_{k=0}^{n} \frac{\Gamma(k - 1/3)}{\Gamma(-1/3)k!} x^{2k}.$$

It is well known that

$$\sqrt[3]{1 - x^2} = \sum_{k=0}^{\infty} \frac{\Gamma(k - 1/3)}{\Gamma(-1/3)k!} x^{2k} \quad \text{for} \quad |x| < 1.$$

Hence

$$\begin{aligned} \|P_n(x, y)\|_E &= \left\| \sum_{k=n+1}^{\infty} \frac{\Gamma(k - 1/3)}{\Gamma(-1/3)k!} x^{2k} \right\|_{[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]} \\ &= \left\| \frac{x^{2+2n} \Gamma\left(\frac{1}{3}(2 + 3n)\right) F\left(1, \frac{2}{3} + n, 2 + n, x^2\right)}{\Gamma\left(-\frac{1}{3}\right) \Gamma(2 + n)} \right\|_{[-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]}, \end{aligned}$$

where F is the hypergeometric function defined for $|z| < 1$ by the power series

$$F(a, b; c; z) = \sum_{\iota=0}^{\infty} \frac{(a)_\iota (b)_\iota}{(c)_\iota} \frac{z^\iota}{\iota!}.$$

Here $(q)_\iota$ is the (rising) Pochhammer symbol. If $x \in [0, 1]$, then the function $F\left(1, \frac{2}{3} + n, 2 + n, x^2\right)$ is the increasing function of x, since its Taylor coefficients are all positive. Therefore, by $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ and $z\Gamma(z) = \Gamma(z + 1)$, we have

$$F\left(1, \frac{2}{3} + n, 2 + n, x^2\right) \leq F\left(1, \frac{2}{3} + n, 2 + n, 1\right) = \frac{\Gamma(2 + n)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{4}{3}\right) \Gamma(n + 1)}$$

$$= 3(1+n).$$

If we recall that $\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)n^{\alpha}} = 1$, then

$$\lim_{n\to\infty} \frac{3\Gamma\left(\frac{2}{3}+n\right)(1+n)}{4\Gamma\left(-\frac{1}{3}\right)\Gamma(2+n)} = 0.$$

We thus may conclude that there exists a constant $C > 0$ (independent of n) for which

$$\|P_n(x, y)\|_E \le C4^{-n}.$$

Consequently for $r > 0$,

$$\lim_{n\to\infty} n^r \|P_n(x, y)\|_E = 0.$$

This gives a contradiction, and the result is established. □

Remark 1 Note that $(x, y) \in E \iff (-x, y) \in E$. On the other hand, if $(x, y) \in E$, then $(x, -y) \notin E$. This is one of the reasons why the set E is a $\mathcal{P}(y)\otimes\mathcal{P}_1(x)$-Markov, but it is not $\mathcal{P}(x) \otimes \mathcal{P}_2(y)$-Markov.

Example 1 illustrates the more general idea.

Example 3 Combining methods used in [2] with method from Example 1, one can provide other examples of $\mathcal{P}(y) \otimes \mathcal{P}_2(x_k)$-Markov sets by considering algebraic sets of the form

$$V = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_k^3 = Q(y)x_k\},$$

where $Q_j \in \mathcal{P}(y)$ and $y = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$.

3 C^∞ Functions

First we introduce the subspace of the space $C^\infty(\mathbb{R}^N)$ related to an algebraic set defined by (1). We define

$$C_V^\infty(\mathbb{R}^N) := \Big\{ f \in C(\mathbb{R}^N) : \forall_{r>0} \lim_{n\to\infty} n^r \operatorname{dist}_I(f, \mathcal{P}_n(y) \otimes \mathcal{P}_{d-1}(x_k)) = 0 \\ \text{for every compact cube } I \text{ in } \mathbb{R}^N \Big\}. \tag{10}$$

Since every cube I is a Markov set, then by Pleśniak's theorem (see [9]) $C_V^\infty(\mathbb{R}^N) \subset C^\infty(\mathbb{R}^N)$. It should be noted that Pleśniak's result, together with the Jackson theorem, implies

$$C^\infty(\mathbb{R}^N) = \Big\{ f \in C(\mathbb{R}^N) : \forall_{r>0} \lim_{n\to\infty} n^r \operatorname{dist}_I (f, \mathcal{P}_n(x_1,\dots,x_N)) = 0 \text{ for every compact cube } I \text{ in } \mathbb{R}^N \Big\}.$$

We say that f is a C_V^∞ function on a compact subset E of V if, there exists a function $\tilde{f} \in C_V^\infty(\mathbb{R}^N)$ with $\tilde{f}_{|E} = f$. We denote by $C_V^\infty(E)$ the space of such functions. Let τ_J be the topology on $C_V^\infty(E)$ determined by the seminorms $\delta_{-1}(f) := \|f\|_E$, $\delta_0(f) := \operatorname{dist}_E(f, \mathcal{P}_0(y) \otimes \mathcal{P}_{d-1}(x_k))$ and

$$\delta_\nu(f) := \sup_{l\ge 1} l^\nu \operatorname{dist}_E(f, \mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k))$$

for $\nu = 1, 2, \dots$ (This idea comes from Zerener's work [11].) The fact that δ_ν's are seminorms on $C_V^\infty(E)$ follows from the definition of the set $C_V^\infty(\mathbb{R}^N)$. It should be noted that this topology need not be complete.

The natural topology τ_0 on the set $C^\infty(\mathbb{R}^N)$ is determined by the seminorms $|\cdot|_K^\nu$, where for each compact set K in $\mathbb{R}^N$ and each $\nu = 0, 1, \dots$,

$$|f|_K^\nu := \max_{|\alpha|\le\nu} \|D^\alpha f\|_K.$$

Therefore, we consider the topology τ_Q on $C_V^\infty(E)$ determined by the seminorms

$$q_{K,\nu}(f) := \inf\Big\{ |\tilde{f}|_K^\nu : f \in C_V^\infty(\mathbb{R}^N),\ \tilde{f}_{|E} = f \Big\}.$$

Then τ_Q coincides with the quotient topology of the space $C_V^\infty(\mathbb{R}^N)/I(E)$, where $C_V^\infty(\mathbb{R}^N)$ is considered with the natural topology τ_0 and $I(E) := \{f \in C_V^\infty(\mathbb{R}^N) : f_{|E} = 0\}$. Notice that the space $(C_V^\infty(\mathbb{R}^N), \tau_0)$ is a closed subspace of the complete space $(C^\infty(\mathbb{R}^N), \tau_0)$. Therefore, the space $(C_V^\infty(\mathbb{R}^N), \tau_0)$ is also complete. In view of the fact that $I(E)$ is a closed subspace of $(C_V^\infty(\mathbb{R}^N), \tau_0)$, the quotient space $C_V^\infty(\mathbb{R}^N)/I(E)$ is complete. Hence $(C_V^\infty(E), \tau_Q)$ is a Fréchet space. To prove the main result, we will need the following lemma (see, e.g., [7], 1.4.2).

Lemma 1 *There are positive constants C_α depending only on $\alpha \in \mathbb{Z}_+^N$ such that for each compact set K in $\mathbb{R}^N$ and each $\epsilon > 0$, one can find a C^∞ function h on $\mathbb{R}^N$ satisfying $0 \le h \le 1$ on $\mathbb{R}^N$, $h = 1$ in a neighborhood of K, $h(x) = 0$ if $\operatorname{dist}(x, K) > \epsilon$, and for all $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{Z}_+^N$, $|D^\alpha h(x)| \le C_\alpha \epsilon^{-|\alpha|}$.*

4 Main Result

Before starting the main result, we prove the following lemma.

Lemma 2 *Let E be a $\mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$-Markov set. Also define*

$$\pi(E) = \Big\{ (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N) \in \mathbb{R}^{N-1} : (x_1, \dots, x_N) \in E,\ x_k \in \mathbb{R} \Big\}.$$

If E is a $\mathcal{P}(y)\otimes\mathcal{P}_{d-1}(x_k)$-Markov set (with M and m), then $\pi(E)$ is a Markov set (as a subset of $\mathbb{R}^{N-1}$), and for every polynomial $P=\sum_{i=0}^{d-1}G_i(y)x_k^i$, there exist constant $C>0$ (depending only on E and d) such that

$$\|G_i\|_{\pi(E)}\leq\frac{C}{i!}(\deg P)^{m(d-1)}\|P\|_E,$$

for every $i=0,1,\ldots,d-1$. Conversely, if $\pi(E)$ is a Markov set (with A and η) and for every polynomial $P=\sum_{i=0}^{d-1}G_i(y)x_k^i$, there exist $B,\lambda>0$ (depending only on E and d) such that

$$\|G_i\|_{\pi(E)}\leq B(\deg P)^{\lambda}\|P\|_E,\quad i=0,1,\ldots,d-1, \tag{11}$$

then E is a $\mathcal{P}(y)\otimes\mathcal{P}_{d-1}(x_k)$-Markov set.

Proof Let E be a $\mathcal{P}(y)\otimes\mathcal{P}_{d-1}(x_k)$-Markov set. The proof starts from the observation that

$$\frac{\partial^{d-1}P}{\partial x_k^{d-1}}=(d-1)!G_{d-1}.$$

Therefore the $\mathcal{P}(y)\otimes\mathcal{P}_{d-1}(x_k)$-Markov property of the set E gives

$$\|G_{d-1}\|_{\pi(E)}\leq\frac{M^{d-1}}{(d-1)!}(\deg P)^{m(d-1)}\|P\|_E.$$

If $i=d-2$, then

$$(d-2)!G_{d-2}=\frac{\partial^{d-2}P}{\partial x_k^{d-2}}-(d-1)G_{d-1}x_k.$$

Hence, there exists constant $C>0$ (depending only on the set E) such that

$$\|G_{d-2}\|_{\pi(E)}\leq\frac{(C+1)M^{d-1}}{(d-2)!}(\deg P)^{m(d-1)}\|P\|_E.$$

Continuing this process, one can show that there exists a constant $C_1>0$ (depending only on the set E and d) such that

$$\|G_i\|_{\pi(E)}\leq\frac{C_1}{i!}(\deg P)^{m(d-1)}\|P\|_E.$$

To prove the converse direction, assume that $\pi(E)$ is a Markov set and (11) holds. Then, for every polynomial $P=\sum_{i=0}^{d-1}G_i(y)x_k^i$, we have

$$\left\|\frac{\partial P}{\partial x_j}\right\|_E\leq\sum_{i=0}^{d-1}\left\|\frac{\partial G_i}{\partial x_j}x_k^i+G_i\frac{\partial x_k^i}{\partial x_j}\right\|_E.$$

Since E is compact, there exists $K > 0$, depending only on the set E, such that

$$\left\| \frac{\partial G_i}{\partial x_j} x_k^i + G_i \frac{\partial x_k^i}{\partial x_j} \right\|_E \leq K \left(\left\| \frac{\partial G_i}{\partial x_j} \right\|_{\pi(E)} + \|G_i\|_{\pi(E)} \right),$$

for every $j = 1, 2, \ldots, N$ and $i = 0, 1, \ldots, d-1$. Therefore,

$$\left\| \frac{\partial P}{\partial x_j} \right\|_E \leq K \left(\sum_{i=0}^{d-1} \left\| \frac{\partial G_i}{\partial x_j} \right\|_{\pi(E)} + \|G_i\|_{\pi(E)} \right).$$

Then, using the fact that $\pi(E)$ is a Markov set, there exists constants $A > 0$ and $\eta > 0$ such that

$$\left\| \frac{\partial P}{\partial x_j} \right\|_E \leq K \left(\sum_{i=0}^{d-1} A(\deg G_i)^\eta \|G_i\|_{\pi(E)} + \|G_i\|_{\pi(E)} \right).$$

Finally, we use (11) to see that

$$\left\| \frac{\partial P}{\partial x_j} \right\|_E \leq Kd \left(AB(\deg P)^{\eta+\lambda} + B(\deg P)^\lambda \right) \|P\|_E.$$

That concludes the proof. □

We say that the set $E \subset V$ is C_V^∞ determining if for each $f \in C_V^\infty(\mathbb{R}^N)$, $f_{|E} = 0$ implies $D^\alpha f_{|E} = 0$, for all $\alpha \in \mathbb{Z}_+^N$. Now we are ready to state our main result.

Theorem 1 *If E is a C_V^∞ determining compact subset of V, then the following statements are equivalent:*

(i) *($\mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$-Markov Inequality) There exist positive constants M and r such that for each polynomial $P \in \mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$ and each $\alpha \in \mathbb{Z}_+^N$,*

$$\|D^\alpha P\|_E \leq M(\deg P)^{r|\alpha|}\|P\|_E.$$

(ii) *There exist positive constants M and r such that for every $P \in \mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$ of degree at most n, $n = 1, 2, \ldots$,*

$$|P(x)| \leq M\|P\|_E \quad \text{if} \quad x \in E_n := \{x \in \mathbb{R}^N : \operatorname{dist}(x, E) \leq 1/n^r\}.$$

(iii) *(Bernstein's Theorem) For every function $f : E \to \mathbb{R}$, if the sequence $\{\operatorname{dist}_E(f, \mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k))\}$ is rapidly decreasing, then there is a $C_V^\infty(\mathbb{R}^N)$ function $\tilde{f}$ on $\mathbb{R}^N$ such that $\tilde{f}_{|E} = f$.*

(iv) *The space $(C_V^\infty(E), \tau_J)$ is complete and $C_V^\infty(E) = C^\infty(E)$.*

(v) *The topologies τ_J and τ_Q for $C_V^\infty(E)$ coincide.*

Proof The proof of equivalence of *(i)* and *(ii)* is almost the same as in [9], and we omit the details.

Next we show that ((*i*) *and* (*ii*)) $\Leftrightarrow$ (*iii*). Suppose that we have function $f : E \to \mathbb{R}$ such that for each $s > 0$,

$$\lim_{l\to\infty} l^s \|f - P_l\|_E = 0.$$

Here $P_l = \sum_{i=0}^{d-1} G_{l,i}(y)x_k^i$ is a metric projection of f onto $\mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k)$ ($l = 0, 1, \ldots$). Set, as in Lemma 2,

$$\pi(E) = \{y = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1} : (x_1, \ldots, x_N) \in E \\ \text{for some } x_k \in \mathbb{R}\}.$$

We assume that r is an integer so large that both *(i)* and *(ii)* are valid for E. Let $\epsilon_l = 1/l^r$ and for $l = 1, 2, \ldots$ take a function $h_l \in C^\infty(\mathbb{R}^{N-1})$ of Lemma 1 corresponding to ϵ_l and $\pi(E)$. We will show that

$$\tilde{f}(x_1, \ldots, x_N) := \sum_{i=0}^{d-1} G_{0,i}(y)x_k^i + \sum_{l=1}^{\infty}\sum_{i=0}^{d-1} h_l(y)(G_{l,i}(y) - G_{l-1,i}(y))x_k^i$$

determines a function from $C_V^\infty(\mathbb{R}^N)$ such that $\tilde{f}_{|E} = f$. In order to prove that $\tilde{f} \in C_V^\infty(\mathbb{R}^N)$, it suffices to check that

$$G_{0,i}(y) + \sum_{l=1}^{\infty} h_l(y)(G_{l,i}(y) - G_{l-1,i}(y)) \in C^\infty(\mathbb{R}^{N-1}),$$

for every $i = 0, 1, \ldots, d-1$. Thus, if $\gamma \in \mathbb{Z}_+^{N-1}$, then, by *(i)* and *(ii)*,

$$\begin{aligned}\sup_{\mathbb{R}^{N-1}} |D^\gamma(h_l(G_{l,i} - G_{l-1,i}))| &\le \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \sup_{\pi(E)_l} |D^\beta h_l D^{\gamma-\beta}(G_{l,i} - G_{l-1,i})| \\ &\le M \sum_{\beta\le\gamma} \binom{\gamma}{\beta} C_\beta l^{r|\beta|} \|D^{\gamma-\beta}(G_{l,i} - G_{l-1,i})\|_{\pi(E)} \\ &\le M_1 l^{r|\gamma|} \|G_{l,i} - G_{l-1,i}\|_{\pi(E)}\end{aligned}$$

where $\pi(E)_l := \{y \in \mathbb{R}^{N-1} : \operatorname{dist}(y, \pi(E)) \le \epsilon_l\}$. From Lemma 2, there is a constant $C > 0$ so that

$$\sup_{\mathbb{R}^{N-1}} |D^\gamma(h_l(G_{l,i} - G_{l-1,i}))| \le C(\deg P_l)^{r(|\gamma|+d-1)} \|P_l - P_{l-1}\|_E.$$

Now if $l \geq \max\{2, d\}$, then

$$\sup_{\mathbb{R}^{N-1}} |D^{\gamma}(h_l(G_{l,i} - G_{l-1,i}))| \leq 2Cl^{-2}\delta_{2r(|\gamma|+d-1)+2}(f),$$

with a constant C independent of l. Taking into account that $\delta_{2r(|\gamma|+d-1)+2}(f)$ is independent of l and the series $\sum_{l=1}^{\infty} l^{-2}$ is convergent, one sees that

$$\sum_{l=1}^{\infty} D^{\gamma}\left(h_l(G_{l,i} - G_{l-1,i})\right)$$

converges uniformly for every $i = 0, 1, \ldots, d-1$.

Next we shall show that *(iii)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(v)*. If Bernstein's theorem holds, then it is clear that $C_V^{\infty}(E) = C^{\infty}(E)$. From this and the fact that the map $C(E) \ni f \to \mathrm{dist}_E(f, \mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k)) \in \mathbb{R}$ is continuous, we have *(iv)*. Now suppose that $(C_V^{\infty}(E), \tau_J)$ is complete. Let I be a cube which contains E in its interior. Applying the Jackson theorem on the cube, for every ν, there exists a constant $C_{\nu} > 0$ so that

$$\delta_{\nu}(f) \leq C_{\nu} q_{I,\nu+1}(f)$$

for all f in $C_V^{\infty}(\mathbb{R}^N)$. Hence by Banach's isomorphism theorem (for Fréchet spaces), we have *(v)*.

The final step of the proof is to show that *(v)* implies *(i)* . If topologies τ_J and τ_Q coincide, there are a positive constant M and an integer $\mu \geq -1$ such that $q_{E,1}(f) \leq M\delta_{\mu}(f)$ for every $f \in C_V^{\infty}(E)$. Since $\pi(E)$ is C^{∞} determining and $\delta_0(f) \leq \|f\|$, we conclude that $\mu \geq 1$. Hence if $f \in \mathcal{P}_{\lambda}(y) \otimes \mathcal{P}_{d-1}(x_k)$, then

$$\left\|\frac{\partial f}{\partial x_j}\right\|_E \leq M \sup_{1\leq l\leq \lambda} l^{\mu}\mathrm{dist}_E(f, \mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k)) \leq M\lambda^{\mu}\|f\|_E$$

for $j = 1, 2, \ldots, N$. This implies that E is a $\mathcal{P}(y) \otimes \mathcal{P}_{d-1}(x_k)$-Markov set. (It is essential here that E is C_V^{∞} determining.) □

Remark 2 Let E be compact subset of V. If E satisfies *(i)*, above, then E is C_V^{∞} determining set.

To see this, take a compact cube I in $\mathbb{R}^N$ containing E in its interior. We let $f \in C_V^{\infty}(\mathbb{R}^N)$ such that $f = 0$ on E. It follows from the definition of $C_V^{\infty}(\mathbb{R}^N)$ that

$$\epsilon_l := \mathrm{dist}_I(f, \mathcal{P}_l(y) \otimes \mathcal{P}_{d-1}(x_k)) = \|f - P_l\|_I = \|f - \sum_{i=0}^{d-1} G_{l,i}(y)x_k^i\|_I$$

is rapidly decreasing. Hence by Markov's inequality, we have

$$D^{\alpha}f = \sum_{i=0}^{d-1} D^{\alpha}\{G_{0,i}(y)x_k^i\} + \sum_{l=1}^{\infty}\sum_{i=0}^{d-1} D^{\alpha}\{(G_{l,i}(y) - G_{l-1,i}(y))x_k^i\} \quad \text{on } I,$$

for all $\alpha \in \mathbb{Z}_+^N$. Finally, by *(i)*, we obtain that $D^\alpha f(x) = \lim_{l\to\infty} D^\alpha P_l(x) = 0$ for every $x \in E$.

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