Bull. Malays. Math. Sci. Soc. (2020) 43:4273–4287
https://doi.org/10.1007/s40840-020-00921-y

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Graphs with Large Italian Domination Number

Teresa W. Haynes[1,2] · Michael A. Henning[2] · Lutz Volkmann[3]

Received: 11 June 2019 / Revised: 21 February 2020 / Published online: 6 March 2020

Abstract

An Italian dominating function on a graph G with vertex set $V(G)$ is a function $f : V(G) \to \{0, 1, 2\}$ having the property that for every vertex v with $f(v) = 0$, at least two neighbors of v are assigned 1 under f or at least one neighbor of v is assigned 2 under f. The weight of an Italian dominating function f is the sum of the values assigned to all the vertices under f. The Italian domination number of G, denoted by $\gamma_I(G)$, is the minimum weight of an Italian dominating of G. It is known that if G is a connected graph of order $n \geq 3$, then $\gamma_I(G) \leq \frac{3}{4}n$. Further, if G has minimum degree at least 2, then $\gamma_I(G) \leq \frac{2}{3}n$. In this paper, we characterize the connected graphs achieving equality in these bounds. In addition, we prove Nordhaus–Gaddum inequalities for the Italian domination number.

Keywords Domination · Italian domination · Roman domination · Roman {2}-domination

AMS subject classification: 05C69

Communicated by Xueliang Li.

✉ Teresa W. Haynes
haynes@etsu.edu

Michael A. Henning
mahenning@uj.ac.za

Lutz Volkmann
volkm@math2.rwth-aachen.de

1 Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614-0002, USA

2 Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa

3 Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany

1 Introduction

Cockayne et al. [3] first introduced Roman domination as a graphical invariant in 2004, following a series of papers (see [13–16]) on defense strategies of the ancient Roman Empire. Since its introduction, over 100 papers have been published on Roman domination and its variants. We refer the reader to [1,7,9,10] for some recent papers on Roman domination.

In this paper, we consider Italian domination, a variant of Roman domination. Italian domination was introduced as Roman {2}-domination in [2] and was renamed and studied further in [5]. See also [8,12]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E(G)$. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted by $N_G(v)$, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function*, abbreviated RD-function, on G if every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of a RD-function is the value $\mathrm{w}(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of a RD-function on G, and a RD-function with weight $\gamma_R(G)$ is called a γ_R-function of G.

One may view Roman domination as graph labeling problem in which each vertex labeled 0 must be adjacent to at least one vertex labeled 2. An *Italian dominating function*, abbreviated an ID-function, of G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V(G)$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. That is, either v is adjacent to at least one vertex u with $f(u) = 2$, or to at least two vertices x and y with $f(x) = f(y) = 1$. Viewed as a graph labeling problem, each vertex labeled 0 must have the labels of the vertices in its closed neighborhood sum to at least 2. The *weight* of a ID-function is the value $\mathrm{w}(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Italian domination number*, denoted by $\gamma_I(G)$, is the minimum weight of an ID-function in G, and an ID-function of G with weight $\gamma_I(G)$ is called a γ_I-function of G.

Our aim in this paper to continue the study of Italian dominating functions in graphs. We characterize connected graphs of order at least 3 with maximum possible Italian domination number. Further we characterize connected graphs with minimum degree at least two with maximum possible Italian domination number. We establish Nordhaus–Gaddum-type inequalities on the Italian domination number.

1.1 Terminology and Notation

For notation and graph theory terminology, we in general follow [4,6]. Specifically, let $G = (V, E)$ be a graph with vertex set $V(G) = V$ of *order* $n(G) = |V|$ and edge set $E(G) = E$ of *size* $m(G) = |E|$, and let v be a vertex in V. The *degree* of v in G, denoted by $d_G(v)$, is the cardinality of its open neighborhood $N_G(v)$, where recall that $N_G(v) = \{u \in V \mid uv \in E\}$. The *minimum* and *maximum degrees* among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S, denoted by $N_G(S)$, whereas the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. For a set $S \subseteq V(G)$, the subgraph induced by S in G is denoted by $G[S]$. Further, the graph obtained from G by deleting the vertices in S and all edges incident with S is denoted by $G - S$.

A set S of vertices in a graph G is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A vertex of degree one is a *leaf*, and its neighbor a *support vertex*. A graph G is r-*regular* if $\delta(G) = \Delta(G) = r$, and G is *regular* if it is r-*regular* for some r. We denote a *path* and *cycle* on n vertices by P_n and C_n, respectively, and we denote a *complete graph* on n vertices by K_n. We denote a *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$, by $K_{p,q}$. The graph $K_{1,q}$ is called a *star*. For $p, q \geq 1$, a *double star* $DS_{p,q}$ is the tree with exactly two vertices that are not leaves, one of which has p leaf neighbors and the other q leaf neighbors. A *daisy* with $k \geq 2$ *petals* is a connected graph that can be constructed from $k \geq 2$ disjoint cycles by identifying a set of k vertices, one from each cycle, into one vertex. If the k cycles have lengths $n_1, n_2, \ldots, n_k$, we denote the daisy by $D(n_1, n_2, \ldots, n_k)$. A *linear forest* is a forest in which every component is a path. We use the standard notation $[k] = \{1, \ldots, k\}$.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. A *diametral path* of G is a shortest path whose length is equal to the diameter of G. A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding the edges uw and vw. The *complement* of a graph G is denoted by $\overline{G}$.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v in T. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. A *grandchild* of v is a descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$ and is denoted by T_v.

2 Main Result

In this paper, we characterize the graphs with largest possible Italian domination number. For this purpose, we shall prove the following results, where $\mathcal{G}$ and $\mathcal{G}_{\geq 2}$ are families of graphs defined in Sect. 4. Proofs of Theorem 1 and 2 are given in Sects. 5 and 6.

Theorem 1 *If G is a connected graph of order $n \geq 3$, then $\gamma_I(G) \leq \frac{3}{4}n$ with equality if and only if $G \in \mathcal{G}$.*

Theorem 2 *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_I(G) \leq \frac{2}{3}n$ with equality if and only if $G \in \mathcal{G}_{\geq 2}$.*

Next, we establish Nordhaus–Gaddum-type result for the Italian domination number. We shall prove the following result, a proof of which is given in Sect. 7.

Theorem 3 *If G is a graph of order $n \geq 3$, then*

$$5 \leq \gamma_I(G) + \gamma_I(\overline{G}) \leq n + 2,$$

and these bounds are tight. Further if $\gamma_I(G) \leq \gamma_I(\overline{G})$, then $\gamma_I(G) + \gamma_I(\overline{G}) = 5$ if and only if there exists a vertex in G of degree $n - 1$ with a neighbor of degree 1 in G or with two adjacent neighbors of degree 2 in G.

We following result shows that the upper bound in Theorem 3 can be improved slightly if the graph G has no small components. More precisely, we prove the following result, a proof of which is given in Sect. 7.

Theorem 4 *If G is a graph of order $n \geq 16$ and having no component with fewer than three vertices, then $\gamma_I(G) + \gamma_I(\overline{G}) \leq n - 1$.*

3 Known Results and Observations

The following theorem summarizes bounds relating domination and Roman domination-type parameters.

Theorem 5 ([2]) *For every graph G, $\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

If G is a graph of order n, then assigning to every vertex the weight 1 produces an ID-function of G, implying that $\gamma_I(G) \leq n$. In addition, in [17] it was proved that $\gamma_I(G) = n$ if and only if $\Delta(G) \leq 1$.

The Italian domination number of a path or a cycle is easy to compute (or see [8]). If $C : v_1, v_2 \dots v_n v_1$ is a cycle of order $n \geq 3$, then the function that assigns weight 1 to the vertices of odd subscript and weight 0 to the vertices of even subscript is an example of a γ_I-function of the cycle.

Observation 1 ([8]) *For $n \geq 1$, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$, while for $n \geq 3$, $\gamma_I(C_n) = \lfloor \frac{n+1}{2} \rfloor$.*

The Italian domination number of a daisy is also easy to compute and follows readily from Observation 1 noting that there exists a γ_I-function of the cycle that assigns to any given selected vertex the weight 1.

Observation 2 *If G is a daisy of order n, then $\gamma_I(G) \leq \lfloor \frac{n+1}{2} \rfloor$.*

Klostermeyer and MacGillivray [8] proved the following upper bounds on the Italian domination number in terms its order.

Theorem 6 ([8]) *If G is a graph of order $n \geq 3$, then*

$$\gamma_I(G) \leq \begin{cases} \frac{3}{4}n \text{ if } G \text{ is connected3} \\ \frac{2}{3}n \text{ if } \delta(G) \geq 2 \\ \frac{1}{2}n \text{ if } \delta(G) \geq 3. \end{cases}$$

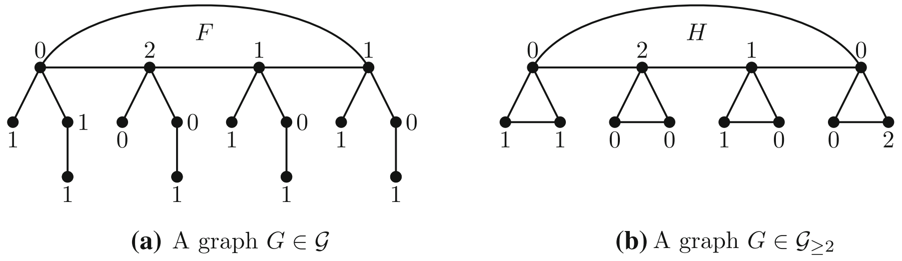

Fig. 1 Graphs in the families $\mathcal{G}$ and $\mathcal{G}_{\geq 2}$

Results of Nordhaus–Gaddum-type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [11], Nordhaus and Gaddum discussed this problem for the chromatic number. We use the following Nordhaus–Gaddum results on the Roman domination of a graph and its complement.

Theorem 7 ([1]) *If G is a graph of order $n \geq 3$, then*

$$5 \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

Furthermore, equality holds in the upper bound only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

Proposition 8 ([17]) *If G is a graph of order n, then $\gamma_I(G) = n$ if and only if $\Delta(G) \leq 1$.*

Proposition 9 ([17]) *If G is a graph of order $n \geq 2$, then $\gamma_I(G) = 2$ if and only if $\Delta(G) = n - 1$ or there exist two different vertices u and v such that $N(u) \cap N(v) = V(G) \setminus \{u, v\}$.*

4 The Families $\mathcal{G}$, $\mathcal{T}$, $\mathcal{G}_{\geq 2}$ and $\mathcal{G}_{\geq 2}^{\min}$

Let F be an arbitrary connected graph of order $n_F \geq 1$, and let G be the graph of order $n = 4n_F$ obtained from F by adding to each vertex v of F three new vertices u, w and x and the edges uv, vw and wx. Thus, $uvwx$ is a path in G, where u is a leaf neighbor of v, w is a neighbor of v of degree 2 and x is a leaf neighbor of w. We call the graph F the *underlying graph* of the graph G. Let $\mathcal{G}$ be the family of all such graphs G. Further, let $\mathcal{T}$ be the family of all such graphs G whose underlying graph is a tree. We note that $\mathcal{T}$ is a subfamily of $\mathcal{G}$. When the underlying graph F is a cycle C_4 on four vertices, the graph G is illustrated in Fig. 1, where the assignment of weights is an example of a γ_I-function of G. We show that every graph in the family $\mathcal{G}$ has Italian domination number three-fourths its order.

Proposition 10 *If $G \in \mathcal{G}$ has order n, then $\gamma_I(G) = \frac{3}{4}n$.*

Proof Let G be an arbitrary graph in the family $\mathcal{G}$, and let G have order n. Let F be the underlying graph of G of order n_F, and so $n = 4n_F$. Adopting our earlier notation, let v be an arbitrary vertex of F, and let u, w and x be the three new vertices and uv, vw

and wx the three new edges added to F when constructing G. Let f be a γ_I-function of G. We show that $f(u)+f(v)+f(w)+f(x) \geq 3$. If $f(v)=0$, then $f(u) \geq 1$ and $f(w)+f(x) \geq 2$. If $f(v)=1$, then $f(u) \geq 1$ and $f(w)+f(x) \geq 1$. If $f(v)=2$, then $f(u)=0$ and $f(w)+f(x) \geq 1$. In all three cases, $f(u)+f(v)+f(w)+f(x) \geq 3$, as claimed. Since this is true for every vertex v of F, we note that $\gamma_I(G) = \mathrm{w}(f) \geq 3n_F$. The function f^* that assigns the value 2 to every vertex of F, the value 1 to every vertex in $V(G) \setminus V(F)$ that has no neighbor in F, and the value 0 to all other vertices of G is an ID-function of G of weight $3n_F$, implying that $\gamma_I(G) \leq \mathrm{w}(f^*) = 3n_F$. Consequently, $\gamma_I(G) = 3n_F = \frac{3}{4}n$. □

We next construct a class $\mathcal{G}_{\geq 2}$ of graphs with minimum degree two as follows. Let H be an arbitrary connected graph of order $n_H \geq 1$, and let G be the graph of order $n = 3n_H$ obtained from H by adding to each vertex v of H two new vertices u and w and the edges uv, vw and uw. Thus, $G[\{u, v, w\}] = K_3$, where both u and w have degree 2 in G. We call the graph H the *underlying graph* of the graph G. Further, we call a triangle of G that contains exactly one vertex of H a *core triangle* of G. For example, the above triangle $G[\{u, v, w\}]$ is a core triangle. Let $\mathcal{G}_{\geq 2}$ be the family of all such graphs G. Further, let $\mathcal{G}^{\min}_{\geq 2}$ be the family of all such graphs G whose underlying graph is a tree. We note that $\mathcal{G}^{\min}_{\geq 2}$ is a (proper) subfamily of $\mathcal{G}_{\geq 2}$. When the underlying graph H is a cycle C_4 on four vertices, the graph G is illustrated in Fig. 1b, where the assignment of weights is an example of a γ_I-function of G. We establish next properties of the graphs in the family $\mathcal{G}_{\geq 2}$.

Proposition 11 *If $G \in \mathcal{G}_{\geq 2}$ has order n, and x and y are arbitrary vertices of G, then the following holds.*

(a) $\gamma_I(G) = \frac{2}{3}n$.
(b) *There exists a γ_I-function f_i of G such that $f_i(x) = i$ for $i \in \{0, 1, 2\}$.*
(c) *There exists a γ_I-function f of G such that $f(x) = f(y) = 1$.*

Proof (a) Let H be the underlying graph of $G \in \mathcal{G}_{\geq 2}$ of order n_H, and so $n = 3n_H$. Adopting our earlier notation, let v be an arbitrary vertex of H, and let u and w be the two new vertices and uv, vw and uw the three new edges added to H when constructing G. Let f be a γ_I-function of G. We show that $f(u)+f(v)+f(w) \geq 2$. If $f(v)=0$, then $f(u)+f(w) \geq 2$. If $f(v)=1$, then $f(u)+f(w) \geq 1$. If $f(v)=2$, then $f(u)+f(w) \geq 0$. In all three cases, $f(u)+f(v)+f(w) \geq 2$, as claimed. Since this is true for every vertex v of H, we note that $\gamma_I(G) = \mathrm{w}(f) \geq 2n_H$. The function f^* that assigns the value 2 to every vertex of H and the value 0 to all vertices of $V(G) \setminus V(H)$ is an ID-function of G of weight $2n_H$, implying that $\gamma_I(G) \leq \mathrm{w}(f^*) = 2n_H$. Consequently, $\gamma_I(G) = 2n_H = \frac{2}{3}n$.

(b) Let x be an arbitrary vertex of G and let G_x be the core triangle in $G \in \mathcal{G}_{\geq 2}$ that contains x. Let f^* be the γ_I-function of G defined in Part (a). For $i \in \{0, 1, 2\}$, let f_i be the γ_I-function of G defined as follows: let $f_i(v) = f^*(v)$ for all $v \in V(G) \setminus V(G_x)$. If $i = 0$, let f_i assign the weight 0 to x and the weight 1 to the remaining two vertices of G_x. If $i = 1$, let f_i assign the weight 1 to x and to exactly one other vertex of G_x, and the weight 0 to the remaining vertex of G_x. If $i = 2$, let f_i assign the weight 2 to x and the weight 0 to the remaining two vertices of G_x. The resulting function f_i is a γ_I-function of G such that $f_i(x) = i$ for $i \in \{0, 1, 2\}$.

(c) Let f^* be the γ_I-function of G defined in Part (a). Let x and y be arbitrary vertices of G. Let G_x and G_y be the core triangles in $G \in \mathcal{G}_{\geq 2}$ that contains x and y, respectively. If $G_x = G_y$, let f be the γ_I-function of G defined as follows: let $f(v) = f^*(v)$ for all $v \in V(G) \setminus V(G_x)$, let $f(x) = f(y) = 1$ and let f assign the weight 0 to the remaining vertex of G_x. If $G_x \neq G_y$, let f be the γ_I-function of G defined as follows: let $f(v) = f^*(v)$ for all $v \in V(G) \setminus (V(G_x) \cup V(G_y))$, let f assign the weight 1 to x and one other vertex of G_x, the weight 1 to y and one other vertex of G_y, and the weight 0 to the remaining two vertices of $G_x \cup G_y$. The resulting function f is a γ_I-function of G such that $f(x) = f(y) = 1$. □

5 Proof of Theorem 1

In this section, we present a proof of Theorem 1. We remark that although Theorem 6 gives that $\gamma_I(G) \leq \frac{3}{4}n$ for connected graphs G of order $n \geq 3$, the extremal graphs are not characterized in [8]. Toward that end, we present a different proof of the $\frac{3}{4}$-bound here that leads to the characterizations of trees and connected graphs achieving equality in the bound. Since deleting an edge cannot decrease the Italian domination number, it suffices to first prove the bound for trees.

Theorem 12 *If T is a tree of order $n \geq 3$, then $\gamma_I(T) \leq \frac{3}{4}n$ with equality if and only if $T \in \mathcal{T}$.*

Proof If $T \in \mathcal{T}$ has order n, then by Proposition 10, $\gamma_I(T) = \frac{3}{4}n$. To prove the necessity, we proceed by induction on the order n of a tree T. Since $n \geq 3$, $\mathrm{diam}(T) \geq 2$. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,n-1}$ for $n \geq 3$ and $\gamma_I(T) = 2 < \frac{3}{4}n$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{r,s}$ for $1 \leq r \leq s$. Hence, $n = r + s + 2 \geq 4$. If $r = 1$, then $\gamma_I(T) = 3 \leq \frac{3}{4}n$ with equality if and only if $s = 1$ and thus $n = 4$ and $T = P_4 \in \mathcal{T}$. If $r \geq 2$, then $n \geq 6$ and $\gamma_I(T) = 4 < \frac{3}{4}n$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$, for otherwise the desired result follows. Thus, $n \geq 5$. Given a subtree T' with n' vertices, where $3 \leq n' < n$, the induction hypothesis yields a γ_I-function f' of T' with weight $\gamma_I(T') = \mathrm{w}(f') \leq \frac{3}{4}n'$. Let $P: v_1 v_2 \ldots v_k$ ($k \geq 5$) be a diametral path of T such that $d_T(v_2)$ is as large as possible. We now root the tree T at the vertex v_k. We proceed further with the following two claims. □

Claim 1 *If $d_T(v_2) \geq 3$, then $\gamma_I(T) < \frac{3}{4}n$.*

Proof Suppose that $d_T(v_2) \geq 3$. Let T' be obtained from T by deleting v_2 and its leaf neighbors. Since $\mathrm{diam}(T) \geq 4$, we have $n' \geq 3$. Applying the inductive hypothesis to T', we have $\mathrm{w}(f') \leq \frac{3}{4}n'$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$, $f(v_2) = 2$ and $f(x) = 0$ for each leaf neighbor x of v_2. The resulting function f is an ID-function on T of weight $\mathrm{w}(f) = \mathrm{w}(f') + 2 \leq \frac{3}{4}n' + 2 \leq \frac{3}{4}(n-3) + 2 < \frac{3}{4}n$. □

Claim 2 *If $d_T(v_2) = d_T(v_3) = 2$, then $\gamma_I(T) < \frac{3}{4}n$.*

Proof Suppose that $d_T(v_2) = d_T(v_3) = 2$. Let T' be obtained from T by deleting v_1, v_2 and v_3. Since $\mathrm{diam}(T) \geq 4$, we have $n' \geq 2$. If $n' = 2$, then T is the path

P_5, and so $\gamma_I(T) = \gamma_I(P_5) = 3 < \frac{3}{4}n$. Hence, we may assume that $n' \geq 3$, for otherwise the desired result follows. Applying the inductive hypothesis to T', we have $\mathrm{w}(f') \leq \frac{3}{4}n'$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$, $f(v_2) = 2$ and $f(v_1) = f(v_3) = 0$. The resulting function f is an ID-function on T of weight $\mathrm{w}(f) = \mathrm{w}(f') + 2 \leq \frac{3}{4}n' + 2 \leq \frac{3}{4}(n-3) + 2 < \frac{3}{4}n$. □

By Claim 1 and Claim 2, we may assume that $d_T(v_2) = 2$ and $d_T(v_3) \geq 3$, for otherwise the desired result follows. By the choice of v_2, every child of v_3 is either a leaf or a vertex similar to v_2, that is, a support vertex of degree 2 with exactly one leaf neighbor.

Claim 3 *If the parent of v_3 is a support vertex of degree 2 in T, then $\gamma_I(T) < \frac{3}{4}n$.*

Proof Suppose that the parent, v_4, of v_3 is a support vertex of degree 2 in T. Thus, T is obtained from a star by subdividing t edges, where $t \geq 2$. If v_3 has no leaf neighbors, then the function f that assigns the weight 1 to v_3 and to every vertex at distance 2 from v_3 and the weight 0 to the remaining vertices of T is a ID-function of T of weight $t+1$, implying that

$$\gamma_I(T) \leq \mathrm{w}(f) = t + 1 < \frac{3}{4}(2t+1) = \frac{3}{4}n.$$

If v_3 has at least one leaf neighbor, then the function f that assigns the weight 2 to v_3, the weight 1 to every vertex at distance 2 from v_3 and the weight 0 to the remaining vertices of T is a ID-function of T of weight $t+2$, implying that

$$\gamma_I(T) \leq \mathrm{w}(f) = t + 2 < \frac{3}{4}(2t+2) \leq \frac{3}{4}n.$$

This completes the proof of Claim 3. □

By Claim 3, we may assume that the parent, v_4, of v_3 is not a support vertex of degree 2 in T. Let T' be obtained from T by deleting v_3 and all its descendants. Thus, $T' = T - V(T_{v_3})$ and $n' \geq 3$. Applying the inductive hypothesis to T', we have $\mathrm{w}(f') \leq \frac{3}{4}n'$. As in the proof of Claim 3, we observe that T_{v_3} is obtained from a star by subdividing $t \geq 1$ edges.

If v_3 has no leaf neighbors, then let f be the ID-function of T defined as follows: let $f(x) = f'(x)$ for all $x \in V(T')$, and let f assign the weight 1 to v_3 and to every grandchild of v_3 and the weight 0 to every child of v_3. The resulting ID-function of T has weight $\mathrm{w}(f) = \mathrm{w}(f') + t + 1$, implying that

$$\gamma_I(T) \leq \mathrm{w}(f) = \mathrm{w}(f') + t + 1 \leq \frac{3}{4}n' + t + 1 = \frac{3}{4}(n - 2t - 1) + t + 1 < \frac{3}{4}n.$$

Hence, we may assume that v_3 has $r \geq 1$ leaf neighbors. In this case, let f be the ID-function of T defined as follows: let $f(x) = f'(x)$ for all $x \in V(T')$, and let f assign the weight 2 to v_3, the weight 1 to every grandchild of v_3 and the weight 0 to every child of v_3. The resulting ID-function of T has weight $\mathrm{w}(f) = \mathrm{w}(f') + t + 2$, implying that

$$\gamma_I(T) \le \mathrm{w}(f) = \mathrm{w}(f') + t + 2 \le \frac{3}{4}n' + t + 2 = \frac{3}{4}(n - 2t - 1 - r) + t + 2 \le \frac{3}{4}n. \quad (1)$$

This establishes the desired upper bound. Suppose that $\gamma_I(T) = \frac{3}{4}n$. Thus, we must have equality throughout the Inequality Chain (1). In particular, this implies that $\gamma_I(T) = \mathrm{w}(f)$ and $\mathrm{w}(f') = \frac{3}{4}n'$. Further, $t = 1$ and $r = 1$, and so T_{v_3} is the path $v_1v_2v_3u_2$ where u_2 is the child of v_3 different from v_2. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. We note that n' is a multiple of 4.

Suppose that $n' = 4$, and so $n = 8$ and T' is a path P_4. If v_4 is a leaf of T', then by our choice of the diametral path P, the subtree T' is the path $v_4v_5v_6v_7$. The function that assigns the weight 2 to v_3, the weight 1 to v_1, v_5 and v_7, and the weight 0 to the remaining vertices of T is a ID-function of T of weight $5 < \frac{3}{4}n = \gamma_I(T)$, a contradiction. Hence, v_4 is a vertex of degree 2 in T', implying that $T \in \mathcal{T}$, as desired. Hence, we may assume that $n' \ge 8$.

Let F' be the underlying tree of $T' \in \mathcal{T}$. By our earlier assumptions, $\gamma_I(T') = \mathrm{w}(f') = \frac{3}{4}n'$. By Proposition 10, we may choose f' so that $f'(x) = 2$ for every vertex $x \in V(F')$, $f'(x) = 0$ for every vertex in $V(T') \setminus V(F')$ that has a neighbor in F', and $f'(x) = 1$ for the remaining vertices of T' that are not in F' and have no neighbor in F'. Let $x_1x_2x_3x_4$ be the copy of P_4 in the construction of T' that contains v_4, where $x_2 \in V(F')$. Thus, x_1 is a leaf neighbor of x_2, the vertex x_3 is a neighbor of x_2 of degree 2 in T' and x_4 is a leaf neighbor of x_3 in T'. Since $n' \ge 8$, the vertex x_2 has at least one neighbor, say y_2, in F'. Let $X = \{x_1, x_2, x_3, x_4\}$ and let $W = \{v_1, v_2, v_3, u_2\}$.

We show that $x_2 = v_4$. Suppose, to the contrary, that $v_4 \in \{x_1, x_3, x_4\}$, and so $v_4 \notin V(F')$. Let f^* be the ID-function of T defined as follows: let $f^*(x) = f'(x)$ for all $x \in V(T') \setminus X$. If $v_4 = x_1$, let f^* assign the weight 2 to v_3 and x_3, the weight 1 to v_1, and the weight 0 to remaining vertices in $X \cup W$. If $v_4 = x_3$, let f^* assign the weight 2 to v_3, the weight 1 to v_1, x_1 and x_4, and the weight 0 to remaining vertices in $X \cup W$. If $v_4 = x_4$, let f^* assign the weight 2 to v_3 and x_2, the weight 1 to v_1, and the weight 0 to remaining vertices in $X \cup W$. In all three cases, we note that the resulting ID-function f^* of T has weight

$$\mathrm{w}(f^*) = (\mathrm{w}(f') - 3) + 5 = \frac{3}{4}n' + 2 = \frac{3}{4}(n - 4) + 2 < \frac{3}{4}n = \gamma_I(T),$$

a contradiction. Hence, $x_2 = v_4$, implying that $T \in \mathcal{T}$. □

We are now in a position to present a proof of Theorem 1. Recall its statement.

Theorem 1. *If G is a connected graph of order $n \ge 3$, then $\gamma_I(G) \le \frac{3}{4}n$ with equality if and only if $G \in \mathcal{G}$.*

Proof If $G \in \mathcal{G}$ has order n, then by Proposition 10, $\gamma_I(G) = \frac{3}{4}n$. To prove the necessity, let G be a connected graph of order $n \ge 3$ and let T be an arbitrary spanning tree of G. Since deleting edges cannot decrease the Italian domination number, Theorem 12 implies that $\gamma_I(G) \le \gamma_I(T) \le \frac{3}{4}n$. It remains for us to show that if $\gamma_I(G) = \frac{3}{4}n$, then $G \in \mathcal{G}$. Suppose that $\gamma_I(G) = \frac{3}{4}n$, implying that $\gamma_I(T) = \frac{3}{4}n$. By Theorem 12, the spanning tree $T \in \mathcal{T}$. Let F be the underlying tree of T. We note that $n = 4k$ for some integer $k \geq 1$. Let $T_1, \ldots, T_k$ be the subtrees of T used to build the tree T, as described in Sect. 4, where each T_i is isomorphic to a path P_4. Let $V_i = V(T_i)$ for $i \in [k]$. If $G[V_i]$ is not a path for some $i \in [k]$, then it is easy to see that there exists an ID-function of T of weight less than $\frac{3}{4}n$, a contradiction. Hence, $G[V_i] = T_i \cong P_4$ for all $i \in [k]$. If an edge uv of G joins vertices of V_i and V_j for $i \neq j$ such that $u \notin V(F)$ or $v \notin V(F)$, then an ID-function of weight less than $\frac{3}{4}n$ can be found analogously as in the last paragraph of the proof of Theorem 12, a contradiction. Hence, if uv is an edge of G that joins vertices of V_i and V_j for $i \neq j$, then both u and v belong to F, implying that $G \in \mathcal{G}$. □

6 Proof of Theorem 2

We remark that although Theorem 6 gives that $\gamma_I(G) \leq \frac{2}{3}n$ for connected graphs G of order n and minimum degree at least 2, the extremal graphs are not characterized in [8]. Toward that end, we present a different proof of the $\frac{2}{3}$-bound here that leads to the characterizations of the connected graphs achieving equality in the bound. For this purpose, we present a proof of Theorem 2. We refer to a graph G as an *edge-minimal graph* if G is edge minimal with respect to satisfying the conditions that $\delta(G) \geq 2$ and G is connected. Since deleting edges cannot decrease the Italian domination number, it suffices to first prove the following result for edge-minimal graphs. In what follows, we define a vertex of a graph G with minimum degree at least two as *small* if it has degree 2 in G, and *large* if it has degree more than 2 in G.

Theorem 13 *If G is an edge-minimal graph of order n, then $\gamma_I(G) \leq \frac{2}{3}n$ with equality if and only if $G \in \mathcal{G}^{\min}_{\geq 2}$.*

Proof If $G \in \mathcal{G}^{\min}_{\geq 2}$ has order n, then by Proposition 11, $\gamma_I(G) = \frac{2}{3}n$. To prove the necessity, we proceed by induction on the order n of an edge-minimal graph G of order n. If $n = 3$, then $G = C_3 \in \mathcal{G}^{\min}_{\geq 2}$. If $n = 4$, then $G = C_4$ and $\gamma_I(G) = 2 < \frac{2}{3}n$. If $n = 5$, then either $G = C_5$ or $G = K_{2,3}$ or G is the daisy $D(3, 3)$. In all three cases, $\gamma_I(G) \leq 3 < \frac{2}{3}n$. This establishes the base cases. Let $n \geq 6$ and suppose that if G' is an edge-minimal graph of order n' where $n' < n$, then $\gamma_I(G') \leq \frac{2}{3}n'$ with equality if and only if $G' \in \mathcal{G}^{\min}_{\geq 2}$. Let G be an edge-minimal graph of order n.

If G is a cycle or a daisy, then by Observations 1 and 2, $\gamma_I(G) = \lfloor \frac{n+1}{2} \rfloor < \frac{2}{3}n$ noting that here $n \geq 6$. Hence, we may assume that G is neither a cycle nor a daisy, for otherwise the desired result follows. Thus, G contains at least two large vertices. Let $\mathcal{L}$ be set of all large vertices of G, i.e., $\mathcal{L} = \{v \in V(G) \mid d_G(v) \geq 3\}$. Let $|\mathcal{L}| = \ell$. By assumption, $\ell \geq 2$. We proceed further with the following series of claims. □

Claim 4 *If the set $\mathcal{L}$ is not an independent set, then $\gamma_I(G) \leq \frac{2}{3}n$ with equality only if $G \in \mathcal{G}^{\min}_{\geq 2}$.*

Proof Suppose that $e = v_1v_2$ is an edge of G, where $v_1, v_2 \in \mathcal{L}$. By the edge minimality of G, the edge e is a bridge of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $v_i \in V_i$ for $i \in [2]$. For $i \in [2]$, let $|V_i| = n_i$, and so $n = n_1 + n_2$. We note that $\gamma_I(G) \le \gamma_I(G_1) + \gamma_I(G_2)$. For $i \in [2]$, the graph G_i is an edge-minimal graph. Applying the inductive hypothesis to G_i, $\gamma_I(G_i) \le \frac{2}{3}n_i$ with equality if and only if $G_i \in \mathcal{G}^{\min}_{\ge 2}$. In particular, $\gamma_I(G) \le \frac{2}{3}n_1 + \frac{2}{3}n_2 = \frac{2}{3}n$. This establishes the desired upper bound on $\gamma_I(G)$.

Suppose, next, that $\gamma_I(G) = \frac{2}{3}n$. Thus, $\frac{2}{3}n = \gamma_I(G) \le \frac{2}{3}n_1 + \frac{2}{3}n_2 = \frac{2}{3}n$. Hence, we must have equality throughout this inequality chain. Thus, $\gamma_I(G_i) = \frac{2}{3}n_i$ for $i \in [2]$, and so $G_i \in \mathcal{G}^{\min}_{\ge 2}$. Let T_i be the underlying tree of G_i for $i \in [2]$. If $G_1 = C_3$ and $G_2 = C_3$, then $G \in \mathcal{G}^{\min}_{\ge 2}$, as desired. Hence, we may assume that $n_1 \ge 6$. If v_1 does not belong to the underlying tree T_1, then removing the edge joining v_1 to its neighbor in T_1 contradicts the edge minimality of G. Hence, $v_1 \in V(T_1)$. If $G_2 = C_3$, then $G \in \mathcal{G}^{\min}_{\ge 2}$. If $G_2 \ne C_3$, then $n_2 \ge 6$, implying by the edge minimality of G that $v_2 \in V(T_2)$, and therefore that $G \in \mathcal{G}^{\min}_{\ge 2}$. □

By Claim 4, we may assume that the set $\mathcal{L}$ of large vertices is independent, for otherwise the desired result follows. By our earlier assumptions, $\ell \ge 2$. For $k \ge 3$, we define a *k-handle* of a graph G as a $(k+1)$-cycle with exactly one vertex of degree at least 3 in G. Further, we define a *handle* of G to be a k-handle for some $k \ge 3$. For $k \ge 1$, we define a *k-linkage* in G as a path on $k+2$ vertices that starts and ends at distinct large vertices and with k internal vertices of degree 2 in G.

Claim 5 *If the graph G contains a k-linkage for some $k \ge 3$ or a k-handle for some $k \ge 5$, then $\gamma_I(G) < \frac{2}{3}n$.*

Proof Suppose that G contains a k-linkage $L\colon x v_1 v_2 \dots v_k y$ for some $k \ge 3$ that joins large vertices x and y. By definition, the k internal vertices of L all have degree 2 in G. Let G^* be the graph of order $n' = n - 3 \ge 4$ obtained from G by deleting the set of vertices $\{v_1, v_2, v_3\}$. If $k \ge 4$, let $w = v_4$, while if $k = 3$, let $w = y$. If $k \ge 4$, let $G' = G^* + xw$. If $k = 3$ and G^* is connected, let $G' = G^*$. If $k = 3$ and G^* is disconnected, let $G' = G^* + xw$. In all cases, the resulting graph G' is an edge-minimal graph. Applying the inductive hypothesis to G', $\gamma_I(G') \le \frac{2}{3}n'$ with equality if and only if $G' \in \mathcal{G}^{\min}_{\ge 2}$. Let f' be a γ_I-function of G'.

Let f be the ID-function of G defined as follows: let $f(v) = f'(v)$ for all $v \in V(G')$. If $f'(x) \ge 1$ and $f'(w) \ge 1$, let $f(v_2) = 1$ and $f(v_1) = f(v_3) = 0$. If $f'(x) = f'(w) = 0$, let $f(v_2) = 2$ and $f(v_1) = f(v_3) = 0$. If $f'(x) = 0$ and $f'(w) = 1$ or $f'(x) = 1$ and $f'(w) = 0$, let $f(v_1) = f(v_3) = 1$ and $f(v_2) = 0$. If $f'(x) = 0$ and $f'(w) = 2$, let $f(v_1) = 2$ and $f(v_2) = f(v_3) = 0$. If $f'(x) = 2$ and $f'(w) = 0$, let $f(v_3) = 2$ and $f(v_1) = f(v_2) = 0$. In all cases, the resulting ID-function f of G has weight

$$\mathrm{w}(f) \le \mathrm{w}(f') + 2 \le \frac{2}{3}n' + 2 = \frac{2}{3}n. \tag{2}$$

Suppose that $\mathrm{w}(f) = \frac{2}{3}n$. Thus, we must have equality throughout the Inequality Chain (2). In particular, $\mathrm{w}(f') = \frac{2}{3}n'$, implying that $G' \in \mathcal{G}^{\min}_{\ge 2}$. By Proposition 11, the γ_I-function f' of G' can be chosen so that $f(x) = f(w) = 1$. In this case, $f(v_2) = 1$

and $f(v_1) = f(v_3) = 0$, implying that $\gamma_I(G) \le \mathrm{w}(f) = \mathrm{w}(f') + 1 = \frac{2}{3}n - 1$. Hence, if G contains a k-linkage for some $k \ge 3$, then $\gamma_I(G) < \frac{2}{3}n$. If G contains a k-handle $xv_1v_2 \dots v_kx$ for some $k \ge 5$ and we let $w = v_4$ and let G' be the graph obtained from G by deleting the set of vertices $\{v_1, v_2, v_3\}$ and adding the edge xw, then identical arguments as in the previous case when G contains a k-linkage show that $\gamma_I(G) < \frac{2}{3}n$. □

By Claim 5, we may assume that the graph G contains no k-linkage where $k \ge 3$ and no k-handle where $k \ge 5$, for otherwise the desired result follows. We now consider the graph $F = G - \mathcal{L}$. By our earlier assumptions, F is a linear forest and every component of F is a path of order 1, 2, 3 or 4. Further, every component of F is joined by exactly two edges to vertices of $\mathcal{L}$ in G. Let $\mathcal{S}$ be a set of all small vertices (of degree 2) in G, and let $|\mathcal{S}| = s$. We note that $F = G[\mathcal{S}]$. Let s_i be the number of components in F of order i for $i \in [4]$. Thus, $n = \ell + s = \ell + s_1 + 2s_2 + 3s_3 + 4s_4$. Let f be the ID-function of G defined as follows. Let f assign the weight 1 to every vertex of $\mathcal{L}$, the weight 1 to every vertex of $\mathcal{S}$ that has no neighbor in $\mathcal{L}$, the weight 1 to exactly one vertex from every component of F of order 2, and the weight 0 to the remaining vertices of G. Thus, $\gamma_I(G) < \frac{2}{3}n$ holds if the following is true:

$$\begin{array}{rrl} & \mathrm{w}(f) & < \frac{2}{3}n \\ \Leftrightarrow & \ell + s_2 + s_3 + 2s_4 & < \frac{2}{3}(\ell + s_1 + 2s_2 + 3s_3 + 4s_4) \\ \Leftrightarrow & \ell & < 2s_1 + s_2 + 3s_3 + 2s_4. \end{array}$$

Counting the edges $[\mathcal{L}, \mathcal{S}]$ between $\mathcal{L}$ and $\mathcal{S}$, we note that $3\ell \le |[\mathcal{L}, \mathcal{S}]| = 2(s_1 + s_2 + s_3 + s_4)$, and so

$$\ell \le \frac{2}{3}(s_1 + s_2 + s_3 + s_4) < 2s_1 + s_2 + 3s_3 + 2s_4.$$

Therefore, $\gamma_I(G) < \frac{2}{3}n$ holds. This completes the proof of Theorem 13. □

We are now in a position to present a proof of Theorem 2. Recall its statement.
Theorem 2. *If G is a connected graph of order n with $\delta(G) \ge 2$, then $\gamma_I(G) \le \frac{2}{3}n$ with equality if and only if $G \in \mathcal{G}_{\ge 2}$.*

Proof If $G \in \mathcal{G}_{\ge 2}$ has order n, then by Proposition 11(a), $\gamma_I(G) = \frac{2}{3}n$. To prove the necessity, let G be a connected graph of order n with $\delta(G) \ge 2$. Let G' be an arbitrary spanning graph of G obtained by deleting edges, if necessary, until the resulting graph is an edge-minimal graph (with respect to satisfying the conditions minimum degree at least 2 and the graph being connected). By Theorem 13, $\gamma_I(G') \le \frac{2}{3}n$ with equality if and only if $G' \in \mathcal{G}_{\ge 2}^{\min}$. Since deleting edges cannot decrease the Italian domination number, we note that $\gamma_I(G) \le \gamma_I(G') \le \frac{2}{3}n$. This establishes the desired upper bound. Suppose that $\gamma_I(G) = \frac{2}{3}n$, implying that $\gamma_I(G') = \frac{2}{3}n$ and $G' \in \mathcal{G}_{\ge 2}^{\min}$. If $G = G'$, then the desired result follows. Hence, we may assume that G' is a proper subgraph of G, implying that $n = 3k$ for some $k \ge 2$. Thus, G' has k core triangles, and so kK_3 is a spanning subgraph of G'. Let T be the underlying tree of G'.

Suppose that two core triangles of G' are joined by two or more edges. Let H be the subgraph of G induced by the vertices of these two triangles. We note that H contains a vertex of degree 4 or 5 or H contains a 6-cycle as a subgraph. In both cases, $\gamma_I(H) \le 3$, implying that $\gamma_I(G) \le \gamma_I(H)+(k-2)\gamma_I(K_3) \le 3+2(k-2) < 2k = \frac{2}{3}n$, a contradiction. Hence, two (distinct) core triangles of G' are joined by at most one edge in G.

Let $e = xy$ be an arbitrary edge in G that was deleted when constructing G'. We show that both x and y belong to the underlying tree T of G'. Suppose, to the contrary, that $x \notin V(T')$. Let G_x and G_y be the core triangles in G' that contain x and y, respectively. By our earlier observations, the edge e is the only edge joining G_x and G_y. Let x' be the vertex of G_x that belongs to T, and let w be a neighbor of x' in T. Further, let G_w be the core triangle in G' that contains w. Let F be the subgraph of G induced by the vertices of these three core triangles, G_w, G_x and G_y. We note that F contains a path P_9 as a subgraph, implying that $\gamma_I(G) \le \gamma_I(F) + (k-3)\gamma_I(K_3) \le 5+2(k-3) < 2k = \frac{2}{3}n$, a contradiction. Hence, both x and y belong to the underlying tree T of G'. Since $e = xy$ be an arbitrary edge in G, this implies that every edge of $E(G) \setminus E(G')$ joins two vertices of T and, therefore, $G \in \mathcal{G}_{\ge 2}$. □

7 Nordhaus–Gaddum-Type Bounds

In this section, we establish Nordhaus–Gaddum-type results for the Italian domination number. Recall the statement of Theorem 3.

Theorem 3. *If G is a graph of order $n \ge 3$, then*

$$5 \le \gamma_I(G) + \gamma_I(\overline{G}) \le n + 2,$$

and these bounds are tight. Further if $\gamma_I(G) \le \gamma_I(\overline{G})$, then $\gamma_I(G) + \gamma_I(\overline{G}) = 5$ if and only if there exists a vertex in G of degree $n-1$ with a neighbor of degree 1 in G or with two adjacent neighbors of degree 2 in G.

Proof If G is the cycle C_5 of order 5, then $\overline{G}$ is also the cycle C_5 of order 5, and we observe that $\gamma_I(G) + \gamma_I(\overline{G}) = 3 + 3 = 6 = n + 1$. If $G = \frac{n}{2}K_2$, then Propositions 8 and 9 imply $\gamma_I(G) + \gamma_I(\overline{G}) = n + 2$. Using these observations, the fact that $\gamma_I(G) \le \gamma_R(G)$ and Theorem 7, we obtain the desired upper bound. If G is a graph of order $n \ge 3$ with $\Delta(G) \le 1$, then we deduce from Propositions 8 and 9 that $\gamma_I(G)+\gamma_I(\overline{G}) = n+2$. Therefore, the upper bound in Theorem 3 is sharp.

For the lower bound assume, without loss of generality, that $\gamma_I(G) \le \gamma_I(\overline{G})$. If $\gamma_I(G) \ge 3$, then $\gamma_I(G) + \gamma_I(\overline{G}) \ge 6$. Thus, let now $\gamma_I(G) = 2$. Then $\Delta(G) = n - 1$ or there exist two different vertices u and v such that $N(u) \cap N(v) = V(G) \setminus \{u, v\}$ by Proposition 9. Therefore, $\overline{G}$ has a component of order 1 or a component of order 2. Since $n \ge 3$, this implies $\gamma_I(\overline{G}) \ge 3$, and so $\gamma_I(G) + \gamma_I(\overline{G}) \ge 5$.

Next, we characterize the graphs G of order $n \ge 3$ with $\gamma_I(G) + \gamma_I(\overline{G}) = 5$. Assume, without loss of generality, that $\gamma_I(G) \le \gamma_I(\overline{G})$. Suppose that there exists a vertex w in G of degree $n - 1$. By Proposition 9, we note that $\gamma_I(G) = 2$. If the vertex w has a neighbor x of degree 1 in G, then let $\overline{f}$ be the ID-function of $\overline{G}$ that assigns the weight 1 to w, the weight 2 to x, and the weight 0 to the remaining vertices of $\overline{G}$. If the vertex w has two adjacent neighbors u and v of degree 2 in G, then let $\overline{f}$ be the ID-function of $\overline{G}$ that assigns the weight 1 to each of u, v and w, and the weight 0 to the remaining vertices of $\overline{G}$. In both cases, $\gamma_I(\overline{G}) \le \mathrm{w}(\overline{f}) = 3$, implying that $\gamma_I(G) + \gamma_I(\overline{G}) \le 5$. As shown earlier, $\gamma_I(G) + \gamma_I(\overline{G}) \ge 5$. Consequently, $\gamma_I(G) + \gamma_I(\overline{G}) = 5$.

Conversely, suppose that $\gamma_I(G) + \gamma_I(\overline{G}) = 5$. Since $\gamma_I(G) \le \gamma_I(\overline{G})$, it follows that $\gamma_I(G) = 2$ and $\gamma_I(\overline{G}) = 3$. Suppose that $\Delta(G) < n - 1$. By Proposition 9, there exist two different vertices y and z such that $N(y) \cap N(z) = V(G) \setminus \{y, z\}$. By supposition, y and z are not adjacent. If $n = 3$, then the common neighbor of y and z has degree $n - 1$, a contradiction. Hence, $n \ge 4$. Since the vertices y and z belong to a component of order 2 in $\overline{G}$, we note that $\gamma_I(\overline{G}) = 2 + \gamma_I(\overline{G} - \{y, z\}) \ge 2 + 2 = 4$, a contradiction. Hence, $\Delta(G) = n - 1$. Let w be a vertex of degree $n - 1$ in G. Let $\overline{f}$ be a γ_I-function of $\overline{G}$. Let $H = \overline{G} - w$ and note that H has order $n - 1$. Since w is isolated in $\overline{G}$, we note that $\overline{f}(w) = 1$, implying that $\gamma_I(H) = 2$. If $\Delta(H) = n - 2$, then there is a vertex x of degree $n - 2$ in H. Such a vertex x is a neighbor of w of degree 1 in G. If $\Delta(H) < n - 2$, then by Proposition 9, there exist two different vertices u and v such that $N_H(u) \cap N_H(v) = V(H) \setminus \{u, v\}$. Such vertices u and v are adjacent neighbors of degree 2 in G that have w as their common neighbor. □

We prove next Theorem 4. Recall its statement.

Theorem 4. *If G is a graph of order $n \ge 16$ and having no component with fewer than three vertices, then $\gamma_I(G) + \gamma_I(\overline{G}) \le n - 1$.*

Proof Let G be a graph of order $n \ge 16$ and having no component with fewer than three vertices. First, assume that $\delta(G) = 1$, and let x be a vertex of degree 1 and y be its neighbor in G. The function $\overline{f}$ that assigns the weight 2 to x, the weight 1 to y, and the weight 0 to the remaining vertices of $\overline{G}$ is an ID-function of $\overline{G}$, implying that $\gamma_I(\overline{G}) \le \mathrm{w}(\overline{f}) = 3$. By Theorem 6, each component of G has Italian domination number at most three-fourths its order, and so $\gamma_I(G) \le \frac{3}{4}n$. Hence, $\gamma_I(G) + \gamma_I(\overline{G}) \le \frac{3}{4}n + 3 \le n - 1$ since $n \ge 16$.

Thus, we may assume that $\delta(G) \ge 2$ and $\delta(\overline{G}) \ge 2$, for otherwise the desired result holds. By Theorem 6, $\gamma_I(G) \le \frac{2}{3}n$. If G is disconnected or $\mathrm{diam}(G) \ge 3$, then two vertices at distance 3 or more apart in G form a dominating set in $\overline{G}$. In this case, the function $\overline{f}$ that assigns the weight 2 to these two vertices and the weight 0 to the remaining vertices of $\overline{G}$ is an ID-function of $\overline{G}$, implying that $\gamma_I(\overline{G}) \le \mathrm{w}(\overline{f}) = 4$. Hence, $\gamma_I(G) + \gamma_I(\overline{G}) \le \frac{2}{3}n + 4 < n - 1$ since $n \ge 16$. Thus, we may assume that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$, for otherwise, the result holds. Let v be a vertex of minimum degree $\delta(G)$ in G. Since $\mathrm{diam}(G) = 2$, every vertex in $V(G) \setminus N[v]$ has at least one neighbor in $N(v)$.

If $\delta(G) = 2$, then the function $\overline{f}$ that assigns the weight 2 to the vertex v, the weight 1 to the two neighbors of v in G, and the weight 0 to the remaining vertices of $\overline{G}$ is an ID-function of $\overline{G}$, implying that $\gamma_I(\overline{G}) \le \mathrm{w}(\overline{f}) = 4$ and again, $\gamma_I(G) + \gamma_I(\overline{G}) \le \frac{2}{3}n + 4 \le n - 1$. Similarly, the result holds if $\delta(\overline{G}) = 2$.

Thus, we may assume that $\delta(G) \ge 3$ and $\delta(\overline{G}) \ge 3$. By Theorem 6, $\gamma_I(G) \le \frac{1}{2}n$ and $\gamma_I(\overline{G}) \le \frac{1}{2}n$. If some vertex $u \in V(G) \setminus N[v]$ has exactly one neighbor, say x, in

$N(v)$, then the function $\overline{f}$ that assigns the weight 2 to both u and v, the weight 1 to x, and the weight 0 to the remaining vertices of $\overline{G}$ is an ID-function of $\overline{G}$, implying that $\gamma_I(\overline{G}) \leq \mathrm{w}(\overline{f}) = 5$. Hence, $\gamma_I(G) + \gamma_I(\overline{G}) \leq \frac{1}{2}n + 5 < n - 1$ since $n \geq 16$. The analogous result holds if such a vertex exists in $\overline{G}$.

Thus, we may assume that every vertex in $V(G) \setminus N[v]$ has at least two neighbors in $N(v)$. The function f that assigns the weight 1 to every vertex in $N_G(v)$ and the weight 0 to the remaining vertices of G is an ID-function of G, implying that $\gamma_I(G) \leq \mathrm{w}(f) = \delta(G)$. Similarly, $\gamma_I(\overline{G}) \leq \delta(\overline{G})$. Hence, $\gamma_I(G) + \gamma_I(\overline{G}) \leq \delta(G) + \delta(\overline{G}) = \delta(G) + n - \Delta(G) - 1 \leq n - 1$. $\square$

Acknowledgements The research of the first and second authors was supported in part by the University of Johannesburg.

References

1. Chambers, E.W., Kinnersley, B., Prince, N., West, D.B.: Extremal problems for Roman domination. SIAM J. Discrete Math. **23**, 1575–1586 (2009)
2. Chellali, M., Haynes, T.W., Hedetniemi, S.T., McRae, A.A.: Roman {2}-domination. Discrete Appl. Math. **204**, 22–28 (2016)
3. Cockayne, E.J., Dreyer, P.A., Hedetniemi, S.M., Hedetniemi, S.T.: Roman domination in graphs. Discrete Math. **278**, 11–22 (2004)
4. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of Domination in Graphs. Marcel Dekker Inc, New York (1998)
5. Henning, M.A., Klostermeyer, W.F.: Italian domination in trees. Discrete Appl. Math. **217**, 557–564 (2017)
6. Henning, M. A., Yeo, A.: Total Domination in Graphs (Springer Monographs in Mathematics). ISBN-13: 978-1461465249 (2013)
7. Henning, M.A., Klostermeyer, W.F., MacGillivray, G.: Perfect Roman domination in trees. Discrete Appl. Math. **236**, 235–245 (2018)
8. Klostermeyer, W.F., MacGillivray, G.: Roman, Italian, and 2-domination. manuscript (2018)
9. Liu, C.-H., Chang, G.J.: Roman domination on 2-connected graphs. SIAM J. Discrete Math. **26**, 193–205 (2012)
10. Liu, C.-H., Chang, G.J.: Upper bounds on Roman domination numbers of graphs. Discrete Math. **312**, 1386–1391 (2012)
11. Nordhaus, E.A., Gaddum, J.W.: On complementary graphs. Am. Math. Monthly **63**, 175–177 (1956)
12. Rahmouni, A., Chellali, M.: Independent Roman {2}-domination in graphs. Discrete Appl. Math. **236**, 408–414 (2018)
13. ReVelle, C.S.: Can you protect the Roman Empire? Johns Hopkins Mag. **49**(2), 40 (1997)
14. ReVelle, C.S.: Test your solution to Can you protect the Roman Empire? Johns Hopkins Mag. **49**(3), 70 (1997)
15. ReVelle, C.S., Rosing, K.E.: Defendens Imperium Romanum: a classical problem in military. Am. Math. Monthly **107**(7), 585–594 (2000)
16. Stewart, I.: Defend the Roman Empire!, Scientific American, pp. 136–138 (1999)
17. Volkmann, L.: Italian domination in digraphs. J. Combin. Math. Combin. Comput., to appear