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A Modified Spanne–Peetre Inequality on Mixed Morrey Spaces

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Abstract
We obtain a modified version of the Spanne–Peetre inequality in the context of Morrey spaces with mixed norm. The geometric structure of the mixed Morrey spaces under consideration, dictates the new definition of Morrey–Lipschitz space. The Spanne–Peetre inequality that we find ensures that if a function belongs to a suitable Morrey space with mixed norm, then the modified integral operator which characterizes the Spanne–Peetre inequality, belongs to a suitable Morrey–Lipschitz space.

Keywords Integral operators · Morrey spaces · Mixed norm

Mathematics Subject Classification 42B20 · 42B35

1 Introduction

The study of the integral operators is a fundamental aspect of harmonic analysis in view of its applications to the theory of applied partial differential equations (see, for instance, [15,30,35]). The basic aspects of the theory have been performed on the Lebesgue spaces and are deeply treated in [32].

Besides the well-known L^p-theory, in the last decades, a lot of studies have been made on Lebesgue spaces with variable exponents, mixed norm Lebesgue spaces and Morrey-type spaces that, roughly speaking, represent a refinement of the classical L^p-spaces.

Nowadays, it is usual to discuss on *Morrey-type* spaces because, in addition to the definition presented by Morrey in [22], were investigated other function spaces whose

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structure is very close to the one of Morrey spaces. For example, Mizuhara in [21] introduced the *generalized* Morrey spaces, and later this definition was modified in several ways in order to obtain other functional Morrey classes that are substantially different from the other ones. Although it is impossible to be exhaustive, we mention the generalized (local) Morrey spaces [5,11], modified local generalized Morrey-type spaces ([12] and the references therein), Morrey spaces related to nonnegative potentials that arise from the study of Schrödinger-type equations ([2,8,9] and the references therein), vanishing Morrey spaces [33], and mixed Morrey spaces [23,24,26,27].

Many authors study strong and weak boundedness problems for some classical and also non-standard integral operators on Morrey-type spaces. For instance, a lot of studies deal with boundedness properties of maximal function, fractional maximal function, Riesz potential, Calderón–Zygmund singular integral operators and commutators between integral operators and locally integrable functions [28]. On the other hand, there is an increasing interest in the study of integral operators with rough kernels (see, for instance, [6,7,10] and the references therein). This new direction, weakening some key assumptions that arise in the classical theory, constitutes a challenging aspect of some recent studies in Harmonic Analysis.

The study contained in this note is placed in the fruitful framework of Morrey-type spaces, and precisely, we will work on mixed Morrey spaces (for the definition, we refer the reader to the next section), in which we derive the modified Spanne–Peetre inequality.

We emphasize that in the last decades several studies have been performed in the context of function spaces with mixed norm. Barza et al. [3] prove embeddings between all the classical, multidimensional and mixed norm Lorentz spaces. Karapetyants and Samko [17] present a new general approach to the definition of a class of mixed norm spaces of analytic functions, and in [18–20], the authors study Bergman-type spaces on the unit disk with mixed norm.

The paper is organized as follows. After this Introduction, in Sect. 2, we collect some basic definitions and we recall some results. Section 3 is devoted to the main result, i.e., the proof of the modified Spanne–Peetre inequality.

Throughout this paper, we adopt the following notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and for a set $E \subseteq \mathbb{R}^n$, we denote its Lebesgue measure by $|E|$, χ_E denotes the characteristic function of E, $B = B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ denotes an open ball centered at x_0, and with radius R, $\lambda B = \{x \in \mathbb{R}^n : |x - x_0| < \lambda R\}$ for any $\lambda > 0$, $B^c = \mathbb{R}^n \setminus B$, we write $A \lesssim B$ to mean that there exists a constant $C > 0$ such that $A \leq CB$ and p' denotes the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$ and, formally, if $p = \infty$ then $p' = 1$ and vice versa.

2 Mathematical Background

Let f be a real-valued measurable function on $\mathbb{R}^n$, $n \geq 1$, and let $0 < \alpha < n$. The *fractional integral* or *Riesz potential of f of order* α is defined as follows:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n$$

provided the integral above exists.

The mapping defined by

$$I_\alpha : f \mapsto I_\alpha f,$$

that is, the convolution operator with kernel $|x|^{\alpha-n}$, is called the *fractional integral operator of order* α.

The case $\alpha = 1$ plays an important role because it deals with some subrepresentation formulas that allow the study of the behavior of L^q norms of $I_\alpha f$ when $f \in L^p$ [34]. Moreover, when $n > 1$, by combining the subrepresentation formulas above with norm estimates for $I_\alpha f$, it is possible to bound $L^q(B)$ norms of $f - f_B$ (f_B stands for the integral average of f over the sphere B) by $L^p(B)$ norms of $|\nabla f|$ for suitable values of p and q. The inequalities obtained are called Poincaré-Sobolev estimates. On the other hand, the theory for general α, with $0 < \alpha < n$, was extensively studied by a lot of authors and attracts a lot of researchers nowadays.

The next well-known theorem states that I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. As usual, we use the notation $\|f\|_p$ for the $L^p(\mathbb{R}^n)$ norm of f, $1 \le p \le \infty$.

Theorem 1 (Hardy–Littlewood, Sobolev, [34]) *Let*

$$0 < \alpha < n, \quad 1 \le p < \frac{n}{\alpha} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then, for every $f \in L^p(\mathbb{R}^n)$, I_α exists a.e. and is measurable in $\mathbb{R}^n$. Moreover,

1. *if $1 < p < \frac{n}{\alpha}$, then*

$$\|I_\alpha f\|_q \le c\|f\|_p$$

for a constant c that depends only on α, n and p.

2. *if $p = 1$, then*

$$\sup_{\lambda>0} \lambda \left|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}\right|^{\frac{1}{q}} \le c\|f\|_1, \quad \left(q = \frac{n}{n-\alpha}\right),$$

for a constant c that depends only on α and n.

Hardy and Littlewood considered the case $n = 1$ [13,14] and Sobolev the case $n > 1$ [29]. When $p > 1$, Thorin obtained estimates [31], and the case $p = 1$ was studied by Zygmund [36].

It is possible to obtain some estimates in the case $p = \frac{n}{\alpha}$. Precisely, in [34] are shown some variants of Theorem 1 for the case $p = \frac{n}{\alpha}$ either by restricting I_α to the subspace of compactly supported $f \in L^{n\alpha}(\mathbb{R}^n)$ or modifying the definition of I_α for general $f \in L^{n\alpha}(\mathbb{R}^n)$. These results have been extensively studied and are often called *Trudinger estimates* or *Moser–Trudinger-type estimates*.

However, the norm inequality

$$\|I_\alpha f\|_q \le c\|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n) \tag{1}$$

for some constant c independent of f, holds only for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. We refer the reader to [34] for some comments and examples that explain why the restriction on p and q mentioned above are necessary for the validity of (1).

In the previous discussion, we have collected some basic results related to the L^p-theory.

Morrey spaces have been introduced by Morrey in 1938 in his work on systems of second-order elliptic partial differential equations [22]. Morrey spaces constitute a very useful family of spaces in the study of the regularity of solutions to various partial differential equations.

Definition 1 ([22]) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter $0 < \operatorname{diam}\Omega < \infty$. For $1 \le p < \infty$ and $\lambda \ge 0$, the Morrey space $L^{p,\lambda}(\Omega)$ is the subspace of $L^p(\Omega)$ defined as

$$L^{p,\lambda}(\Omega) = \left\{u \in L^p(\Omega) : \|u\|_{L^{p,\lambda}(\Omega)} < \infty\right\},$$

where

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left(\sup_{\substack{x\in\Omega \\ 0<\rho\le \operatorname{diam}\Omega}} \rho^{-\lambda} \int_{\Omega\cap B(x,r)} |u(y)|^p \, dy \right)^{\frac{1}{p}}. \tag{2}$$

Using standard arguments, it is easy to see that the quantity defined by (2) defines a norm on $L^{p,\lambda}$ and that the resulting normed space is complete, that is, it is a Banach space.

With obvious modification, Definition 2 works also if $\Omega = \mathbb{R}^n$.

In [1], the author extends Theorem 1 to Morrey spaces.

Theorem 2 (Adams inequality) *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $1 < p < \frac{n}{\alpha}(1-\lambda)$. *If* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^{q,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Now we introduce the definition of mixed Morrey space. (We refer the reader to [26,27] for further details.)

Definition 2 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter $0 < \operatorname{diam}\Omega < \infty$, $1 < p, q < +\infty$, $0 < \lambda < n$, $0 < \mu < 1$ and fix $T > 0$. We define the set $L^{q,\mu}(0, T, L^{p,\lambda}(\Omega))$ as the class of functions $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\|u\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} := \left(\sup_{\substack{t_0\in(0,T)\\ \rho>0}} \frac{1}{\rho^{\mu}} \int\limits_{(0,T)\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^{\lambda}} \int\limits_{\Omega\cap B(x,\rho)} |u(y,t)|^p \, \mathrm{d}y \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}},$$

is finite.

The same definition holds if $\Omega = \mathbb{R}^n$.

In [26], the authors obtained the Adams inequality in the context of Morrey spaces with mixed norm, considering for a fixed $T > 0$ and $0 < \alpha < n$, the following fractional integral operator of order α

$$I_\alpha f(x,t) = \int\limits_{\mathbb{R}^n} \frac{f(y,t)}{|x-y|^{n-\alpha}} \, \mathrm{d}y, \quad \text{a.e. in } \mathbb{R}^n, \quad \text{for } t \in (0,T).$$

Theorem 3 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, $1 < q' < +\infty$, $0 < \mu' < 1$ *and* $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. *Then,*

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

The results contained in [26,27] are extensions of the classical boundedness results of some standard integral operators.

The classical Adams inequality holds for a precise range of values of p. If $p \geq \frac{n}{\alpha}(1-\lambda)$, Spanne and Peetre [25] obtained a *modified* version of the Adams inequality—the Spanne–Peetre inequality—in which appears a new version of the classical Riesz potential in place of I_α and plays an important role the Lipschitz space that we define below.

Precisely, let $0 < \alpha < n$. We define the *modified integral operator* $\widetilde{I}_\alpha$ as

$$\widetilde{I}_\alpha f(x,t) = \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \chi_{\{|y|\geq 1\}}(y) \right) f(y,t) \, \mathrm{d}y, \quad t \in (0,T).$$

In order to obtain a norm inequality for $\widetilde{I}_\alpha$—that is, a Spanne–Peetre inequality—, as announced before, we need of suitable additional function spaces. We start with the classical definitions of Lipschitz and BMO spaces.

Definition 3 (*Lipschitz and BMO spaces*) Let $0 \leq \varepsilon < 1$. We set

$$\mathrm{Lip}_\varepsilon(\mathbb{R}^n) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)} < \infty\},$$

where

$$\|u\|_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)} = \sup_{B\subseteq\mathbb{R}^n} \inf_{c\in\mathbb{C}} \frac{1}{|B|^{1+\frac{\varepsilon}{n}}} \int_B |f(x) - c| \, \mathrm{d}x,$$

being B a ball in $\mathbb{R}^n$. Furthermore, we set $BMO(\mathbb{R}^n) = \mathrm{Lip}_0(\mathbb{R}^n)$.

BMO stands for *bounded mean oscillation*. This class was defined by John and Nirenberg [16] in 1961.

Following [4] and taking into account the anisotropic structure of the mixed Morrey spaces, we can construct a more suitable version of the spaces mentioned in Definition 3.

Definition 4 (*Morrey–Lipschitz and Morrey-BMO spaces*) Let $0 \le \varepsilon < 1$, $1 < q < \infty$, $0 < \mu < 1$ and fix $T > 0$. We define the set $L^{q,\mu}(0,T,\mathrm{Lip}_\varepsilon(\mathbb{R}^n))$ as the class of functions $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\|u\|_{L^{q,\mu}(0,T,\mathrm{Lip}_\varepsilon(\mathbb{R}^n))} = \left(\sup_{\substack{t_0 \in (0,T) \\ \rho > 0}} \frac{1}{\rho^\mu} \int_{(0,T)\cap(t_0-\rho,t_0+\rho)} \|f(\cdot,t)\|^q_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)} \,\mathrm{d}t \right)^{\frac{1}{q}}.$$

Furthermore, we set $L^{q,\mu}(0,T,BMO(\mathbb{R}^n)) = L^{q,\mu}(0,T,\mathrm{Lip}_0(\mathbb{R}^n))$.

3 Main Result

Theorem 4 (Modified Spanne–Peetre inequality) *Let* $0 < \alpha < n$, $0 < \lambda, \mu < 1$, $\frac{n}{\alpha}(1-\lambda) \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $f \in L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))$. *Then the following inequality holds:*

$$\|\widetilde{I}_\alpha f\|_{L^{q,\mu}(0,T,\mathrm{Lip}_\varepsilon(\mathbb{R}^n))} \lesssim \|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}$$

where $0 < \varepsilon = \alpha - \frac{n}{p}(1-\lambda) < 1$. *In particular, if* $p = \frac{n}{\alpha}(1-\lambda)$, *we have*

$$\|\widetilde{I}_\alpha f\|_{L^{q,\mu}(0,T,BMO(\mathbb{R}^n))} \lesssim \|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Proof For every $x_0 \in \mathbb{R}^n$ and $r > 0$, let $B = B(x_0, r)$ the sphere with center x_0 and radius r. The following decomposition of f holds:

$$\begin{aligned} f(x,t) &= f(x,t)\chi_{2B}(x) + f(x,t)\chi_{(2B)^c}(x) \\ &\equiv f_0(x,t) + f_\infty(x,t), \quad \forall x \in \mathbb{R}^n,\ t \in (0,T). \end{aligned}$$

Moreover, we set

$$c_0 = -\int_{|y|\ge 1} \frac{f_0(y,t)}{|y|^{n-\alpha}}\,\mathrm{d}y, \quad c_1 = \widetilde{I}_\alpha(f_\infty)(x_0,t), \quad c = c_0 + c_1.$$

For any $x \in \mathbb{R}^n$ and $t \in (0, T)$, we have the following pointwise inequality:

$$|\widetilde{I}_\alpha f(x,t) - c| \le |I_\alpha f_0(x,t)| + \int_{\mathbb{R}^n} \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |f_\infty(y,t)| \, \mathrm{d}y \equiv I + II.$$

First we estimate I. For any $t \in (0, T)$, we have:

$$\begin{aligned}
\int_B I \, \mathrm{d}x &= \int_B |I_\alpha f_0(x,t)| \, \mathrm{d}x \\
&\le \left(\int_B |I_\alpha f_0(x,t)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_B 1 \, \mathrm{d}x \right)^{\frac{1}{q'}} \\
&\lesssim \left(\int_{2B} |f(x,t)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_B 1 \, \mathrm{d}x \right)^{\frac{1}{q'}} \\
&= \left(\int_{2B} |f(x,t)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} |B|^{\frac{1}{q'}} \\
&\lesssim \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)} |B|^{\frac{1}{q'}} |B|^{\frac{\lambda}{p}} \\
&= \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)} |B|^{1+\frac{\varepsilon}{n}}.
\end{aligned}$$

Next we estimate II. Let $x \in B$. Then, for any $t \in (0, T)$, we have:

$$\begin{aligned}
II &\lesssim \sum_{j=1}^{\infty} \int_{2^j r \le |x_0-y| < 2^{j+1}} \frac{|x-x_0|}{|x_0-y|^{n-\alpha+1}} |f(y,t)| \, \mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \frac{r}{(2^j r)^{n-\alpha+1}} \int_{2^{j+1}B} |f(y,t)| \, \mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \frac{r}{(2^j r)^{n-\alpha+1}} \left(\int_{2^{j+1}B} |f(y,t)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \left(\int_{2^{j+1}B} 1 \, \mathrm{d}y \right)^{\frac{1}{p'}} \\
&\lesssim \sum_{j=1}^{\infty} \frac{r}{(2^j r)^{n-\alpha+1}} \left(\frac{1}{|2^{j+1}B|^\lambda} \int_{2^{j+1}B} |f(y,t)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \cdot |2^{j+1}B|^{\frac{\lambda}{p}} \cdot |2^{j+1}B|^{\frac{1}{p'}} \\
&\lesssim \sum_{j=0}^{\infty} \frac{r}{(2^j r)^{n-\alpha+1}} \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)} |2^{j+1}B|^{\frac{\lambda}{p}+\frac{1}{p'}} \cdot \\
&\quad \cdot \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} 1 \, \mathrm{d}y \right)^{\frac{\lambda}{p}} \cdot \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} 1 \, \mathrm{d}y \right)^{\frac{1}{p'}} \\
&\lesssim \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)} |B|^{\frac{\alpha}{n}-1+\frac{\lambda}{p}+1-\frac{1}{p}} \sum_{j=1}^{\infty} 2^{-jn\left(1-\frac{\alpha}{n}+\frac{1}{n}-\frac{\lambda}{p}-1+\frac{1}{p}\right)} \\
&\lesssim |B|^{\frac{\varepsilon}{n}} \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)},
\end{aligned}$$

since $-\frac{\alpha}{n}+\frac{1}{n}-\frac{\lambda}{p}+\frac{1}{p}>0$. Thus we have

$$\begin{aligned}\|\widetilde{I}_\alpha f\|_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)} &\lesssim \sup_{B\subseteq\mathbb{R}^n}\frac{1}{|B|^{1+\frac{\varepsilon}{n}}}\int_B|\widetilde{I}_\alpha-c|\,\mathrm{d}x\\ &\lesssim \sup_{B\subseteq\mathbb{R}^n}\frac{1}{|B|^{1+\frac{\varepsilon}{n}}}\left(\int_B I\,\mathrm{d}x+\int_B II\,\mathrm{d}x\right)\\ &\lesssim \|f(\cdot,t)\|_{L^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

From this inequality, it follows that

$$\|\widetilde{I}_\alpha f(\cdot,t)\|^q_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)}\lesssim\|f(\cdot,t)\|^q_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Now, integrating both sides on $(0,T)\cap(t_0-\rho,t_0+\rho)$ and multiplying by $\frac{1}{\rho^\mu}$, we obtain

$$\begin{aligned}&\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\|\widetilde{I}_\alpha f(\cdot,t)\|^q_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)}\,\mathrm{d}t\\ &\quad\lesssim\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\|f(\cdot,t)\|^q_{L^{p,\lambda}(\mathbb{R}^n)}\,\mathrm{d}t.\end{aligned}$$

Then, taking the supremum for $t_0\in(0,T)$ and $\rho>0$ and elevating both sides to $\frac{1}{q}$, we have

$$\begin{aligned}&\left(\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\|\widetilde{I}_\alpha f(\cdot,t)\|^q_{\mathrm{Lip}_\varepsilon(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{1}{q}}\lesssim\\ &\lesssim\left(\sup_{\substack{t_0\in(0,T)\\ \rho>0}}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\|f(\cdot,t)\|^q_{L^{p,\lambda}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Then the theorem is proved. □

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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