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Constructions of K-g-Frames and Tight K-g-Frames in Hilbert Spaces

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Abstract

In this paper, we mainly discuss the constructions of some new K-g-frames which differ from the existing methods. Meanwhile, we use the relation between a positive operator and the frame operator of a K-g-frame to yield a new K-g-frame. We also obtain a necessary and sufficient condition to generate a new K-g-frame. In addition, we correct some recent results which were obtained by Huang and Leng. In the end, we give an equivalent characterization to construct some new tight K-g-frames by two given g-Bessel sequences. Our results generalize and improve some remarkable results.

Keywords K-g-frame · Positive operator · Tight K-g-frame · G-Bessel sequence

Mathematics Subject Classification 42C15

1 Introduction

A frame as a generalization of an orthonormal basis, appeared first in the late 1940s and early 1950s, provides us with a powerful theoretical tool because of its redundancy and flexibility. Now a frame plays an important role in sampling theory [1], compressed sensing [2] and a number of other fields. We refer the readers to [3–5] for an introduction to frame theory and its applications. In [6], Sun proposed the notion of g-frame, which generalized the concept of frame extensively. We know that though many basic properties of g-frame can be shared with frame, not all the properties

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between them are same. For example, an exact frame is equivalent to a Riesz basis, but an exact g-frame is not equivalent to a g-Riesz basis. We refer the reader to the papers [6–11] for more information about g-frames.

Being an extension of frame, the concept of K-frame was introduced by Găvruţa [12], which allows an atomic decomposition of elements in the range of K. In fact, a K-frame is a more general version of frame. There are many differences between a K-frame and a frame. For instance, the sequence $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$ if and only if $\{f_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}$ and the corresponding synthesis operator is surjective, but the sequence $\{f_j\}_{j\in J}$ is a K-frame for $\mathcal{H}$ if and only if $\{f_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}$ and the range of K is involved in the range of the corresponding synthesis operator. For more details on K-frames, see references in [12–18].

Recently, Xiao et al. [19] put forward the notion of K-g-frame, which is more general than g-frame and K-frame in Hilbert spaces. Naturally, K-g-frame attracts many scholars' attention. Now it has been a hot topic to make full use of various conditions to construct a new K-g-frame (see [20–23]). Hua and others gave several methods to generate tight K-g-frames and tight g-frames (see [24]). For more details on K-g-frame, readers can consult [19–24].

In this paper, we first construct a K-g-frame from a given K-g-frame and a g-Bessel sequence. Next, we adopt a novel way to generate a new K-g-frame from two existing K-g-frames. We also give a necessary and sufficient condition to yield a K-g-frame. Finally, we give an equivalent characterization of constructing tight K-g-frames by two given g-Bessel sequences. We correct the results of Theorem 3.4 and Corollary 3.17 in [21] and Theorem 3.10 in [22]. We also generalize and improve some remarkable results.

Throughout this paper, we will adopt such notations. $\mathcal{H}$ is a separable Hilbert space, and $I_{\mathcal{H}}$ is the identity operator for $\mathcal{H}$. $\mathbb{C}$ is the set of all complex numbers. $B(\mathcal{H}_1, \mathcal{H}_2)$ is a collection of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, and if $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}$, $B(\mathcal{H}_1, \mathcal{H}_2)$ is denoted by $B(\mathcal{H})$. Let $K \in B(\mathcal{H})$ and $K \neq 0$, the range and the kernel of K are denoted by $R(K)$ and $N(K)$, respectively. $\{\mathcal{V}_j\}_{j\in J}$ is a sequence of closed subspaces of $\mathcal{H}$, where J is a finite or countable index set. $\ell^2(\{\mathcal{V}_j\}_{j\in J})$ is defined by

$$\ell^2(\{\mathcal{V}_j\}_{j\in J}) = \left\{ \{g_j\}_{j\in J} : g_j \in \mathcal{V}_j, \quad j \in J, \quad \sum_{j\in J} \|g_j\|^2 < +\infty \right\}$$

with the inner product

$$\langle \{f_j\}_{j\in J}, \{g_j\}_{j\in J} \rangle = \sum_{j\in J} \langle f_j, g_j \rangle.$$

It is trivial that $\ell^2(\{\mathcal{V}_j\}_{j\in J})$ is a Hilbert space.

Definition 1.1 A sequence $\{\Lambda_j : \Lambda_j \in B(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad (\forall f \in \mathcal{H}). \tag{1.1}$$

The constants A and B are called the lower and upper g-frame bounds, respectively. If only the only right inequality of (1.1) holds, $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bound B.

If $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, we may define the bounded linear operator T_Λ by

$$T_\Lambda : \ell^2(\{\mathcal{V}_j\}_{j\in J}) \to \mathcal{H} : T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j, \quad \{g_j\}_{j\in J} \in \ell^2(\{\mathcal{V}_j\}_{j\in J}.$$

T_Λ is called the synthesis operator. The adjoint operator T_Λ^* is given as follows:

$$T_\Lambda^* f : \mathcal{H} \to \ell^2(\{\mathcal{V}_j\}_{j\in J}) : T_\Lambda^* f = \{\Lambda_j f\}_{j\in J}, \quad f \in \mathcal{H}.$$

T_Λ^* is called the analysis operator. The operator given by

$$S_\Lambda : \mathcal{H} \to \mathcal{H} : S_\Lambda f = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad f \in \mathcal{H}$$

is called the g-frame operator.

Definition 1.2 Let $K \in B(\mathcal{H})$. A sequence $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a K-frame for $\mathcal{H}$ if there exist two positive constants A and B such that

$$A\|K^* f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2, \quad (\forall f \in \mathcal{H}).$$

The constants A and B are called the lower and upper K-frame bounds, respectively.

Definition 1.3 ([19]) Let $K \in B(\mathcal{H})$. A sequence $\{\Lambda_j : \Lambda_j \in B(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is called a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if there exist two positive constants A and B such that

$$A\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad (\forall f \in \mathcal{H}).$$

The constants A and B are called the lower and upper K-g-frame bounds, respectively.

Definition 1.4 ([24]) Let $K \in B(\mathcal{H})$. A sequence $\{\Lambda_j : \Lambda_j \in B(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is called a tight K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, if there exists a positive constant A such that

$$A\|K^* f\|^2 = \sum_{j\in J} \|\Lambda_j f\|^2, \quad (\forall f \in \mathcal{H}).$$

In order to obtain our main results, we need the following lemmas.

Lemma 1.5 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and suppose that $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with closed range $R(U)$. Then, there exists a unique bounded operator $U^+ : \mathcal{H}_2 \to \mathcal{H}_1$ satisfying*

$$N_{U^+} = R(U)^\perp,\ R(U^+) = N_U^\perp,\ UU^+ f = f, \quad (\forall f \in R(U)).$$

The operator U^+ is called the pseudo-inverse operator of U.

Lemma 1.6 ([13]) *Suppose that $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is an operator with closed range, then*

$$\|U^+\|^{-1}\|f\| \le \|U^* f\| \le \|U\|\|f\|, \quad (\forall f \in R(U)).$$

Lemma 1.7 ([19]) *The sequence $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if and only if the synthesis operator T_Λ is well defined and bounded, and $R(K) \subset R(T_\Lambda)$.*

Lemma 1.8 ([24]) *Let $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Then, $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, if and only if there exists a positive constant A such that $S_\Lambda = AKK^*$, where S_Λ is the g-frame operator for $\{\Lambda_j\}_{j\in J}$.*

Lemma 1.9 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be g-Bessel sequences for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. If $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$, then $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bound $(\sqrt{B_1}\|U_1\| + \sqrt{B_2}\|U_2\|)^2$.*

The proof is easy, we omit it. Later, we will need the following important result from operator theory:

Theorem 1.10 (Douglas's theorem [25]) *Let $U_1 \in B(\mathcal{H}_1, \mathcal{H})$, $U_2 \in B(\mathcal{H}_2, \mathcal{H})$. Then, the following are equivalent:*

(1) *$R(U_1) \subseteq R(U_2)$;*
(2) *$U_1 U_1^* \le \alpha^2 U_2 U_2^*$ for some $\alpha > 0$;*
(3) *there exists a bounded operator $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ so that $U_1 = U_2 X$.*

Several ways to generate g-frames have been discussed in [9–11]. After the notion of K-frame was proposed, there are some references to give a number of construction methods about K-frames (see [14–17]). Motivated by recent progress in constructions of some new K-g-frames (see [19–23]), we give two different ways to construct new K-g-frames.

Remark 1.11 In [21, Theorem 3.4], the following statement has been formulated: let $K \in B(\mathcal{H})$ and $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with lower and upper bounds A and B, respectively; if $U \in B(\mathcal{H})$ has closed range and $UK = KU$, then $\{\Lambda_j U^*\}_{j\in J}$ is a K-g-frame for $R(U)$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with lower and upper bounds $A\|U^+\|^{-2}$ and $B\|U\|^2$, respectively. In Example 1.12, we show that this statement is not true in the general case.

Example 1.12 Suppose $\mathcal{H} = \mathbb{C}^3$, $J = \{1, 2, 3\}$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{H}$, and $\mathcal{V}_j = \overline{span}\{e_j\}$. Now define $K \in B(\mathcal{H})$, $U \in B(\mathcal{H})$ and $\{\Lambda_j\}_{j\in J}$ as follows:

$$
\begin{aligned}
K : \mathcal{H} \to \mathcal{H},\ Kf &= \langle f, e_3\rangle e_1 + \langle f, e_1 + e_2\rangle e_2, \quad f \in \mathcal{H},\\
U : \mathcal{H} \to \mathcal{H},\ Uf &= \langle f, e_3\rangle (e_1 - e_2), \quad f \in \mathcal{H},\\
\Lambda_1 : \mathcal{H} \to \mathcal{V}_1,\ \Lambda_1 f &= \langle f, e_1\rangle e_1, \quad f \in \mathcal{H},\\
\Lambda_2 : \mathcal{H} \to \mathcal{V}_2,\ \Lambda_2 f &= \langle f, e_2\rangle e_2, \quad f \in \mathcal{H},\\
\Lambda_3 : \mathcal{H} \to \mathcal{V}_3,\ \Lambda_3 f &= \langle f, e_2\rangle e_3, \quad f \in \mathcal{H}.
\end{aligned}
$$

Now we show that $K^* f = \langle f, e_1\rangle e_3 + \langle f, e_2\rangle (e_1 + e_2),\ f \in \mathcal{H}$. In fact, for any $f, m \in \mathcal{H}$, we have

$$
\begin{aligned}
\langle K^* f, m\rangle = \langle f, Km\rangle &= \langle f, \langle m, e_3\rangle e_1 + \langle m, e_1 + e_2\rangle e_2\rangle\\
&= \langle f, e_1\rangle \overline{\langle m, e_3\rangle} + \langle f, e_2\rangle \overline{\langle m, e_1 + e_2\rangle}\\
&= \langle f, e_1\rangle \langle e_3, m\rangle + \langle f, e_2\rangle \langle e_1 + e_2, m\rangle\\
&= \langle \langle f, e_1\rangle e_3 + \langle f, e_2\rangle (e_1 + e_2), m\rangle.
\end{aligned}
$$

Thus, for each $f \in \mathcal{H}$, we obtain

$$
\begin{aligned}
\|K^* f\|^2 = \|\langle f, e_1\rangle e_3 + \langle f, e_2\rangle (e_1 + e_2)\|^2 &= |\langle f, e_1\rangle|^2 + 2|\langle f, e_2\rangle|^2\\
&= \sum_{j=1}^{3} \|\Lambda_j f\|^2 \le 3\|f\|^2.
\end{aligned}
$$

This implies that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. It is clear that $U \in B(\mathcal{H})$ has closed range. For all $f \in \mathcal{H}$, we get

$$
\begin{aligned}
UKf &= U(\langle f, e_3\rangle e_1 + \langle f, e_1 + e_2\rangle e_2) = \langle f, e_3\rangle Ue_1 + \langle f, e_1 + e_2\rangle Ue_2 = 0\\
&= \langle f, e_3\rangle (e_2 - e_2) = \langle f, e_3\rangle (Ke_1 - Ke_2) = K(\langle f, e_3\rangle (e_1 - e_2)) = KUf.
\end{aligned}
$$

Then, $UK = KU$.

The adjoint operator of U is U^*, $U^* f = \langle f, e_1 - e_2\rangle e_3,\ f \in \mathcal{H}$. Indeed, for all $f, m \in \mathcal{H}$, we have

$$
\begin{aligned}
\langle U^* f, m\rangle = \langle f, Um\rangle = \langle f, \langle m, e_3\rangle (e_1 - e_2)\rangle &= \langle f, e_1 - e_2\rangle \overline{\langle m, e_3\rangle}\\
&= \langle f, e_1 - e_2\rangle \langle e_3, m\rangle = \langle \langle f, e_1 - e_2\rangle e_3, m\rangle.
\end{aligned}
$$

Choosing $f = e_2 - e_1 \in R(U) = \overline{span}\{e_1 - e_2\}$, we get $\|K^* f\|^2 = 3$ and

$$\sum_{j=1}^{3} \|\Lambda_j U^* f\|^2 = \sum_{j=1}^{3} \|\Lambda_j(\langle f, e_1 - e_2\rangle e_3)\|^2 = \sum_{j=1}^{3} \|\langle f, e_1 - e_2\rangle(\Lambda_j e_3)\|^2 = 0.$$

Hence, $\{\Lambda_j U^*\}_{j\in J}$ is not a K-g-frame for $R(U)$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Furthermore, $\{\Lambda_j U^*\}_{j\in J}$ is not a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ and

$$\overline{span}\{e_1 + e_2, e_3\} = R(K^*) \nsubseteq R(U) = \overline{span}\{e_1 - e_2\}.$$

Remark 1.13 In [22, Theorem 3.10], the following statement has been formulated: let $\{\Lambda_j\}_{j\in J}$ be an atomic system for K, and let S_Λ be the frame operator of $\{\Lambda_j\}_{j\in J}$; let U be a positive operator, then $\{\Lambda_j + \Lambda_j U\}_{j\in J}$ is an atomic system for K. In Example 1.14, we show that this statement is not true in the general case.

Example 1.14 Suppose $\mathcal{H} = \mathbb{C}^3$, $J = \{1, 2, 3\}$. Assume that $\{e_j\}_{j\in J}$ is an orthonormal basis of $\mathcal{H}$, and $\mathcal{V}_1 = \mathcal{V}_2 = \overline{span}\{e_1\}$, $\mathcal{V}_3 = \overline{span}\{e_3\}$. Now define $K \in B(\mathcal{H})$, $U \in B(\mathcal{H})$ and $\{\Lambda_j\}_{j\in J}$ as follows:

$$\begin{aligned}
K &: \mathcal{H} \to \mathcal{H},\ Kf = \langle f, e_1\rangle e_2, \quad f \in \mathcal{H},\\
U &: \mathcal{H} \to \mathcal{H},\ Uf = \langle f, e_1\rangle e_1 + \langle f, 2e_2 - e_3\rangle e_2 + \langle f, e_3 - e_2\rangle e_3, \quad f \in \mathcal{H},\\
\Lambda_1 &: \mathcal{H} \to \mathcal{V}_1,\ \Lambda_1 f = \langle f, e_1\rangle e_1, \quad f \in \mathcal{H},\\
\Lambda_2 &: \mathcal{H} \to \mathcal{V}_2,\ \Lambda_2 f = \langle f, e_1\rangle e_1, \quad f \in \mathcal{H},\\
\Lambda_3 &: \mathcal{H} \to \mathcal{V}_3,\ \Lambda_3 f = \langle f, e_2\rangle e_3, \quad f \in \mathcal{H}.
\end{aligned}$$

Now we show that $K^* f = \langle f, e_2\rangle e_1$, $f \in \mathcal{H}$. In fact, for all $f, m \in \mathcal{H}$, we get

$$\begin{aligned}
\langle K^* f, m\rangle &= \langle f, Km\rangle = \langle f, \langle m, e_1\rangle e_2\rangle = \langle f, e_2\rangle\overline{\langle m, e_1\rangle}\\
&= \langle f, e_2\rangle\langle e_1, m\rangle = \langle\langle f, e_2\rangle e_1, m\rangle.
\end{aligned}$$

Hence, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|K^* f\|^2 &= \|\langle f, e_2\rangle e_1\|^2 = |\langle f, e_2\rangle|^2\\
&\le \sum_{j=1}^{3} \|\Lambda_j f\|^2 = 2|\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2 \le 2\|f\|^2.
\end{aligned}$$

Thus, $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.

By a simple calculation, we can obtain that U is a self-adjoint operator. Then, for each $f \in \mathcal{H}$, we conclude

$$\begin{aligned}
\langle Uf, f\rangle &= \langle\langle f, e_1\rangle e_1 + \langle f, 2e_2 - e_3\rangle e_2 + \langle f, e_3 - e_2\rangle e_3, f\rangle\\
&= \langle f, e_1\rangle\overline{\langle f, e_1\rangle} + 2\langle f, e_2\rangle\overline{\langle f, e_2\rangle} - \langle f, e_3\rangle\overline{\langle f, e_2\rangle}\\
&\quad + \langle f, e_3\rangle\overline{\langle f, e_3\rangle} - \langle f, e_2\rangle\overline{\langle f, e_3\rangle}
\end{aligned}$$

$$= |\langle f, e_1\rangle|^2 + 2|\langle f, e_2\rangle|^2 - 2\text{Re}\{\langle f, e_2\rangle\overline{\langle f, e_3\rangle}\} + |\langle f, e_3\rangle|^2$$
$$= |\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2 + |\langle f, e_2\rangle - \langle f, e_3\rangle|^2 \geq 0.$$

Therefore, U is a positive operator.

It is clear that $\{\Lambda_j + \Lambda_j U\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. By a direct computation, we get

$$(\Lambda_1 + \Lambda_1 U)f = 2\langle f, e_1\rangle e_1; \quad (\Lambda_2 + \Lambda_2 U)f = 2\langle f, e_1\rangle e_1;$$
$$(\Lambda_3 + \Lambda_3 U)f = \langle f, 3e_2 - e_3\rangle e_3.$$

Choosing $f = e_2 + 3e_3 \in \mathcal{H}$, we get $\|K^* f\|^2 = |\langle f, e_2\rangle|^2 = 1$ and

$$\sum_{j=1}^{3} \|(\Lambda_j + \Lambda_j U)f\|^2 = 8|\langle f, e_1\rangle|^2 + |\langle f, 3e_2 - e_3\rangle|^2 = 0.$$

This proves that $\{\Lambda_j + \Lambda_j U\}_{j\in J}$ is not a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Let S_Λ be the frame operator of $\{\Lambda_j\}_{j\in J}$, then for any $f \in \mathcal{H}$, we have

$$S_\Lambda f = \sum_{j=1}^{3} \Lambda_j^* \Lambda_j f = 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2.$$

Now we will show that $US_\Lambda \neq S_\Lambda U$; indeed, for all $f \in \mathcal{H}$, we obtain

$$US_\Lambda f = 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle(2e_2 - e_3); \quad S_\Lambda U f = 2\langle f, e_1\rangle e_1 + \langle f, 2e_2 - e_3\rangle e_2.$$

2 Main Results

Theorem 2.1 *Let $K_1 \in B(\mathcal{H}_1)$ and $K_2 \in B(\mathcal{H}_2)$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a K_1-g-frame and $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with the synthesis operators T_Λ and T_Γ, respectively. Assume $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^* + U_2 T_\Gamma T_\Gamma^* U_2^* \geq 0$. If U_1 has closed range, $U_1 K_1 = K_2 U_1$ and $R(K_2^*) \subset R(U_1)$, then $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.*

Proof Let $\{\Lambda_j\}_{j\in J}$ be a K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with frame bounds A_1 and B_1. Let $\{\Gamma_j\}_{j\in J}$ be a g-Bessel sequence for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with g-Bessel bound B_2. By Lemma 1.9, we conclude that $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bound $(\sqrt{B_1}\|U_1\| + \sqrt{B_2}\|U_2\|)^2$.

For each $g \in \mathcal{H}_2$, we obtain

$$\sum_{j\in J} 2\text{Re}\{\langle \Lambda_j U_1^* g, \Gamma_j U_2^* g\rangle\} = 2\text{Re}\left\langle \sum_{j\in J} \Gamma_j^* \Lambda_j U_1^* g, U_2^* g \right\rangle$$

$$= 2\mathrm{Re}\langle T_\Gamma T_\Lambda^* U_1^* g,\ U_2^* g\rangle$$
$$= \langle (U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*) g,\ g\rangle.$$

Since $U_1 \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, and $U_1 K_1 = K_2 U_1$, it is clear that $K_1^* U_1^* = U_1^* K_2^*$. According to Lemma 1.6, for any $g \in \mathcal{H}_2$, we get

$$\begin{aligned}
&\sum_{j\in J} \|(\Lambda_j U_1^* + \Gamma_j U_2^*) g\|^2 \\
&\quad = \sum_{j\in J} \|\Lambda_j U_1^* g\|^2 + \sum_{j\in J} 2\mathrm{Re}\{\langle \Lambda_j U_1^* g,\ \Gamma_j U_2^* g\rangle\} + \sum_{j\in J} \|\Gamma_j U_2^* g\|^2 \\
&\quad = \sum_{j\in J} \|\Lambda_j U_1^* g\|^2 + \langle (U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*) g,\ g\rangle \\
&\qquad + \langle T_\Gamma T_\Gamma^* U_2^* g,\ U_2^* g\rangle \\
&\quad = \sum_{j\in J} \|\Lambda_j U_1^* g\|^2 + \langle (U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^* + U_2 T_\Gamma T_\Gamma^* U_2^*) g,\ g\rangle \\
&\quad \geq \sum_{j\in J} \|\Lambda_j U_1^* g\|^2 \geq A_1 \|K_1^* U_1^* g\|^2 = A_1 \|U_1^* K_2^* g\|^2 \\
&\quad \geq A_1 \|U_1^+\|^{-2} \|K_2^* g\|^2.
\end{aligned}$$

Thus, for every $g \in \mathcal{H}_2$, we obtain

$$A_1 \|U_1^+\|^{-2} \|K_2^* g\|^2 \leq \sum_{j\in J} \|(\Lambda_j U_1^* + \Gamma_j U_2^*) g\|^2 \leq (\sqrt{B_1}\|U_1\| + \sqrt{B_2}\|U_2\|)^2 \|g\|^2.$$

So $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. □

Corollary 2.2 *Let $K_1 \in B(\mathcal{H}_1)$ and $K_2 \in B(\mathcal{H}_2)$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. If $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $UK_1 = K_2 U$ and $R(K_2^*) \subset R(U)$, then $\{\Lambda_j U^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.*

Corollary 2.3 *Let $K, U \in B(\mathcal{H})$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. If U is a positive operator and $US_\Lambda = S_\Lambda U$, where S_Λ is the frame operator of $\{\Lambda_j\}_{j\in J}$, then $\{\Lambda_j + \Lambda_j U\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.*

Proof Assume that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Since

$$T_\Lambda T_\Lambda^* U^* + U T_\Lambda T_\Lambda^* + U T_\Lambda T_\Lambda^* U^* = S_\Lambda U + U S_\Lambda + U S_\Lambda U,$$

from Theorem 2.1, we need only to prove that $S_\Lambda U + U S_\Lambda + U S_\Lambda U \geq 0$. According to Proposition 4.33 in [26], we obtain that there exists a unique positive operator C

such that $U = C^2$. In addition, since $US_\Lambda = S_\Lambda U$, we have $CS_\Lambda = S_\Lambda C$. It follows that

$$\begin{aligned}\langle (S_\Lambda U + US_\Lambda + US_\Lambda U)f, f\rangle &= 2\langle C^2 S_\Lambda f, f\rangle + \langle UT_\Lambda T_\Lambda^* Uf, f\rangle \\ &= 2\langle CS_\Lambda Cf, f\rangle + \langle UT_\Lambda T_\Lambda^* Uf, f\rangle \\ &= 2\|T_\Lambda^* Cf\|^2 + \|T_\Lambda^* Uf\|^2 \\ &\geq 0\end{aligned}$$

for all $f \in \mathcal{H}$. So from Theorem 2.1, Corollary 2.3 holds. □

Remark 2.4 By taking $U_1 = U$ and $U_2 = 0$, we obtain Corollary 2.2. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $K_1 = K_2 = K$ in Corollary 2.2, we can correct Theorem 3.4 in [21]. In counterexample 1.12, $R(K^*) \subset R(U)$ may not be true. Hence, the condition $R(K^*) \subset R(U)$ is necessary. From Corollary 2.2, we may obtain Corollary 5.32 in [3], Proposition 2.24 in [9] and Theorem 3.3 in [16], and we also correct Proposition 12 in [17]. In counterexample 1.14, the condition $US_\Lambda = S_\Lambda U$ is not true. Hence, this condition is necessary in Corollary 2.3. From Corollary 2.3, we may obtain Theorem 3.11 in [16]. From Theorem 2.1, we improve Theorem 3.2 in [5], Theorem 2.4 in [10], Corollary 4.4 in [11], Theorem 2.12 in [14] and Theorem 3.5 in [22].

Theorem 2.5 *Let $K_1 \in B(\mathcal{H}_1)$ be an operator with closed range, suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are K_1-g-frames for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Assume $K_2 \in B(\mathcal{H}_2)$, $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $U_1T_\Lambda T_\Gamma^* U_2^* + U_2T_\Gamma T_\Lambda^* U_1^* \geq 0$. Then, $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, if one of the following conditions holds:*

(1) $P = U_1 + U_2$, $R(P^*) \subset R(K_1)$, $R(K_2) \subset R(P)$.
(2) $Q = U_1 - U_2$, $R(Q^*) \subset R(K_1)$, $R(K_2) \subset R(Q)$.

Proof Let $\{\Lambda_j\}_{j\in J}$ be a K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with frame bounds A_1 and B_1. Let $\{\Gamma_j\}_{j\in J}$ be a K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with frame bounds A_2 and B_2. From Lemma 1.9, we obtain that $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bound $(\sqrt{B_1}\|U_1\| + \sqrt{B_2}\|U_2\|)^2$.

According to the proof of Theorem 2.1, for all $g \in \mathcal{H}_2$, we get

$$\begin{aligned}&\sum_{j\in J} \|(\Lambda_j U_1^* + \Gamma_j U_2^*)g\|^2 \\ &\quad= \sum_{j\in J} \|\Lambda_j U_1^* g\|^2 + \sum_{j\in J} 2\mathrm{Re}\{\langle \Lambda_j U_1^* g, \Gamma_j U_2^* g\rangle\} + \sum_{j\in J} \|\Gamma_j U_2^* g\|^2 \\ &\quad\geq A_1\|K_1^* U_1^* g\|^2 + \langle (U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*)g, g\rangle + A_2\|K_1^* U_2^* g\|^2 \\ &\quad\geq A_1\|K_1^* U_1^* g\|^2 + A_2\|K_1^* U_2^* g\|^2.\end{aligned}$$

Without loss of generality, assume that statement (1) holds; let $\lambda = \min\{A_1, A_2\}$, by the parallelogram law and Lemma 1.6, for every $g \in \mathcal{H}_2$, we obtain

$$A_1\|K_1^* U_1^* g\|^2 + A_2\|K_1^* U_2^* g\|^2 \geq \lambda(\|K_1^* U_1^* g\|^2 + \|K_1^* U_2^* g\|^2)$$

$$
\begin{aligned}
&= \frac{\lambda}{2}(\|K_1^*(U_1+U_2)^*g\|^2 + \|K_1^*(U_1-U_2)^*g\|^2) \\
&\geq \frac{\lambda}{2}\|K_1^*(U_1+U_2)^*g\|^2 = \frac{\lambda}{2}\|K_1^*P^*g\|^2 \\
&\geq \frac{\lambda}{2}\|K_1^+\|^{-2}\|P^*g\|^2.
\end{aligned}
$$

From $R(K_2) \subseteq R(P)$, we conclude that there exists $\alpha > 0$ such that $K_2K_2^* \leq \alpha^2 PP^*$ by Theorem 1.10. It follows that $\alpha^{-2}\|K_2^*g\|^2 \leq \|P^*g\|^2$ for all $g \in \mathcal{H}_2$. Thus, for each $g \in \mathcal{H}_2$, we get

$$
\begin{aligned}
&\frac{\lambda}{2}\alpha^{-2}\|K_1^+\|^{-2}\|K_2^*g\|^2 \\
&\quad \leq \sum_{j\in J}\|(\Lambda_j U_1^* + \Gamma_j U_2^*)g\|^2 \leq (\sqrt{B_1}\|U_1\| + \sqrt{B_2}\|U_2\|)^2\|g\|^2.
\end{aligned}
$$

This proves that $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. □

Remark 2.6 From Theorem 2.5, we can get Proposition 2.24 in [9], Proposition 3.6 in [15], Theorem 3.5 in [20] and Proposition 3.2 in [23]. It is natural to consider whether the conditions $R(P^*) \subset R(K_1)$ and $R(Q^*) \subset R(K_1)$ are not necessary in Theorem 2.5. Now we give an example to illustrate that the conditions are essential.

Example 2.7 Let $\mathcal{H}_1 = \mathbb{C}^3$ and $J = \{1, 2, 3\}$. Assume that $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{H}_1$ and $\mathcal{V}_j = \overline{span}\{e_j\}$. Let $\{g_j\}_{j=1}^4$ be an orthonormal basis for $\mathcal{H}_2 = \mathbb{C}^4$. Now define $K_1 \in B(\mathcal{H}_1)$, $K_2 \in B(\mathcal{H}_2)$, $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $\{\Lambda_j\}_{j\in J}$ as follows:

$$
\begin{aligned}
&K_1 : \mathcal{H}_1 \to \mathcal{H}_1,\ K_1 f = \langle f, e_1\rangle e_1 + \langle f, e_3\rangle e_2, \quad f \in \mathcal{H}_1, \\
&K_2 : \mathcal{H}_2 \to \mathcal{H}_2,\ K_2 g = \langle g, g_2\rangle g_1, \quad g \in \mathcal{H}_2, \\
&U_1 : \mathcal{H}_1 \to \mathcal{H}_2,\ U_1 f = \langle f, e_2\rangle g_3 + \langle f, e_3\rangle g_1, \quad f \in \mathcal{H}_1, \\
&U_2 : \mathcal{H}_1 \to \mathcal{H}_2,\ U_2 f = \langle f, e_2\rangle g_3, \quad f \in \mathcal{H}_1, \\
&\Lambda_1 : \mathcal{H}_1 \to \mathcal{V}_1,\ \Lambda_1 f = \langle f, e_2\rangle e_1, \quad f \in \mathcal{H}_1, \\
&\Lambda_2 : \mathcal{H}_1 \to \mathcal{V}_2,\ \Lambda_2 f = \langle f, e_1\rangle e_2, \quad f \in \mathcal{H}_1, \\
&\Lambda_3 : \mathcal{H}_1 \to \mathcal{V}_3,\ \Lambda_3 f = \langle f, e_2\rangle e_3, \quad f \in \mathcal{H}_1.
\end{aligned}
$$

Let $\Gamma_j = \Lambda_j,\ j = 1, 2, 3$. Now we prove that $K_1^* f = \langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_3,\ f \in \mathcal{H}_1$. Indeed, for all $f, m \in \mathcal{H}_1$, we have

$$
\begin{aligned}
\langle K_1^* f, m\rangle = \langle f, K_1 m\rangle &= \langle f, \langle m, e_1\rangle e_1 + \langle m, e_3\rangle e_2\rangle \\
&= \langle f, e_1\rangle\overline{\langle m, e_1\rangle} + \langle f, e_2\rangle\overline{\langle m, e_3\rangle} \\
&= \langle f, e_1\rangle\langle e_1, m\rangle + \langle f, e_2\rangle\langle e_3, m\rangle \\
&= \langle\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_3, m\rangle.
\end{aligned}
$$

Hence, for any $f \in \mathcal{H}_1$, we get

$$\begin{aligned}\|K_1^* f\|^2 &= \|\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_3\|^2 = |\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2 \\ &\le \sum_{j=1}^{3} \|\Lambda_j f\|^2 = \sum_{j=1}^{3} \|\Gamma_j f\|^2 = |\langle f, e_1\rangle|^2 + 2|\langle f, e_2\rangle|^2 \le 2\|f\|^2.\end{aligned}$$

It follows that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Let T_Λ and T_Γ be the corresponding synthesis operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, respectively. Since $\Gamma_j = \Lambda_j,\ j = 1, 2, 3$, for every $f \in \mathcal{H}_1$, we obtain

$$T_\Lambda T_\Gamma^* f = T_\Gamma T_\Lambda^* f = T_\Lambda T_\Lambda^* f = \sum_{j=1}^{3} \Lambda_j^* \Lambda_j f = \langle f, e_1\rangle e_1 + 2\langle f, e_2\rangle e_2.$$

Now we show that $U_1^* g = \langle g, g_3\rangle e_2 + \langle g, g_1\rangle e_3,\ g \in \mathcal{H}_2$ and $U_2^* g = \langle g, g_3\rangle e_2,\ g \in \mathcal{H}_2$. In fact, for all $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, we obtain

$$\begin{aligned}\langle U_1^* g, f\rangle &= \langle g, U_1 f\rangle = \langle g, \langle f, e_2\rangle g_3 + \langle f, e_3\rangle g_1\rangle \\ &= \langle g, g_3\rangle \overline{\langle f, e_2\rangle} + \langle g, g_1\rangle \overline{\langle f, e_3\rangle} \\ &= \langle g, g_3\rangle \langle e_2, f\rangle + \langle g, g_1\rangle \langle e_3, f\rangle = \langle \langle g, g_3\rangle e_2 + \langle g, g_1\rangle e_3, f\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle U_2^* g, f\rangle &= \langle g, U_2 f\rangle = \langle g, \langle f, e_2\rangle g_3\rangle = \langle g, g_3\rangle \overline{\langle f, e_2\rangle} \\ &= \langle g, g_3\rangle \langle e_2, f\rangle = \langle \langle g, g_3\rangle e_2, f\rangle.\end{aligned}$$

By a direct calculation, we can conclude

$$\langle (U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*) g, g\rangle = 4|\langle g, g_3\rangle|^2 \ge 0$$

for all $g \in \mathcal{H}_2$. This implies that $U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^* \ge 0$.

Now we prove that $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Indeed, for each $g \in \mathcal{H}_2$, we get

$$\sum_{j=1}^{3} \|(\Lambda_j U_1^* + \Gamma_j U_2^*) g\|^2 = \|2\langle g, g_3\rangle e_1\|^2 + \|2\langle g, g_3\rangle e_3\|^2 = 8|\langle g, g_3\rangle|^2 \le 8\|g\|^2.$$

The adjoint operator of K_2 is K_2^*, $K_2^* g = \langle g, g_1\rangle g_2,\ g \in \mathcal{H}_2$. In fact, for any $g, h \in \mathcal{H}_2$, we obtain

$$\begin{aligned}\langle K_2^* g, h\rangle &= \langle g, K_2 h\rangle = \langle g, \langle h, g_2\rangle g_1\rangle = \langle g, g_1\rangle \overline{\langle h, g_2\rangle} \\ &= \langle g, g_1\rangle \langle g_2, h\rangle = \langle \langle g, g_1\rangle g_2, h\rangle.\end{aligned}$$

We can choose $g = g_1 \in \mathcal{H}_2$, then we obtain $\|K_2^* g\|^2 = |\langle g, g_1\rangle|^2 = 1$ and

$$\sum_{j=1}^{3} \|(\Lambda_j U_1^* + \Gamma_j U_2^*)g\|^2 = 8|\langle g, g_3\rangle|^2 = 0.$$

Therefore, $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is not a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. It is obvious that $K_1 \in B(\mathcal{H}_1)$ has closed range. Let $P = U_1 + U_2$ and $Q = U_1 - U_2$, we have $Pf = 2\langle f, e_2\rangle g_3 + \langle f, e_3\rangle g_1,\ f \in \mathcal{H}_1$ and $Qf = \langle f, e_3\rangle g_1,\ f \in \mathcal{H}_1$. Hence, we get

$$\overline{span}\{g_1\} = R(K_2) \subset R(P) = \overline{span}\{g_1, 2g_3\},$$
$$\overline{span}\{g_1\} = R(K_2) \subset R(Q) = \overline{span}\{g_1\},$$

but

$$\overline{span}\{2e_2, e_3\} = R(P^*) \not\subseteq R(K_1) = \overline{span}\{e_1, e_2\},$$
$$\overline{span}\{e_3\} = R(Q^*) \not\subseteq R(K_1) = \overline{span}\{e_1, e_2\}.$$

In the following, we offer an equivalent characterization of generating K-g-frames.

Remark 2.8 In [21, Corollary 3.17], it was stated that $K \in B(\mathcal{H})$ is an operator with closed range and $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$; suppose that $U \in B(\mathcal{H})$ has closed range and $UK = KU$, then the following conditions are equivalent: (1) U is surjective; (2) $\{\Lambda_j U^*\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. We announce a counterexample in Example 2.9.

Example 2.9 Assume $\mathcal{H} = \mathbb{C}^3$, $J = \{1, 2, 3\}$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{H}$, and $\mathcal{V}_j = \overline{span}\{e_j\}$. Now define $K \in B(\mathcal{H})$, $U \in B(\mathcal{H})$ and $\{\Lambda_j\}_{j\in J}$ as follows:

$$K : \mathcal{H} \to \mathcal{H},\ Kf = \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1, \quad f \in \mathcal{H},$$
$$U : \mathcal{H} \to \mathcal{H},\ Uf = \sum_{j=1}^{2}\langle f, e_j\rangle e_j \quad f \in \mathcal{H},$$
$$\Lambda_1 : \mathcal{H} \to \mathcal{V}_1,\ \Lambda_1 f = \langle f, e_2\rangle e_1, \quad f \in \mathcal{H},$$
$$\Lambda_2 : \mathcal{H} \to \mathcal{V}_2,\ \Lambda_2 f = \langle f, e_1\rangle e_2, \quad f \in \mathcal{H},$$
$$\Lambda_3 : \mathcal{H} \to \mathcal{V}_3,\ \Lambda_3 f = \langle f, e_3\rangle e_3, \quad f \in \mathcal{H}.$$

It is clear that $K \in B(\mathcal{H})$ has closed range. The adjoint operator of K is K^*, $K^* f = \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1,\ f \in \mathcal{H}$. In fact, for all $f, m \in \mathcal{H}$, we have

$$\begin{aligned}\langle K^* f, m\rangle &= \langle f, Km\rangle = \langle f, \langle m, e_1\rangle e_2 + \langle m, e_2\rangle e_1\rangle \\ &= \langle f, e_2\rangle\overline{\langle m, e_1\rangle} + \langle f, e_1\rangle\overline{\langle m, e_2\rangle} \\ &= \langle f, e_2\rangle\langle e_1, m\rangle + \langle f, e_1\rangle\langle e_2, m\rangle = \langle\langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1, m\rangle.\end{aligned}$$

For every $f \in \mathcal{H}$, we get

$$\|K^*f\|^2 = \|\langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1\|^2 = |\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2$$
$$\le \sum_{j=1}^{3} \|\Lambda_j f\|^2 = \sum_{j=1}^{3} |\langle f, e_j\rangle|^2 = \|f\|^2.$$

Thus, $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.

It is obvious that $U \in B(\mathcal{H})$ has closed range. By a direct calculation, for each $f \in \mathcal{H}$, we obtain

$$\begin{aligned} UKf &= U(\langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1) = \langle f, e_1\rangle Ue_2 + \langle f, e_2\rangle Ue_1 \\ &= \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1 \\ &= \langle f, e_1\rangle Ke_1 + \langle f, e_2\rangle Ke_2 = K\sum_{j=1}^{2}\langle f, e_j\rangle e_j = KUf, \end{aligned}$$

then $UK = KU$.

By a simple computation, we get $U^*f = \sum_{j=1}^{2}\langle f, e_j\rangle e_j,\ f \in \mathcal{H}$, then for all $f \in \mathcal{H}$, we have

$$\|K^*f\|^2 = \|\langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_1\|^2 = |\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2$$
$$= \sum_{j=1}^{3} \|\Lambda_j U^* f\|^2 = \sum_{j=1}^{2} |\langle f, e_j\rangle|^2 \le \|f\|^2.$$

Hence, $\{\Lambda_j U^*\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ and it is clear that U is not surjective.

Theorem 2.10 *Let $K_1 \in B(\mathcal{H}_1)$ and $\{\Lambda_j\}_{j\in J}$ be a K_1-g-frame for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Suppose that $K_2 \in B(\mathcal{H}_2)$ is an operator with dense range, $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is an operator with closed range and $UK_1 = K_2U$, then $\{\Lambda_j U^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if and only if U is surjective.*

Proof Suppose that U is surjective, it is obvious that $R(K_2^*) \subset R(U)$. According to Corollary 2.2, $\{\Lambda_j U^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$.

On the other hand, let T_Λ be the synthesis operator of the K_1-g-frame $\{\Lambda_j\}_{j\in J}$ and L be the synthesis operator of the K_2-g-frame $\{\Lambda_j U^*\}_{j\in J}$, then for any $\{g_j\}_{j\in J} \in \ell^2(\{\mathcal{V}_j\}_{j\in J})$, we obtain

$$L\{g_j\}_{j\in J} = \sum_{j\in J}(\Lambda_j U^*)^* g_j = U\sum_{j\in J}\Lambda_j^* g_j = UT_\Lambda\{g_j\}_{j\in J}.$$

Hence, we get $L = UT_\Lambda$.

Since $\{\Lambda_j U^*\}_{j\in J}$ is a K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, we have $R(K_2) \subset R(L)$ by Lemma 1.7. Hence $R(K_2) \subset R(UT_\Lambda) \subset R(U)$. Then, we obtain $\overline{R(K_2)} \subset$

$\overline{R(U)}$. Since $K_2 \in B(\mathcal{H}_2)$ is an operator with dense range and $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is an operator with closed range, then U is surjective. □

Remark 2.11 By taking $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $K_1 = K_2 = K$, we may correct Corollary 3.17 in [21]. In counterexample 2.9, we obtain that statement (2) does not imply statement (1).

In the end, we give a necessary and sufficient condition to yield a series of tight K-g-frames by two existing g-Bessel sequences.

Theorem 2.12 *Suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with synthesis operators T_Λ, T_Γ and frame operators S_Λ, S_Γ, respectively. Let $K \in B(\mathcal{H}_2)$ and $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then, $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a tight K-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if and only if there exists $A > 0$ such that*

$$AKK^* = U_1 S_\Lambda U_1^* + U_2 S_\Gamma U_2^* + U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*.$$

Proof According to Lemma 1.9, we get that $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$. Let L be the synthesis operator of $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$, then for each $\{g_j\}_{j\in J} \in \ell^2(\{\mathcal{V}_j\}_{j\in J})$, we obtain

$$\begin{aligned} L\{g_j\}_{j\in J} &= \sum_{j\in J} (\Lambda_j U_1^* + \Gamma_j U_2^*)^* g_j = U_1 \sum_{j\in J} \Lambda_j^* g_j + U_2 \sum_{j\in J} \Gamma_j^* g_j \\ &= (U_1 T_\Lambda + U_2 T_\Gamma)\{g_j\}_{j\in J}. \end{aligned}$$

Thus $L = U_1 T_\Lambda + U_2 T_\Gamma$. Suppose that S is the frame operator of $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$, then

$$\begin{aligned} S = LL^* &= (U_1 T_\Lambda + U_2 T_\Gamma)(U_1 T_\Lambda + U_2 T_\Gamma)^* \\ &= U_1 T_\Lambda T_\Lambda^* U_1^* + U_2 T_\Gamma T_\Gamma^* U_2^* + U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^* \\ &= U_1 S_\Lambda U_1^* + U_2 S_\Gamma U_2^* + U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*. \end{aligned}$$

By Lemma 1.8, $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a tight K-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if and only if there exists $A > 0$ such that

$$AKK^* = U_1 S_\Lambda U_1^* + U_2 S_\Gamma U_2^* + U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*.$$

This completes the proof. □

From Lemma 1.8 and Theorem 2.12, we have the following corollary.

Corollary 2.13 *Let $K_1 \in B(\mathcal{H}_1)$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be tight K_1-g-frames for $\mathcal{H}_1$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with frame bounds A_1 and A_2. Let $K_2 \in B(\mathcal{H}_2)$ and $U_1, U_2 \in B(\mathcal{H}_1, \mathcal{H}_2)$. Assume that T_Λ and T_Γ are synthesis operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, respectively. Then, $\{\Lambda_j U_1^* + \Gamma_j U_2^*\}_{j\in J}$ is a tight K_2-g-frame for $\mathcal{H}_2$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if and only if there exists $A > 0$ such that*

$$AK_2K_2^* = A_1 U_1 K_1 K_1^* U_1^* + A_2 U_2 K_1 K_1^* U_2^* + U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^*.$$

Remark 2.14 We can obtain Theorem 2.1 in [18] and Theorem 10 in [24] from Theorem 2.12. From Corollary 2.13, we may obtain Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5 in [18]; meanwhile, we can also get Theorem 14, Theorem 16, Theorem 18 and Theorem 20 in [24]. We improve Theorem 2.7 in [10] and Theorem 4.7 in [11] by Corollary 2.13.

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