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Global Lower Bounds on the First Eigenvalue for a Diffusion Operator

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Abstract
We derive global lower bounds for the first eigenvalue of a symmetric diffusion $\Delta_X := \Delta - \nabla_X$ on Riemannian manifolds with the Bakry–Émery–Ricci curvature bounded from below.

Keywords First eigenvalue · Diffusion operator · Bakry–Émery–Ricci curvature

Mathematics Subject Classification 35P15 · 53C21

1 Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold and X be a smooth vector field on M^n. The diffusion operator

$$\Delta_X := \Delta + \nabla_X \tag{1.1}$$

is an important generalization of the Laplacian operator Δ, in particular, the Witten–Laplacian

$$\Delta_f := \Delta - \nabla_{\nabla f} \tag{1.2}$$

is a special case of (1.1) by taking $X = -\nabla f$ for some $f \in C^\infty(M^n)$.

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As in [1,3,9], the m-dimensional Bakry–Émery–Ricci curvature of the diffusion operator Δ_X is defined as

$$Ric_X^m := Ric - \frac{1}{2}L_X g - \frac{X \otimes X}{m-n} \quad (1.3)$$

for any number $m \in (n, \infty)$, where L_X stands for the Lie derivative along the direction X. In particular, the m-dimensional Bakry–Émery–Ricci curvature of the Witten–Laplacian operator Δ_f is defined as

$$Ric_f^m := Ric + Hessf - \frac{df \otimes df}{m-n}, \quad (1.4)$$

where $Hessf$ is the Hessian of f.

For m-dimensional Bakry–Émery–Ricci curvatures, we can allow m to be infinite:

$$Ric_X^\infty := Ric - \frac{1}{2}L_X g, \quad (1.5)$$

and

$$Ric_f^\infty := Ric + Hessf, \quad (1.6)$$

which are called the ∞-dimensional Bakry–Émery–Ricci curvature of the diffusion operator Δ_X (of the Witten–Laplacian operator), respectively. We refer the readers to [5,6,11,12,15,16] for applications of Bakry–Émery–Ricci curvatures.

There are several well-known results on lower bound estimates for the first eigenvalue of Laplacian operator on closed Riemannian manifolds (see Section 5 of [7] for a summary): Lichnerowicz [10] (see also [14]) showed that the first nonzero eigenvalue of the Laplacian on a closed manifold must satisfy $\lambda_1 \geq mK$ if the Ricci curvature is bounded from below by $(m-1)K$. When the Ricci curvature is nonnegative, Li–Yau [8] proved that $\lambda_1 \geq \frac{\pi^2}{(1+a)d^2}$, where $0 \leq a < 1$ is a constant and d is the diameter of the underlying closed manifold. More generally, they [8] also derived a lower bound estimate that depends on the lower bound of the Ricci curvature, the upper bound of the diameter, and the dimension of the manifold alone. In this note, we will prove that these results are still valid for the first eigenvalue of the diffusion operator Δ_X on the closed manifold M^n under the condition of the Barky–Émery–Ricci curvature bounded from below.

Wu [17,18] established upper bounded first nontrivial eigenvalue for the Witten–Laplacian under the condition that the m-dimensional (∞-dimensional) Barky–Émery–Ricci curvature bounded below, respectively. We will consider global lower bounds in the case of the diffusion operator (1.1) via the m-dimensional (∞-dimensional) Bakry–Émery–Ricci curvature of (1.1).

The main theorems only consider m-dimensional Bakry–Émery–Ricci curvatures since the ∞-dimensional cases can be obtained by similar proof. On the closed Riemannian manifold (M^n, g), we derive a series lower bounds for the first eigenvalue of the diffusion operator Δ_X.

Theorem 1.1 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^m \geq -(m-1)Kg$ for some constant $K > 0$. Then, the first nonzero eigenvalue of the diffusion operator Δ_X on M^n must satisfy*

$$\lambda_1 \geq mK.$$

Theorem 1.2 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^m \geq 0$ and d be the diameter of M^n. Then,*

$$\lambda_1 \geq \frac{\pi^2}{(1-a)d^2}$$

for some constant $0 \leq a < 1$.

Theorem 1.3 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_f^m \geq -(m-1)Kg$ for some constant $K > 0$ and d be the diameter of M^n. Then, there exist positive constants $A_1(m)$ and $A_2(m)$ depending only on m so that the first nonzero eigenvalue of Δ_f satisfies*

$$\lambda_1 \geq \frac{A_1}{d^2} \exp\{-A_2 d\sqrt{K}\}.$$

We also prove global lower bound estimate on the first eigenvalue for the diffusion operator Δ_X on a complete noncompact Riemannian manifold.

Theorem 1.4 *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $Ric_X^m \geq -(m-1)Kg$ for some constant $K \geq 0$. Then, there is a global lower bound estimate on the first eigenvalue for the diffusion operator Δ_X*

$$\lambda_1 \geq -\frac{(m-1)^4 K^2}{4}. \tag{1.7}$$

In the rest of this paper, Theorems 1.1–1.3 are proved in Sect. 2, while Theorem 1.4 is established in Sect. 3.

2 The Closed Case

In this section, we will prove Theorems 1.1–1.3 of lower bound estimates for the first eigenvalue of the diffusion operator Δ_X on the closed manifold M^n, which are generalizations of [8,10,14].

Theorem 2.1 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^m \geq -(m-1)Kg$ for some constant $K > 0$. Then, the first nonzero eigenvalue of the diffusion operator Δ_X on M^n must satisfy*

$$\lambda_1 \geq mK. \tag{2.1}$$

Proof This result generalizes the lower bound result by Lichnerowicz [10] (see also Obata [14]). Let u be a nonconstant eigenfunction satisfying

$$\Delta_X u = \lambda u \tag{2.2}$$

with $\lambda > 0$.

Consider the function $A = |\nabla u|^2 - \frac{\lambda}{m} u^2$. A direct computation implies that

$$\begin{aligned}
\frac{1}{2}\Delta_X A &= \nabla_i \left(\nabla_i \nabla_j u \nabla_j u - \frac{\lambda}{m} u \nabla_i u\right) + X^i \left(\nabla_i \nabla_j u \nabla_j u - \frac{\lambda}{m} u \nabla_i u\right) \\
&= |Hessu|^2 + \langle \Delta \nabla u, \nabla u\rangle - \frac{\lambda}{m}|\nabla u|^2 - \frac{\lambda}{m} u \Delta u \\
&\quad + X^i \nabla_i \nabla_j u \nabla_j u - \frac{\lambda}{m} u \nabla_X u \\
&= |Hessu|^2 + \langle \nabla \Delta u, \nabla u\rangle + Ric(\nabla u, \nabla u) \\
&\quad + X^i \nabla_i \nabla_j u \nabla_j u - \frac{\lambda}{m}|\nabla u|^2 - \frac{\lambda}{m} u \Delta_X u \\
&= |Hessu|^2 + \langle \nabla \Delta_X u, \nabla u\rangle + Ric_X^m(\nabla u, \nabla u) \\
&\quad + \frac{(Xu)^2}{m-n} - \frac{\lambda}{m}|\nabla u|^2 - \frac{\lambda}{m} u \Delta_X u,
\end{aligned} \tag{2.3}$$

where $Hessu$ is the Hessian of u, and we used the second Bianchi identity in the third equality.

Since $m > n$, we have

$$\begin{aligned}
0 &\le \left(\sqrt{\frac{m-n}{mn}}\Delta u - \sqrt{\frac{n}{m(m-n)}}\nabla_X u\right)^2 \\
&\le \frac{(\Delta u)^2}{n} - \frac{1}{m}[(\Delta u)^2 + 2\Delta u \nabla_X u + (\nabla_X u)^2] + \frac{(\nabla_X u)^2}{m-n} \\
&\le |Hessu|^2 - \frac{1}{m}(\Delta_X u)^2 + \frac{(\nabla_X u)^2}{m-n},
\end{aligned}$$

i.e.,

$$|Hessu|^2 + \frac{(\nabla_X u)^2}{m-n} \ge \frac{1}{m}(\Delta_X u)^2. \tag{2.4}$$

Plugging the fact of $Ric_X^m \ge -(m-1)Kg$, (2.2) and (2.4) into (2.3), we get

$$\begin{aligned}
\frac{1}{2}\Delta_X A &\ge \frac{\lambda^2 u^2}{m} + \lambda|\nabla u|^2 - (m-1)K|\nabla u|^2 - \frac{\lambda}{m}|\nabla u|^2 - \frac{\lambda^2 u^2}{m} \\
&= (m-1)\left(\frac{\lambda}{m} - K\right)|\nabla u|^2.
\end{aligned} \tag{2.5}$$

If $\lambda \le mK$, then $\Delta_X A \le 0$ on M^n. By the compactness of M^n and the strong maximum principle, A must be identically constant. In particular, the right hand side of (2.5) must be identically 0, i.e., $\lambda \equiv mK$ since u is nonconstant.

To conclude, the first nonzero eigenvalue of the diffusion operator Δ_X on M^n is no less than mK. □

When the ∞-dimensional Bakry–Émery–Ricci curvature of the diffusion operator Δ_X is bounded below, we have

Corollary 2.2 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^\infty \ge -nKg$ for some constant $K > 0$ and $|X|^2 \le \alpha$ for some constant $\alpha \ge 0$. Then, the first nonzero eigenvalue of the diffusion operator Δ_X on M^n must satisfy*

$$\lambda_1 \ge (n+1)\left(K + \frac{\alpha}{n}\right). \tag{2.6}$$

Proof Note that $(Xh)^2 \le \alpha|\nabla h|^2$. Taking $m = n+1$ in the proof of Theorem 2.1, (2.5) becomes

$$\begin{aligned}\frac{1}{2}\Delta_X A &\ge \frac{\lambda^2 u^2}{n+1} + \lambda|\nabla u|^2 - (nK+\alpha)|\nabla u|^2 - \frac{\lambda}{n+1}|\nabla u|^2 - \frac{\lambda^2 u^2}{n+1}\\ &= n\left(\frac{\lambda}{n+1} - K - \frac{\alpha}{n}\right)|\nabla u|^2.\end{aligned}$$

Similarly, $\lambda \equiv (n+1)(K+\frac{\alpha}{n})$ if $\lambda \le (n+1)(K+\frac{\alpha}{n})$. To conclude, the first nonzero eigenvalue of the diffusion operator Δ_X on M^n is no less than $(n+1)(K+\frac{\alpha}{n})$. □

Then, we generalize two lower bound results by Li–Yau [8]. Let λ_1 be the least nontrivial eigenvalue of the diffusion operator Δ_X on the closed manifold M^n and let u be the corresponding eigenfunction. By multiplying with a constant, it is possible to exist a positive constant $a \in [0, 1)$ so that

$$a = \inf_{M^n} u + 1 = \sup_{M^n} u - 1.$$

Theorem 2.3 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^m \ge 0$ and d be the diameter of M^n. Then,*

$$\lambda_1 \ge \frac{\pi^2}{(1-a)d^2}. \tag{2.7}$$

Proof Set $\lambda = \lambda_1$ and $v = u - a$. Then, the equation becomes

$$\Delta_X v = \lambda(v + a).$$

Let $B := |\nabla v|^2 - cv^2$ with $c = \lambda(1-a) > 0$ and x_0 be a maximum point of B. Choose a frame $\{e_i\}_{i=1}^n$ so that $v_1(x_0) = \nabla_{e_1} v(x_0) = |\nabla v(x_0)|$ if $|\nabla v(x_0)| \ne 0$. Note that

$$\frac{1}{2}\nabla_i B = \nabla_i \nabla_j v \nabla_j v - cv\nabla_i v, \tag{2.8}$$

so at x_0

$$0 = v_1(v_{11} - cv), \tag{2.9}$$

and the Hessian of v satisfies

$$|Hessv|^2 \geq v_{11}^2 = c^2 v^2. \tag{2.10}$$

Covariant differentiating (2.8) with respect to e_i again, then using the Bochner formula and (2.10), we can get at x_0

$$\begin{aligned}
0 &\geq \frac{1}{2}\Delta_X B \\
&= |Hessv|^2 + \nabla_i\nabla_i\nabla_j v\nabla_j v - c|\nabla v|^2 - cv\Delta v \\
&\quad + X^i(\nabla_i\nabla_j v\nabla_j v - cv\nabla_i v) \\
&= |Hessv|^2 + \langle \nabla v, \nabla \Delta v\rangle + Ric(\nabla v, \nabla v) \\
&\quad - c|\nabla v|^2 - cv\Delta_X v + X^i\nabla_i\nabla_j v\nabla_j v \\
&\geq c^2v^2 + \langle \nabla v, \nabla \Delta_X v\rangle + Ric_X^m(\nabla v, \nabla v) \\
&\quad + \frac{(Xv)^2}{m-n} - cv_1^2 - c\lambda v(v+a) \\
&\geq c^2v^2 + \lambda v_1^2 - cv_1^2 - c\lambda v(v+a) \\
&= (\lambda - c)(v_1^2 - cv^2) - ac\lambda v \\
&\geq a\lambda B(x_0) - ac\lambda.
\end{aligned} \tag{2.11}$$

Hence, for all $x \in M^n$,

$$|\nabla v(x)|^2 + cv^2(x) = B(x) \leq B(x_0) \leq c,$$

i.e.,

$$|\nabla v(x)|^2 \leq \lambda(1-a)(1 - v^2(x)). \tag{2.12}$$

Also (2.12) is trivially satisfied if $|\nabla v(x_0)| = 0$. Let γ be the shortest geodesic from the minimizing point of v to the maximizing point. The length of γ is at most d. Integrating the gradient estimate (2.12) along this segment with respect to arclength, we obtain

$$\sqrt{\lambda(1-a)} \cdot d \geq \sqrt{\lambda(1-a)}\int_\gamma \mathrm{d}s \geq \int_\gamma \frac{|\nabla v|\mathrm{d}s}{\sqrt{1-v^2}} \geq \int_{-1}^{1} \frac{\mathrm{d}u}{\sqrt{1-v^2}} = \pi.$$

(2.7) follows immediately. □

Trivially, Theorem 2.3 holds for $Ric_X^\infty \geq 0$:

Corollary 2.4 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_X^\infty \geq 0$ and d be the diameter of M^n. Then,*

$$\lambda_1 \geq \frac{\pi^2}{(1-a)d^2}. \tag{2.13}$$

Under the condition of the m-dimensional Bakry–Émery–Ricci curvature of the Witten–Laplacian operator bounded below, we have

Theorem 2.5 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_f^m \geq -(m-1)Kg$ for some constant $K > 0$ and d be the diameter of M^n. Then, there exist positive constants $A_1(m)$ and $A_2(m)$ depending only on m so that the first nonzero eigenvalue of Δ_f satisfies*

$$\lambda_1 \geq \frac{A_1}{d^2}\exp\{-A_2 d\sqrt{K}\}.$$

Proof Let u be a nonconstant eigenfunction satisfying

$$\Delta_f u = \lambda u.$$

By the fact that

$$\lambda \int_{M^n} u e^{-f} = \int_{M^n} \Delta_f u e^{-f} = 0,$$

u must change sign. Hence, we may normalize u to satisfy $\min u = -1$ and $\max u \leq 1$. Let us consider the function

$$w = \log(a+u)$$

for some constant $a > 1$. The function w satisfies

$$\begin{aligned}\Delta_f w &= \frac{\Delta_f u}{a+u} - \frac{|\nabla u|^2}{(a+u)^2}\\ &= \frac{\lambda u}{a+u} - |\nabla w|^2.\end{aligned} \tag{2.14}$$

Calculating directly, we get

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla w|^2 &= |Hessw|^2 + \nabla_i\nabla_i\nabla_j w\nabla_j w - \nabla_i\nabla_j w\nabla_j w\nabla_i f\\ &= |Hessw|^2 + \langle\nabla w, \nabla\Delta w\rangle + Ric(\nabla w, \nabla w) - \nabla_i\nabla_j w\nabla_j w\nabla_i f\\ &= |Hessw|^2 + \langle\nabla w, \nabla\Delta_f w\rangle + Ric_f^m(\nabla w, \nabla w) + \frac{\langle\nabla f, \nabla w\rangle^2}{m-n}\end{aligned}$$

$$
\begin{aligned}
&= |Hessw|^2 + \left\langle \nabla w, \nabla \left(\frac{\lambda u}{a+u} - |\nabla w|^2 \right) \right\rangle + Ric_f^m(\nabla w, \nabla w) \\
&\quad + \frac{\langle \nabla f, \nabla w \rangle^2}{m-n} \\
&\geq |Hessw|^2 + \frac{a\lambda}{a+u}|\nabla w|^2 - \langle \nabla w, \nabla |\nabla w|^2 \rangle - (m-1)K|\nabla w|^2 \\
&\quad + \frac{\langle \nabla f, \nabla w \rangle^2}{m-n},
\end{aligned} \tag{2.15}
$$

where $Hessw$ is the Hessian of w, and we used the Bochner formula in the second equality and (2.14) in the fourth.

Note that

$$
\begin{aligned}
0 &\leq \left(\sqrt{\frac{m-n}{mn}} \Delta w + \sqrt{\frac{n}{m(m-n)}} \langle \nabla w, \nabla f \rangle \right)^2 \\
&= \left(\frac{1}{n} - \frac{1}{m} \right) (\Delta w)^2 + \frac{2}{m} \Delta w \langle \nabla w, \nabla f \rangle + \left(\frac{1}{m-n} - \frac{1}{m} \right) \langle \nabla w, \nabla f \rangle^2 \\
&\leq |Hessw|^2 - \frac{1}{m}\big((\Delta w)^2 - 2\Delta w \langle \nabla w, \nabla f \rangle + \langle \nabla w, \nabla f \rangle^2 \big) + \frac{\langle \nabla w, \nabla f \rangle^2}{m-n} \\
&= |Hessw|^2 - \frac{(\Delta_f w)^2}{m} + \frac{\langle \nabla w, \nabla f \rangle^2}{m-n} \\
&= |Hessw|^2 - \frac{1}{m} \left(\frac{\lambda u}{a+u} - |\nabla w|^2 \right)^2 + \frac{\langle \nabla w, \nabla f \rangle^2}{m-n} \\
&\leq |Hessw|^2 - \frac{1}{m} \left(|\nabla w|^4 - \frac{2\lambda u}{a+u} |\nabla w|^2 \right) + \frac{\langle \nabla w, \nabla f \rangle^2}{m-n},
\end{aligned}
$$

where we used (2.14) in the third equality. Therefore, we have

$$
|Hessw|^2 + \frac{\langle \nabla w, \nabla f \rangle^2}{m-n} \geq \frac{1}{m} \left(|\nabla w|^4 - \frac{2\lambda u}{a+u} |\nabla w|^2 \right). \tag{2.16}
$$

Applying (2.16)–(2.15), we obtain

$$
\begin{aligned}
\frac{1}{2}\Delta_f |\nabla w|^2 \geq \frac{1}{m} \left(|\nabla w|^4 - \frac{2\lambda u}{a+u} |\nabla w|^2 \right) + \frac{a\lambda}{a+u} |\nabla w|^2 \\
- \langle \nabla w, \nabla |\nabla w|^2 \rangle - (m-1)K|\nabla w|^2.
\end{aligned} \tag{2.17}
$$

If $x_1 \in M^n$ is a point where $|\nabla w|^2$ achieves its maximum, the maximum implies that at such point

$$
0 \geq \frac{1}{m} |\nabla w|^4 - \left(\frac{2\lambda u}{m(a+u)} - \frac{a\lambda}{a+u} + (m-1)K \right) |\nabla w|^2, \tag{2.18}
$$

i.e.,

$$\begin{aligned}|\nabla w|^2(x) &\le |\nabla w|^2(x_1)\\ &\le \frac{2\lambda u(x_1)}{a+u(x_1)} - \frac{ma\lambda}{a+u(x_1)} + m(m-1)K\\ &\le \frac{2\lambda}{a-1} + m(m-1)K \qquad (2.19)\end{aligned}$$

for all $x \in M^n$. Integrating $|\nabla w| = |\nabla \log(a+u)|$ along a minimal geodesic γ joining the points at which $u = -1$ and $u = \max u$, we have

$$\begin{aligned}\log\left(\frac{a}{a-1}\right) &\le \log\left(\frac{a+\max u}{a-1}\right)\\ &\le \int_\gamma |\nabla \log(a+u)|\\ &\le d\left(\frac{2\lambda}{a-1} + m(m-1)K\right)^{\frac{1}{2}} \qquad (2.20)\end{aligned}$$

for all $a > 1$. Setting $t = \frac{a-1}{a}$, we have

$$2\lambda \ge \frac{2}{a}\lambda \ge t\left(\frac{1}{d^2}\left(\log\frac{1}{t}\right)^2 - m(m-1)K\right) \qquad (2.21)$$

for all $0 < t < 1$. Maximizing the right hand side as a function of t by setting

$$t = \exp(-1 - \sqrt{1 + m(m-1)Kd^2}),$$

we obtain the estimate

$$\begin{aligned}\lambda &\ge \frac{1}{d^2}(1 + \sqrt{1 + m(m-1)Kd^2})\exp(-1 - \sqrt{1 + m(m-1)Kd^2})\\ &\ge \frac{A_1}{d^2}\exp\{-A_2 d\sqrt{K}\} \qquad (2.22)\end{aligned}$$

as claimed. □

As we obtained Theorem 2.5, we can derive the following result by similar arguments.

Corollary 2.6 *Let (M^n, g) be an n-dimensional closed manifold with $Ric_f^\infty \ge -nKg$ for some constant $K > 0$, $|X|^2 \le \alpha$ for some constant $\alpha \ge 0$ and d be the diameter of M^n. Then, there exist positive constants $A_3(n)$ and $A_4(n)$ depending only on m so that the first nonzero eigenvalue of Δ_f satisfies*

$$\lambda_1 \ge \frac{A_3}{d^2}\exp\{-A_4 d\sqrt{K+\alpha}\}.$$

Proof Note that $(Xh)^2 \leq \alpha|\nabla h|^2$. Taking $m = n+1$ in the proof of Theorem 2.1, (2.17) becomes

$$\frac{1}{2}\Delta_f|\nabla w|^2 \geq \frac{1}{n+1}\left(|\nabla w|^4 - \frac{2\lambda u}{a+u}|\nabla w|^2\right) + \frac{a\lambda}{a+u}|\nabla w|^2 - \langle\nabla w, \nabla|\nabla w|^2\rangle - (nK+\alpha)|\nabla w|^2. \quad (2.23)$$

Furthermore, we can obtain

$$\begin{aligned}\lambda &\geq \frac{1}{d^2}(1+\sqrt{1+(n+1)(nK+\alpha)d^2})\exp(-1-\sqrt{1+(n+1)(nK+\alpha)d^2}) \\ &\geq \frac{A_3}{d^2}\exp\{-A_4 d\sqrt{K+\alpha}\} \quad (2.24)\end{aligned}$$

as claimed. □

3 The Complete Noncompact Case

In this section, we prove a global lower bound for the first eigenvalue of diffusion operator Δ_X by using the technique of gradient estimate as Wu [17,18] did. First of all, we present a smooth cutoff function originated by Calabi [2] (see also [4,13]).

Choose a smooth function $\xi : [0, +\infty) \to [0, 1]$ so that $0 \leq \xi(s) \leq 1$, $\xi(s) = 1$ for $s \leq 1$ and $\xi(s) = 0$ for $s \geq 2$. Moreover, for some constants $C_1 > 0$ and $C_2 > 0$,

$$-C_1 \leq \frac{\xi'(s)}{\sqrt{\xi(s)}} \leq 0$$

and

$$\xi''(s) \geq -C_2.$$

Let (M^n, g) be a Riemannian manifold with $Ric \geq -(n-1)Kg$ for some constant $K := K(2\rho) > 0$ in $B(\bar{x}, 2\rho)$ and $r(x) := d(x, \bar{x})$ be the distance function from a fixed point $\bar{x} \in M^n$. For any $\rho > 0$, we define the cutoff function by

$$\phi(x) = \xi\left(\frac{r(x)}{\rho}\right).$$

We can assume without loss of generality that the function ϕ is smooth in $B(\bar{x}, 2\rho)$ by the arguments of Calabi [2] (see also [4]).

It is clear that $0 \leq \phi \leq 1$ on M^n, $\phi \equiv 1$ on $\overline{B(\bar{x}, \rho)}$ and $\phi \equiv 0$ outside $B(\bar{x}, 2\rho)$. Moreover, we have

$$\frac{|\nabla \phi|^2}{\phi} = \frac{(\xi')^2}{\rho^2 \xi} \leq \frac{C_1^2}{\rho^2} \tag{3.1}$$

in $B(\bar{x}, 2\rho)$.

If $Ric_X^m \geq -(m-1)Kg$ for some constant $K := K(2\rho) > 0$ in $B(\bar{x}, 2\rho)$, the generalized Laplacian comparison theorem (see Corollary 3.3 of [9]) implies that $\Delta_X r \leq (m-1)(\frac{1}{r} + \sqrt{K})$.

To deal with $\Delta\phi$, we divide the arguments into two cases:

- Case 1: $r(x) < \rho$. In this case, $\phi = 1$ around x. Therefore, $\Delta\phi = 0$.
- Case 2: $\rho \leq r(x) < 2\rho$. By direct computations, we have

$$\begin{aligned} \Delta_X \phi &= \frac{\xi'}{\rho} \Delta_X r + \frac{\phi''}{\rho^2} |\nabla r|^2 \\ &\geq -\frac{C_1}{\rho}(m-1)\left(\frac{1}{r(x)} + \sqrt{K}\right) - \frac{C_2}{\rho^2} \\ &\geq -\frac{C_1}{\rho}(m-1)\left(\frac{1}{\rho} + \sqrt{K}\right) - \frac{C_2}{\rho^2}. \end{aligned} \tag{3.2}$$

Therefore, we obtain

$$\Delta_X \phi \geq -\frac{(m-1)C_1(1+\rho\sqrt{K}) + C_2}{\rho^2} \tag{3.3}$$

in $B(\bar{x}, 2\rho)$. Then, we prove the following essential inequality.

Lemma 3.1 *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold. If u is a positive function defined on the geodesic ball $B(\bar{x}, \rho) \in M^n$ satisfying $\Delta_X u = \lambda u$ for some constant λ and $h = \log u$, then we have*

$$\begin{aligned} \frac{1}{2}\Delta_X |\nabla h|^2 \geq{}& \frac{m}{4(m-1)} \cdot \frac{|\nabla |\nabla h|^2|^2}{|\nabla h|^2} + \frac{1}{m-1}(|\nabla h|^2 - \lambda)^2 + Ric_X^m(\nabla h, \nabla h) \\ &- \left[\frac{\lambda}{(m-1)|\nabla h|^2} + \frac{m-2}{m-1}\right]\langle \nabla |\nabla h|^2, \nabla h\rangle. \end{aligned} \tag{3.4}$$

Proof Note that $\nabla h = \frac{\nabla u}{u}$, we have

$$\Delta_X h = \frac{\Delta_X u}{u} - \frac{|\nabla u|^2}{u^2} = \lambda - |\nabla h|^2. \tag{3.5}$$

Moreover,

$$\begin{aligned}\frac{1}{2}\Delta_X|\nabla h|^2 &= |Hessh|^2 + \nabla_i\nabla_i\nabla_j h\nabla_j h + X^i\nabla_j\nabla_j h\nabla_j h\\ &= |Hessh|^2 + \langle\nabla\Delta h, \nabla h\rangle + Ric(\nabla h, \nabla h) + X^i\nabla_j\nabla_j h\nabla_j h\\ &= |Hessh|^2 + \langle\nabla\Delta_X h, \nabla h\rangle + Ric_X^m(\nabla h, \nabla h) + \frac{(Xh)^2}{m-n}\\ &= |Hessh|^2 - \langle\nabla|\nabla h|^2, \nabla h\rangle + Ric_X^m(\nabla h, \nabla h) + \frac{(Xh)^2}{m-n},\end{aligned} \tag{3.6}$$

where $Hessh$ is the Hessian of h, and we used the Bochner formula in the second equality and (3.5) in the last.

As in the proof of Theorem 2.3, we choose a frame $\{e_i\}_{i=1}^n$ so that $|\nabla h| = \nabla_{e_1}h \equiv h_1$. Then, we have

$$|\nabla|\nabla h|^2|^2 = 4|Hessh(\nabla h, \cdot)|^2 = 4h_1^2\sum_{i=1}^n h_{1i}^2 = 4|\nabla h|^2\cdot\sum_{i=1}^n h_{1i}^2,$$

i.e.,

$$\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} = 4\sum_{i=1}^n h_{1i}^2. \tag{3.7}$$

Similar as the estimate (2.5) in [17], we get

$$\begin{aligned}|Hessh|^2 + \frac{(Xh)^2}{m-n} &\geq h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{1}{n-1}(\Delta h - h_{11})^2 + \frac{(Xh)^2}{m-n}\\ &= h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{1}{n-1}(|\nabla h|^2 - \lambda + h_{11} + Xh)^2 + \frac{(Xh)^2}{m-n}\\ &\geq h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{(|\nabla h|^2 - \lambda + h_{11})^2}{m-1}\\ &\geq \frac{m}{4(m-1)}\cdot\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{(|\nabla h|^2-\lambda)^2}{m-1}\\ &\quad + \frac{|\nabla h|^2-\lambda}{m-1}\cdot\frac{\langle\nabla|\nabla h|^2, \nabla h\rangle}{|\nabla h|^2},\end{aligned} \tag{3.8}$$

where we used Cauchy's inequality in the first line, (3.5) in the second, the fact of $\frac{(a+b)^2}{n-1} + \frac{b^2}{m-n} \geq \frac{a^2}{m-1}$ in the third and (3.7) in the last.

(3.4) follows by applying (3.8)–(3.6). □

Now we are ready to derive the global lower estimates for the first eigenvalue of Δ_X in the noncompact case.

Theorem 3.2 *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $Ric_X^m \geq -(m-1)Kg$ for some constant $K \geq 0$. Then, there is a global lower bound estimate on the first eigenvalue for the diffusion operator Δ_X*

$$\lambda_1 \geq -\frac{(m-1)^4 K^2}{4}. \tag{3.9}$$

Proof Define $H = \phi|\nabla h|^2$. Then, in $B(\bar{x}, 2\rho)$, we have

$$\begin{aligned}
\frac{1}{2}\phi\Delta_X H \geq & \frac{m}{4(m-1)} \cdot \frac{\phi^2|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{1}{m-1}(H-\phi\lambda)^2 - (m-1)KH \\
& -\phi^2\left[\frac{\lambda}{(m-1)|\nabla h|^2} + \frac{m-2}{m-1}\right]\langle\nabla|\nabla h|^2, \nabla h\rangle + \phi\langle\nabla\phi, \nabla|\nabla h|^2\rangle \\
& +H\Delta_X\phi \\
\geq & \frac{m}{4(m-1)} \cdot \frac{|\nabla H|^2 - |\nabla h|^2|\nabla\phi|^2}{|\nabla h|^2} + \frac{1}{m-1}(H-\phi\lambda)^2 - (m-1)KH \\
& -\phi\left[\frac{\lambda}{(m-1)|\nabla h|^2} + \frac{m-2}{m-1}\right]\langle\nabla H, \nabla h\rangle \\
& +H\left[\frac{\lambda}{(m-1)|\nabla h|^2} + \frac{m-2}{m-1}\right]\langle\nabla\phi, \nabla h\rangle \\
& +\langle\nabla\phi, \nabla H\rangle - \frac{|\nabla\phi|^2}{\phi}H - \frac{(m-1)C_1(1+\rho\sqrt{K}) + C_2}{\rho^2}H,
\end{aligned} \tag{3.10}$$

where we used Lemma 3.1 in the first inequality and (3.3) in the second.

Suppose that $x_1 \in B(\bar{x}, 2\rho) \subset M^n$ is a maximum of G. Applying the maximum principle to (3.10), we get

$$\begin{aligned}
0 \geq & -\frac{m|\nabla\phi|^2}{4(m-1)} + \frac{1}{m-1}(H-\phi\lambda)^2 - (m-1)KH \\
& +H\left[\frac{\lambda}{(m-1)|\nabla h|^2} + \frac{m-2}{m-1}\right]\langle\nabla\phi, \nabla h\rangle \\
& -\frac{|\nabla\phi|^2}{\phi}H - \frac{(m-1)C_1(1+\rho\sqrt{K}) + C_2}{\rho^2}H \\
\geq & -\frac{m+C_1^2}{4(m-1)\rho^2} + \frac{H^2 - 2\phi\lambda + (\phi\lambda)^2}{m-1} \\
& -(m-1)KH - \frac{|\lambda|\phi C_1}{(m-1)\rho}H^{\frac{1}{2}} - \frac{(m-2)C_1}{(m-1)\rho}H^{\frac{3}{2}} \\
& -\frac{(m-1)C_1(1+\rho\sqrt{K}) + C_1^2 + C_2}{\rho^2}H,
\end{aligned} \tag{3.11}$$

where we used (3.1) and Cauchy's inequality. Note that

$$\frac{(\phi\lambda)^2}{m-1} - \frac{|\lambda|\phi C_1}{(m-1)\rho} H^{\frac{1}{2}} \geq -\frac{C_1^2}{4(m-1)\rho^2} H,$$

and

$$-\frac{(m-2)C_1}{(m-1)\rho} H^{\frac{3}{2}} \geq -\frac{1}{2(m-1)} H^2 - \frac{(m-2)^2 C_1^2}{2(m-1)\rho^2} H.$$

Hence, (3.11) reduces to

$$\begin{aligned} 0 \geq & \frac{H^2}{2(m-1)} - \frac{2\phi\lambda}{m-1} - \frac{m + C_1^2}{4(m-1)\rho^2} \\ & - \left[(m-1)K + \frac{C_1^2}{4(m-1)\rho^2} + \frac{(m-2)^2 C_1^2}{2(m-1)\rho^2} \right. \\ & \left. + \frac{(m-1)C_1(1+\rho\sqrt{K}) + C_1^2 + C_2}{\rho^2} \right] H. \end{aligned} \tag{3.12}$$

Therefore, we get

$$\begin{aligned} & \left[(m-1)K + \frac{C_1^2}{4(m-1)\rho^2} + \frac{(m-2)^2 C_1^2}{2(m-1)\rho^2} + \frac{(m-1)C_1(1+\rho\sqrt{K}) + C_1^2 + C_2}{\rho^2} \right]^2 \\ & + \frac{2}{m-1} \cdot \left[\frac{2\phi\lambda}{m-1} + \frac{m + C_1^2}{4(m-1)\rho^2} \right] \geq 0. \end{aligned} \tag{3.13}$$

Taking $\rho \to \infty$ in (3.13), it becomes

$$(m-1)^2 K^2 + \frac{4\lambda}{(m-1)^2} \geq 0, \tag{3.14}$$

i.e.,

$$\lambda \geq -\frac{(m-1)^4 K^2}{4}. \tag{3.15}$$

This completes the proof. □

We have similar result via ∞-dimensional Bakry–Émery–Ricci curvature of the diffusion operator Δ_X.

Corollary 3.3 *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $Ric_X^\infty \geq -nKg$ for some constant $K \geq 0$ and $|X|^2 \leq \alpha$. Then, there is a global lower bound estimate on the first eigenvalue for the diffusion operator Δ_X*

$$\lambda_1 \geq -\frac{n^2(nK+\alpha)^2}{4}. \tag{3.16}$$

Proof Taking $m = n+1$ in Lemma 3.1, we can obtain

$$\begin{aligned}\frac{1}{2}\Delta_X|\nabla h|^2 \geq \frac{n+1}{4n}\cdot\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{1}{n}(|\nabla h|^2-\lambda)^2 + Ric_X^\infty(\nabla h, \nabla h)\\ -(Xh)^2 - \left(\frac{\lambda}{n|\nabla h|^2} + \frac{n-1}{n}\right)\langle\nabla|\nabla h|^2, \nabla h\rangle.\end{aligned} \tag{3.17}$$

Since $Ric_X^\infty \geq -nKg$ and $|X|^2 \leq \alpha$, it is clear that

$$Ric_X^{n+1} \geq -(nK+\alpha)g. \tag{3.18}$$

Then, the generalized Laplacian comparison theorem (see Corollary 3.3 of [9]) implies that $\Delta_X r \leq n(\frac{1}{r} + \sqrt{K+\frac{\alpha}{n}})$.

Using the same method as in the proof of (3.3), we can get

$$\Delta_X\phi \geq -\frac{nC_1(1+\rho\sqrt{K+\frac{\alpha}{n}})+C_2}{\rho^2} \tag{3.19}$$

Furthermore, (3.13) becomes

$$\begin{aligned}\left[nK+\alpha+\frac{C_1^2}{4n\rho^2}+\frac{(n-1)^2C_1^2}{2n\rho^2}+\frac{(nC_1(1+\rho\sqrt{K+\frac{\alpha}{n}})+C_1^2+C_2}{\rho^2}\right]^2\\ +\frac{2}{n}\cdot\left[\frac{2\phi\lambda}{n}+\frac{n+1+C_1^2}{4n\rho^2}\right] \geq 0.\end{aligned} \tag{3.20}$$

Taking $\rho \to \infty$ in (3.20), we get

$$(nK+\alpha)^2 + \frac{4\lambda}{n^2} \geq 0, \tag{3.21}$$

i.e.,

$$\lambda \geq -\frac{n^2(nK+\alpha)^2}{4} \tag{3.22}$$

as desired. □

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