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Multiplication Alteration by Two-Cocycles: The Non-associative Version

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Abstract

In this paper, we introduce the theory of multiplication alteration by two-cocycles for non-associative structures like non-associative bimonoids with left (right) division. Also, we explore the connections between Yetter–Drinfeld modules for Hopf quasigroups, projections of Hopf quasigroups, skew pairings and quasitriangular structures, obtaining the non-associative version of the main results proved by Doi and Takeuchi in the Hopf algebra setting.

Keywords Non-associative bimonoid · Hopf quasigroup · Two-cocycle · Skew pairing · Double cross product · Strong projection · Quasitriangular Hopf quasigroup

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1 Introduction and Preliminaries

Let R be a commutative ring with a unit and denote the tensor product over R by $\otimes$. In [34], we can find one of the first interesting examples of multiplication alteration by a two-cocycle for R-algebras. In this case, Sweedler proved that, if U is an associative unitary R-algebra with a commutative subalgebra A and $\sigma = \sum a_i \otimes b_i \otimes c_i \in A \otimes A \otimes A$ is an Amistur two-cocycle, then U admits a new associative and unitary product defined by $u \bullet v = \sum a_i \; ub_i \; vc_i$ for all $u, v \in U$. Moreover, if U is central separable, U with the new product is still central separable and is isomorphic to the Rosenberg–Zelinsky central separable algebra obtained from the two-cocycle σ^{-1} (see [33]). Later, Doi discovered in [11] a new construction to modify the algebra structure of a bialgebra A over a field $\mathbb{F}$ using an invertible two-cocycle σ in A. In this case, if $\sigma : A \otimes A \to \mathbb{F}$ is the two-cocycle, the new product on A is defined by

$$a * b = \sum \sigma(a_1 \otimes b_1)a_2 b_2 \sigma^{-1}(a_3 \otimes b_3)$$

for $a, b \in A$. With the new algebra structure and the original coalgebra structure, A is a new bialgebra denoted by A^σ, and if A is a Hopf algebra with antipode λ_A, so is A^σ with antipode given by

$$\lambda_{A^\sigma}(a) = \sum \sigma(a_1 \otimes \lambda_A(a_2))\lambda_A(a_3)\sigma^{-1}(\lambda_A(a_4) \otimes a_5)$$

for $a \in A$. One of the main remarkable examples of this construction is the Drinfeld double of a Hopf algebra H. If H^* is the dual of H and $A = H^{*cop} \otimes H$, the Drinfeld double $D(H)$ can be obtained as A^σ where σ is defined by $\sigma((x \otimes g) \otimes (y \otimes h)) = x(1_H)y(g)\varepsilon_H(h)$ for $x, y \in H^*$ and $g, h \in H$. As was pointed by Doi and Takeuchi in [12], "this will be the shortest description of the multiplication of $D(H)$."

A particular case of alterations of products by two-cocycles is provided by invertible skew pairings on bialgebras. If A and H are bialgebras and $\tau : A \otimes H \to \mathbb{F}$ is an invertible skew pairing, Doi and Takeuchi defined in [12] a new bialgebra $A \bowtie_\tau H$ in the following way: The morphism $\omega : A \otimes H \otimes A \otimes H \to \mathbb{F}$ defined by $\omega((a \otimes g) \otimes (b \otimes h)) = \varepsilon_A(a)\varepsilon_H(h)\tau(b \otimes g)$, for $a, b \in A$ and $g, h \in H$, is a two-cocycle in $A \otimes H$ and $A \bowtie_\tau H = (A \otimes H)^\omega$. The construction of $A \bowtie_\tau H$ also generalizes the Drinfeld double because H^{*cop} and H are skew-paired. Moreover, $A \bowtie_\tau H$ is an example of Majid's double cross product $A \bowtie H$ (see [23,25]) where the left H-module structure of A, denoted by φ_A, and the right A-module structure of H, denoted by ϕ_H, are defined by

$$\varphi_A(h \otimes a) = \sum \tau(a_1 \otimes h_1)a_2 \tau^{-1}(a_3 \otimes h_2),$$
$$\phi_H(h \otimes a) = \sum \tau(a_1 \otimes h_1)h_2 \tau^{-1}(a_2 \otimes h_3),$$

for $a \in A$ and $h \in H$.

On the other hand, a relevant class of Hopf algebras are quasitriangular Hopf algebras. This kind of Hopf algebraic objects was introduced by Drinfeld [13] and provides solutions of the quantum Yang–Baxter equation: If H is quasitriangular with morphism $R : \mathbb{F} \to H \otimes H$ and N is a left H-module with action φ_N, the endomorphism $T : N \otimes N \to N \otimes N$ defined by $T(n \otimes n') = \sum \varphi_M(R^1 \otimes n) \otimes \varphi_M(R^2 \otimes n')$ is a solution of the quantum Yang–Baxter equation. If, moreover, for a Hopf algebra A there exists an invertible skew pairing $\tau : A \otimes H \to \mathbb{F}$, by Proposition 2.5 of [12], we have that $g : A \bowtie_\tau H \to H$, defined by $g(a \otimes h) = \sum \tau(a \otimes R^1)R^2 h$ for $a \in A$, $h \in H$, is a Hopf algebra projection. Thus, skew pairings and quasitriangular Hopf algebras give relevant examples of Hopf algebra projections.

The theory of Hopf algebra projections was established by Radford in [30]. In this work, we can find the conditions than permit to obtain a Hopf algebra structure on the tensor product of two Hopf algebras A and H, where the product is the smash product algebra and the coproduct is the smash coproduct coalgebra. Moreover, the results proved by Radford also allow to characterize these kinds of objects in terms of bialgebra projections. By using the bosonization process, Radford's results were later generalized to the braided context by Majid [24], who established a one-to-one correspondence between Hopf algebras in the category of left–left Yetter–Drinfeld modules and Hopf algebras with a projection. Therefore, if we come back to the Hopf algebra projection induced by two Hopf algebras A and H, such that H is quasitriangular, and a skew pairing $\tau : A \otimes H \to \mathbb{F}$, we obtain by Majid's bijection a Hopf algebra in ${}^H_H\mathcal{YD}$. As was proved in [1], this Hopf algebra (or braided Hopf algebra) is A with a modified product and antipode. If $A \rtimes H$ denotes the bosonization of A, $A \rtimes H$ and $A \bowtie_\tau H$ are isomorphic as Hopf algebras.

A relevant generalization of Hopf algebras is the non-associative Hopf algebras introduced by Pérez-Izquierdo in [29]. A particular and interesting example of non-associative Hopf algebras is the Hopf quasigroups considered by Klim and Majid in [21]. These kinds of objects allow to understand the structure of the algebraic 7-sphere and also the structure of the enveloping algebra of a Malcev algebra. Moreover, non-associative Hopf algebras are related to other non-associative algebraic structures, and in the last years, an increasing research in this area has been developed (see [2–4,27,28,35]).

The main motivation of this paper is to introduce the theory of alteration multiplication, in the sense of Doi, for non-associative algebraic structures in monoidal categories. An outline of the paper is as follows. In Sect. 2, we recall some definitions and we prove some useful results for the next sections. In the third section, we prove that for a non-associative bimonoid A with a left (right) division, if there exists an invertible two-cocycle σ, it is possible to define a new non-associative bimonoid A^σ with a left (right) division. Then, if A is a Hopf quasigroup in the sense of Klim and Majid, A^σ is a Hopf quasigroup, and if A is a Hopf algebra, we recover Doi's construction. In Sect. 4, we introduce the notion of skew pairing and prove that, as in the associative Hopf algebra setting, if there exists a skew pairing, for two non-associative bimonoids A and H with a left (right) division, we can define a new non-associative bialgebra $A \bowtie_\tau H$ with a left (right) division such that $A \bowtie_\tau H = (A \otimes H)^\omega$ for some two-cocycle ω induced by τ. This implies a similar result for Hopf quasigroups, and as in the Hopf world, we prove in Sect. 5 that $A \bowtie_\tau H$ can be described in terms of double cross products. Finally, using the theory of Hopf quasigroup projections developed in [2], we show that for a Hopf quasigroup A and a quasitriangular Hopf quasigroup H, if there exists an invertible skew pairing τ, it is possible to obtain a strong Hopf quasigroup projection, and as a consequence of the results proved in [2], we obtain that A admits a structure of Hopf quasigroup in the category ${}^{H}_{H}\mathcal{YD}$ introduced in [2].

In this paper, we will work in a monoidal setting. Following [22], recall that a monoidal category is a category $\mathcal{C}$ equipped with a tensor product functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a unit object K of $\mathcal{C}$ and a family of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$
$$r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in $\mathcal{C}$ (called associativity, right unit and left unit constraints, respectively) satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$
$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where id_X denotes the identity morphism for each object X in $\mathcal{C}$. A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. Taking into account that every non-strict monoidal category is monoidal equivalent to a strict one (see [19]), we can assume without loss of generality that the category is strict and, as a consequence, our results remain valid for every non-strict symmetric monoidal category, which would include, for example, the categories of vector spaces over a field $\mathbb{F}$, or the one of left modules over a commutative ring R.

In what follows, for simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

A strict monoidal category $\mathcal{C}$ is braided (see [16,17]) if it has a braiding, i.e., a natural family of isomorphisms $t_{M,N} : M \otimes N \to N \otimes M$ such that the equalities

$$t_{M,N\otimes P} = (N \otimes t_{M,P}) \circ (t_{M,N} \otimes P), \quad t_{M\otimes N,P} = (t_{M,P} \otimes N) \circ (M \otimes t_{N,P}),$$

hold. In this case, it is obvious for all object M of $\mathcal{C}$ that $t_{M,K} = t_{K,M} = id_M$. Moreover, we will say that the category is symmetric if $t_{N,M} \circ t_{M,N} = id_{M\otimes N}$ for all M, N in $\mathcal{C}$.

From now on, $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and symmetry c. Also, inspired by the work of Bespalov et al. (see, e.g., [5]), we will assume that every idempotent morphism $q : X \to X$ in the category $\mathcal{C}$ admits a factorization $q = i \circ p$ where $i : Z \to X$ and $p : X \to Z$ are called the injection and the projection associated with q and Z is the image object in $\mathcal{C}$ of q. The family of categories where every idempotent morphism splits includes the categories with epi–monic factorization, the categories with equalizers and the categories with coequalizers. For example, the category of complete bornological spaces is symmetric, closed and not abelian, but it does have coequalizers (see [26]). On the other hand, the category of complex Hilbert spaces, denoted by **Hilb**, is an example of not abelian (and not closed) symmetric monoidal category with coequalizers (see [18]).

A magma in $\mathcal{C}$ is a pair $A = (A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ (product) is a morphism in $\mathcal{C}$. A unital magma in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where (A, μ_A) is a magma in $\mathcal{C}$ and $\eta_A : K \to A$ (unit) is a morphism in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. A monoid in $\mathcal{C}$ is a unital magma $A = (A, \eta_A, \mu_A)$ in $\mathcal{C}$ satisfying $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, i.e., the product μ_A is associative. Given two unital magmas (monoids) A and B, $f : A \to B$ is a morphism of unital magmas (monoids) if $f \circ \eta_A = \eta_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

Also, if A, B are unital magmas (monoids) in $\mathcal{C}$, the object $A \otimes B$ is a unital magma (monoid) in $\mathcal{C}$ where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. If $A = (A, \eta_A, \mu_A)$ is a unital magma, so is $A^{op} = (A, \eta_A, \mu_A \circ c_{A,A})$.

A comagma in $\mathcal{C}$ is a pair $D = (D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in $\mathcal{C}$. A counital comagma in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where (D, δ_D) is a comagma in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit) is a morphism in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in $\mathcal{C}$ is a counital comagma in $\mathcal{C}$ satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct δ_D is coassociative. If D and E are counital comagmas (comonoids) in $\mathcal{C}$, $f : D \to E$ is a morphism of counital comagmas (comonoids) if $\varepsilon_E \circ f = \varepsilon_D$, and $(f \otimes f) \circ \delta_D = \delta_E \circ f$.

Moreover, if D, E are counital comagmas (comonoids) in $\mathcal{C}$, the object $D \otimes E$ is a counital comagma (comonoid) in $\mathcal{C}$ where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. If $D = (D, \varepsilon_D, \delta_D)$ is a counital comagma so is $D^{cop} = (D, \varepsilon_D, c_{D,D} \circ \delta_D)$.

Let $f : B \to A$ and $g : B \to A$ be morphisms between a comagma B and a magma A. We define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$. If A is unital and B counital, we will say that f is convolution invertible if there exists $f^{-1} : B \to A$ such that $f * f^{-1} = f^{-1} * f = \varepsilon_B \otimes \eta_A$. Note that if $B = K$ we have that $f * g = \mu_A \circ (f \otimes g)$ and f is convolution invertible if there exists $f^{-1} : K \to A$ such that $f * f^{-1} = f^{-1} * f = \eta_A$.

2 Non-associative Bimonoids

In this section, we introduce the definition of non-associative bimonoid with left (right) division. We give some properties and establish the relation with left (right) Hopf quasigroups.

Definition 2.1 A non-associative bimonoid in the category $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are morphisms of unital magmas. (Equivalently, η_H and μ_H are morphisms of counital comagmas.) Then, the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \tag{1}$$

$$\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H, \tag{2}$$

$$\delta_H \circ \eta_H = \eta_H \otimes \eta_H, \tag{3}$$

$$\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}. \tag{4}$$

We say that H has a left division if moreover there exists a morphism $l_H : H \otimes H \to H$ in $\mathcal{C}$ (called the left division of H) such that

$$l_H \circ h = \varepsilon_H \otimes H = \mu_H \circ (H \otimes l_H) \circ (\delta_H \otimes H), \tag{5}$$

where $h = (H \otimes \mu_H) \circ (\delta_H \otimes H)$.

A morphism $f : H \to B$ between non-associative bimonoids H and B is a morphism of unital magmas and comonoids.

We say that a non-associative bimonoid H in the category $\mathcal{C}$ is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$.

Remark 2.2 There is a similar notion of non-associative bimonoid with right division, replacing l_H by a morphism $r_H : H \otimes H \to H$ that, instead of (5), satisfies:

$$r_H \circ d = H \otimes \varepsilon_H = \mu_H \circ (r_H \otimes H) \circ (H \otimes \delta_H), \tag{6}$$

where $d = (\mu_H \otimes H) \circ (H \otimes \delta_H)$.

Note that, if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, the notion of non-associative bimonoid with left and a right division is the one introduced by Pérez-Izquierdo in [29] with the name of unital H-bialgebra.

Moreover, the morphisms h and d are the same that the ones called by Iyer et al. (see [15]) left composite and right composite, respectively.

Now, we give some properties about non-associative bimonoids with left (right) division.

Proposition 2.3 *Let H be a non-associative bimonoid. There exists a left division l_H if and only if the morphism h is an isomorphism. As a consequence, a left division l_H is uniquely determined.*

Similarly, there exists a right division r_H if and only if the morphism d is an isomorphism. As a consequence, a right division r_H is uniquely determined.

Proof Let $l_H : H \otimes H \to H$ be a left division. Define $h' = (H \otimes l_H) \circ (\delta_H \otimes H)$. Then, by (5) and the coassociativity of δ_H, we obtain that h' is the inverse of h.

On the other hand, if h is an isomorphism, using the coassociativity of δ_H, we obtain that

$$\begin{aligned}(\delta_H \otimes H) \circ h^{-1} \circ h = \delta_H \otimes H &= (H \otimes (h^{-1} \circ h)) \circ (\delta_H \otimes H) \\ &= (H \otimes h^{-1}) \circ (\delta_H \otimes H) \circ h\end{aligned}$$

and the equality

$$(\delta_H \otimes H) \circ h^{-1} = (H \otimes h^{-1}) \circ (\delta_H \otimes H) \tag{7}$$

holds.

Then, the morphism $l_H = (\varepsilon_H \otimes H) \circ h^{-1}$ is a left division for H. Indeed, trivially $l_H \circ h = \varepsilon_H \otimes H$ and, by (7), we have that

$$\mu_H \circ (H \otimes l_H) \circ (\delta_H \otimes H) = \mu_H \circ (H \otimes \varepsilon_H \otimes H) \circ (\delta_H \otimes H) \circ h^{-1}$$

$$= (\varepsilon_H \otimes H) \circ h \circ h^{-1} = \varepsilon_H \otimes H.$$

The proof for the right side is similar, and we leave the details to the reader. Note that in this case $d^{-1} = (r_H \otimes H) \circ (H \otimes \delta_H)$ and $r_H = (H \otimes \varepsilon_H) \circ d^{-1}$. □

Remark 2.4 In the conditions of the previous proposition, if h is an isomorphism, we obtain

$$h^{-1} \circ \delta_H = H \otimes \eta_H, \tag{8}$$
$$\mu_H \circ h^{-1} = \varepsilon_H \otimes H. \tag{9}$$

In a similar way, if d is an isomorphism,

$$d^{-1} \circ \delta_H = \eta_H \otimes H, \tag{10}$$
$$\mu_H \circ d^{-1} = H \otimes \varepsilon_H. \tag{11}$$

Also, composing with $\varepsilon_H \otimes H$ in (8),

$$l_H \circ \delta_H = \varepsilon_H \otimes \eta_H. \tag{12}$$

Composing with $H \otimes \eta_H$ in (5),

$$id_H * \lambda_H = \varepsilon_H \otimes \eta_H \tag{13}$$

for $\lambda_H = l_H \circ (H \otimes \eta_H)$. Similarly, composing with $\eta_H \otimes H$ in (6) we have

$$\varrho_H * id_H = \varepsilon_H \otimes \eta_H \tag{14}$$

for $\varrho_H = r_H \circ (\eta_H \otimes H)$.

On the other hand, by (1), (3) and (5)

$$l_H \circ (\eta_H \otimes H) = id_H. \tag{15}$$

Also, for right divisions we have

$$r_H \circ (H \otimes \eta_H) = id_H. \tag{16}$$

Finally, by (2) and (5)

$$\varepsilon_H \circ l_H = \varepsilon_H \otimes \varepsilon_H. \tag{17}$$

Therefore,

$$\varepsilon_H \circ \lambda_H = \varepsilon_H, \tag{18}$$

and

$$\lambda_H \circ \eta_H = \eta_H. \tag{19}$$

Of course, for a right division we have

$$\varepsilon_H \circ r_H = \varepsilon_H \otimes \varepsilon_H, \tag{20}$$

$$\varepsilon_H \circ \varrho_H = \varepsilon_H, \tag{21}$$

$$\varrho_H \circ \eta_H = \eta_H. \tag{22}$$

The following result was proved in ([29], Proposition 6) for unital H-bialgebras. In this paper, we give an alternative proof based in Proposition 2.3.

Proposition 2.5 *Let H be a non-associative bimonoid with left division l_H. It holds that*

$$\delta_H \circ l_H = (l_H \otimes l_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H). \tag{23}$$

As a consequence, if $\lambda_H = l_H \circ (H \otimes \eta_H)$, we have that λ_H is anticomultiplicative, i.e.,

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H. \tag{24}$$

If r_H is a right division for H, the equality

$$\delta_H \circ r_H = (r_H \otimes r_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H)) \tag{25}$$

holds. Then, if $\varrho_H = r_H \circ (\eta_H \otimes H)$, we have that

$$\delta_H \circ \varrho_H = (\varrho_H \otimes \varrho_H) \circ c_{H,H} \circ \delta_H. \tag{26}$$

Proof Indeed, if we compose in the first term of (23) with the isomorphism $h = (H \otimes \mu_H) \circ (\delta_H \otimes H)$, we obtain

$$\delta_H \circ l_H \circ h = \varepsilon_H \otimes \delta_H$$

and, on the other hand, composing in the second term,

$$\begin{aligned}
&(l_H \otimes l_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H) \circ h \\
&= (l_H \otimes l_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \\
&\quad \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H\otimes H})) \circ (\delta_H \otimes H) \ (\text{by } (4)) \\
&= ((l_H \circ h) \otimes l_H) \circ (H \otimes c_{H,H} \otimes \mu_H) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \\
&\quad \circ (((H \otimes \delta_H) \circ \delta_H) \otimes \delta_H) \ (\text{by naturality of } c \text{ and coassociativity}) \\
&= (H \otimes l_H) \circ (c_{H,H} \otimes \mu_H) \circ \delta_{H\otimes H} \ (\text{by naturality of } c \text{ and properties of the counit}) \\
&= (H \otimes (l_H \circ h)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \ (\text{by naturality of } c) \\
&= \varepsilon_H \otimes \delta_H \ (\text{by naturality of } c).
\end{aligned}$$

Therefore, (23) holds. Finally, equality (24) follows by (23) and (3). Similarly, we can prove identities (25) and (26). □

Example 2.6 An interesting example of a non-associative bimonoid arises from Sabinin algebras. Following [29], a vector space V over a field of characteristic zero is called a Sabinin algebra if it is endowed with multilinear operations

$$\langle x_1, x_2, \ldots, x_m; y, z\rangle,\ \ m \geq 0,$$
$$\Phi(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n),\ \ m \geq 1, n \geq 2,$$

satisfying the equalities

$$\langle x_1, x_2, \ldots, x_m; y, z\rangle = -\langle x_1, x_2, \ldots, x_m; z, y\rangle,$$
$$\langle x_1, x_2, \ldots, x_r, a, b, x_{r+1}, \ldots, x_m; y, z\rangle - \langle x_1, x_2, \ldots, x_r, b, a, x_{r+1}, \ldots, x_m; y, z\rangle$$
$$+\sum_{k=0}^{r}\sum_{\omega}\langle x_{\omega_1}, \ldots, x_{\omega_k}; \langle x_{\omega_{k+1}}, \ldots x_{\omega_r}; a, b\rangle, \ldots, x_m; y, z\rangle = 0,$$
$$\sigma_{x,y,z}\langle x_1, x_2, \ldots, x_r, x; y, z\rangle + \sum_{k=0}^{r}\sum_{\omega}\langle x_{\omega_1}, \ldots, x_{\omega_k}; \langle x_{\omega_{k+1}}, \ldots x_{\omega_r}; y, z\rangle, x\rangle = 0$$

and

$$\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Phi(x_{\tau(1)}, \ldots, x_{\tau(m)}; y_{\upsilon(1)}, \ldots, y_{\upsilon(n)}),$$

where ω runs the set of all bijections of the type $\omega : \{1, 2, \ldots, r\} \rightarrow \{1, 2, \ldots, r\}$, $i \mapsto \omega_i$, $\omega_1 < \omega_2 < \cdots < \omega_k$, $\omega_{k+1} < \cdots < \omega_r$, $k = 0, 1, \ldots, r$, $r \geq 0$, $\sigma_{x,y,z}$ denotes the cyclic sum by x, y, z; $\tau \in S_m$, $\upsilon \in S_n$ and S_l is the symmetric group.

The universal enveloping algebra of a Sabinin algebra was constructed in [29], where it was also proved that it can be provided with a cocommutative non-associative bimonoid structure with left and right division. Moreover, as was pointed out in [29] and [27], when we take a finite set of independent operations and $\Phi = 0$, the notion of Sabinin algebra includes as examples Lie, Malcev and Bol algebras.

Now, we introduce the notion of left Hopf quasigroup.

Definition 2.7 A left Hopf quasigroup H in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \rightarrow H$ in $\mathcal{C}$ (called the left antipode of H) satisfying:

$$\mu_H\circ(\lambda_H\otimes\mu_H)\circ(\delta_H\otimes H) = \varepsilon_H\otimes H = \mu_H\circ(H\otimes\mu_H)\circ(H\otimes\lambda_H\otimes H)\circ(\delta_H\otimes H). \tag{27}$$

Note that composing with $H \otimes \eta_H$ in (27) we obtain

$$\lambda_H * id_H = \varepsilon_H \otimes \eta_H. \tag{28}$$

Obviously, there is a similar definition for the right side, i.e., H is a right Hopf quasigroup if there is a morphism $\varrho_H : H \rightarrow H$ in $\mathcal{C}$ (called the right antipode of H) such that

$$\mu_H\circ(\mu_H\otimes H)\circ(H\otimes\varrho_H\otimes H)\circ(H\otimes\delta_H) = H\otimes\varepsilon_H = \mu_H\circ(\mu_H\otimes\varrho_H)\circ(H\otimes\delta_H). \tag{29}$$

Then, composing with $\eta_H \otimes H$ in (29) we obtain

$$id_H * \varrho_H = \varepsilon_H \otimes \eta_H. \quad (30)$$

The above definition is a generalization of the notion of Hopf quasigroup (also called non-associative Hopf algebra with the inverse property, or non-associative IP Hopf algebra) introduced in [21]. (In this case, $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$.) We recall this definition in a monoidal setting (see [2,3]). Note that a Hopf quasigroup is associative if an only if it is a Hopf algebra.

Definition 2.8 A Hopf quasigroup H in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying (27) and (29). If H is a Hopf quasigroup in $\mathcal{C}$, the antipode λ_H is unique and antimultiplicative, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad (31)$$

([21], Proposition 4.2). A morphism between Hopf quasigroups H and B is a morphism $f : H \to B$ of unital magmas and comonoids. Then (see Lemma 1.4 of [2]), the equality

$$\lambda_B \circ f = f \circ \lambda_H \quad (32)$$

holds.

Remark 2.9 Note that if H is both left and right Hopf quasigroup, the left and right antipodes are the same. In effect, denote by λ_H and ϱ_H the left and the right antipodes, respectively. Then, taking into account (28), the coassociativity of δ_H and condition (29),

$$\varrho_H = (\lambda_H * id_H) * \varrho_H = \mu_H \circ (\mu_H \otimes \varrho_H) \circ (\lambda_H \otimes \delta_H) \circ \delta_H = \lambda_H.$$

As a consequence, H is a Hopf quasigroup if and only if H is a left and right Hopf quasigroup.

Example 2.10 A loop $(L, \cdot, \diagup, \diagdown)$ is a quadruple where L is a set, $\cdot$ (multiplication), $\diagup$ (right division) and $\diagdown$ (left division) are binary operations, satisfying the identities

$$v \diagdown (v \cdot u) = u, \quad (33)$$
$$u = (u \cdot v) \diagup v, \quad (34)$$
$$v \cdot (v \diagdown u) = u, \quad (35)$$
$$u = (u \diagup v) \cdot v, \quad (36)$$

and such that it contains an identity element e_L (i.e., $e_L \cdot x = x = x \cdot e_L$ hold for all x in L). In what follows, multiplications on L will be expressed by juxtaposition. If N is a non-empty subset of L, we say that N is a subloop of L if it is closed under the three binary operations. Then, under these conditions, $e_L = e_N$.

A loop (iso)morphism is a (bijective) map $h : L_1 \to L_2$ such that $h(uv) = h(u)h(v)$, $h(u \diagup v) = h(u) \diagup h(v)$ and $h(u \diagdown v) = h(u) \diagdown h(v)$ for all $u, v \in L_1$. It is easy to see that the equality $h(e_{L_1}) = e_{L_2}$ holds for all loop morphism h.

Let R be a commutative ring and L a loop. Then, the loop algebra

$$RL = \bigoplus_{u \in L} Ru$$

is a cocommutative non-associative bimonoid with product and left and right division defined by linear extensions of those defined in L and

$$\delta_{RL}(u) = u \otimes u, \quad \varepsilon_{RL}(u) = 1_R$$

on the basis elements (see [29]).

Now, we give the relation between non-associative bimonoids with left division and left Hopf quasigroups.

Proposition 2.11 *The following assertions are equivalent:*

(i) H is a non-associative bimonoid with left division l_H such that

$$l_H = \mu_H \circ (\lambda_H \otimes H), \tag{37}$$

where $\lambda_H = l_H \circ (H \otimes \eta_H)$.

(ii) H is a non-associative bimonoid with left division l_H such that

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H, \tag{38}$$

where $\lambda_H = l_H \circ (H \otimes \eta_H)$.

(iii) H is a left Hopf quasigroup.

Proof By (5), (i) implies (ii). Moreover, composing in (38) with $(H \otimes l_H) \circ (\delta_H \otimes H)$ and using coassociativity, we get that (ii) implies (i). Now, assume (i). Then, by (37) and (5),

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = l_H \circ (H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H,$$

and in a similar way $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H$. Finally, if H is a left Hopf quasigroup, the morphism $l_H = \mu_H \circ (\lambda_H \otimes H)$ is a left division and satisfies (37). □

The relation between non-associative bimonoids with right division and right Hopf quasigroups is the following (the proof is similar to the one used for left divisions):

Proposition 2.12 *The following assertions are equivalent:*

(i) H is a non-associative bimonoid with right division r_H such that

$$r_H = \mu_H \circ (H \otimes \varrho_H), \tag{39}$$

where $\varrho_H = r_H \circ (\eta_H \otimes H)$.

(ii) H is a non-associative bimonoid with right division r_H such that

$$\mu_H \circ (\mu_H \otimes \varrho_H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H, \tag{40}$$

where $\varrho_H = r_H \circ (\eta_H \otimes H)$.

(iii) H is a right Hopf quasigroup.

Example 2.13 Let L be a loop. If for every element $u \in L$, there exists an element $u^{-1} \in L$ (the inverse of u) such that the equalities

$$u^{-1}(uv) = v = (vu)u^{-1}, \tag{41}$$

hold for every $v \in L$, we will say that L is a loop with the inverse property (for brevity an IP loop).

As a consequence, it is easy to show that, if L is an IP loop, for all $u \in L$ the element u^{-1} is unique and

$$u^{-1}u = e_L = uu^{-1} \tag{42}$$

hold. Moreover,

$$(uv)^{-1} = v^{-1}u^{-1}, \tag{43}$$

holds for any pair of elements $u, v \in L$.

Now, let R be a commutative ring and let L be and IP loop. Then, by Proposition 4.7 of [21], the non-associative bimonoid

$$RL = \bigoplus_{u \in L} Ru$$

defined in Example 2.10 is a cocommutative Hopf quasigroup where the antipode is defined by $\lambda_{RL}(u) = u^{-1}$.

Note that Moufang loops provided examples of IP loops, and loop algebras of Moufang loops correspond to Moufang–Hopf algebras. This fact suggests that there is a correspondence between groups with triality and Hopf algebras with triality (see [4]).

Example 2.14 Let R be a commutative ring with $\frac{1}{2}$ and $\frac{1}{3}$ in R. A Malcev algebra $(M, [,])$ over R is a free R-module M equipped with a bilinear and anticommutative operation [,] such that:

$$[J(a, b, c), a] = J(a, b, [a, c]),$$

where $J(a, b, c) = [[a, b], c] - [[a, c], b] - [a, [b, c]]$ denotes the Jacobian in a, b, c (see [28]). Then, every Lie algebra is a Malcev algebra with $J = 0$. The universal enveloping algebra $U(M)$ can be provided with a Hopf quasigroup structure as a particularization of the construction alluded in Example 2.6.

Remark 2.15 Any Hopf quasigroup is a particular instance of a non-associative bimonoid with left and right division. In this case, it suffices to take

$$l_H := \mu_H \circ (\lambda_H \otimes H),$$

$$r_H := \mu_H \circ (H \otimes \lambda_H).$$

In any case, the notion of a non-associative bimonoid is wider because the loop algebra RL associated with a loop L and the universal algebra $U(V)$ of a Sabinin algebra V falls under its definition (see [27,29]).

3 Product Alterations by Two-Cocycles for Non-associative Bimonoids

In this section, we prove that two-cocycles provide a deformation way of altering the product of a non-associative bimonoid to produce other non-associative bimonoids. These kinds of cocycle deformations were introduced in the Hopf algebra setting by Doi in [11].

Definition 3.1 Let H be a non-associative bimonoid, and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. We say that σ is a two-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma) \tag{44}$$

holds, where $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$ and $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$.

Equivalently, σ is a two-cocycle if

$$\sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H})) = \sigma \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H). \tag{45}$$

Note that, if we compose in (45) with $\eta_H \otimes \eta_H \otimes H$, we obtain

$$((\sigma \circ (\eta_H \otimes H)) \otimes (\sigma \circ (\eta_H \otimes H))) \circ \delta_H = (\sigma \circ (\eta_H \otimes \eta_H)) \otimes (\sigma \circ (\eta_H \otimes H)), \tag{46}$$

and, if we compose with $H \otimes \eta_H \otimes \eta_H$, we get that

$$((\sigma \circ (H \otimes \eta_H)) \otimes (\sigma \circ (H \otimes \eta_H))) \circ \delta_H = (\sigma \circ (H \otimes \eta_H)) \otimes (\sigma \circ (\eta_H \otimes \eta_H)). \tag{47}$$

A two-cocycle σ is called normal if further

$$\sigma \circ (\eta_H \otimes H) = \varepsilon_H = \sigma \circ (H \otimes \varepsilon_H), \tag{48}$$

and it is easy to see that if σ is normal so is σ^{-1} because

$$\begin{aligned}\sigma^{-1} \circ (\eta_H \otimes H) &= \varepsilon_H * (\sigma^{-1} \circ (\eta_H \otimes H)) = (\sigma \circ (\eta_H \otimes H)) * (\sigma^{-1} \circ (\eta_H \otimes H)) \\ &= (\sigma * \sigma^{-1}) \circ (\eta_H \otimes H) = \varepsilon_H,\end{aligned}$$

and similarly $\sigma^{-1} \circ (H \otimes \eta_H) = \varepsilon_H$. Analogously, if σ^{-1} is normal so is σ.

Remark 3.2 It is not difficult to show that, if σ is a two-cocycle, $\tau = (\sigma^{-1} \circ (\eta_H \otimes \eta_H)) \otimes \sigma$ is a normal two-cocycle (see [36]). The inverse of τ is $\tau^{-1} = (\sigma \circ (\eta_H \otimes \eta_H)) \otimes \sigma^{-1}$ and the normal condition for τ follows from the identities $(\tau \circ (\eta_H \otimes H)) * (\tau \circ (\eta_H \otimes H)) = \tau \circ (\eta_H \otimes H)$ and $(\tau \circ (H \otimes \eta_H)) * (\tau \circ (H \otimes \eta_H)) = \tau \circ (H \otimes \eta_H)$. (This identities are consequence of (46) and (47), respectively.) As a consequence, in the following we assume all two-cocycles are normal.

On the other hand, the morphisms $\partial^i(\sigma)$, $i \in \{1, 2, 3, 4\}$, are convolution invertible with inverses $\partial^i(\sigma^{-1})$, $i \in \{1, 2, 3, 4\}$, respectively. Then, the equalities

$$\partial^3(\sigma) * \partial^2(\sigma^{-1}) = \partial^1(\sigma^{-1}) * \partial^4(\sigma), \tag{49}$$

$$\partial^4(\sigma^{-1}) * \partial^1(\sigma) = \partial^2(\sigma) * \partial^3(\sigma^{-1}), \tag{50}$$

$$\partial^3(\sigma^{-1}) * \partial^1(\sigma^{-1}) = \partial^2(\sigma^{-1}) * \partial^4(\sigma^{-1}), \tag{51}$$

hold. Moreover, (49), (50) and (51) are equivalent to

$$(\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_H \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H \otimes \delta_H)$$
$$= (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H), \tag{52}$$

$$((\sigma \circ (\mu_H \otimes H)) \otimes (\sigma^{-1} \circ (H \otimes \mu_H))) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes \delta_H \otimes \delta_H) = (\sigma^{-1} \otimes \sigma) \circ (H \otimes \delta_H \otimes H) \tag{53}$$

and

$$\sigma^{-1} \circ (H \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) = \sigma^{-1} \circ (((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H}) \otimes H), \tag{54}$$

respectively.

Proposition 3.3 *Let H be a non-associative bimonoid. Let σ be a two-cocycle. Define the product μ_{H^σ} as*

$$\mu_{H^\sigma} = (\sigma \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes \delta_{H\otimes H}) \circ \delta_{H\otimes H}.$$

Then, $H^\sigma = (H, \eta_{H^\sigma} = \eta_H, \mu_{H^\sigma}, \varepsilon_{H^\sigma} = \varepsilon_H, \delta_{H^\sigma} = \delta_H)$ is a non-associative bimonoid.

Proof Equalities (1) and (3) hold trivially. Using that H is a non-associative bimonoid and (48), we get that $\mu_{H^\sigma} \circ (\eta_H \otimes H) = id_H = \mu_{H^\sigma} \circ (H \otimes \eta_H)$. Moreover, by (2),

$$\varepsilon_H \circ \mu_{H^\sigma} = \sigma * \sigma^{-1} = \varepsilon_H \otimes \varepsilon_H.$$

Finally, by the naturality of c, the coassociativity of δ_H and the properties of the counit,

$$(\mu_{H^\sigma} \otimes \mu_{H^\sigma}) \circ \delta_{H\otimes H}$$
$$= (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes (\sigma^{-1} * \sigma) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H}))$$
$$\circ (H \otimes ((c_{H,H} \otimes c_{H,H}) \circ \delta_{H\otimes H}) \otimes H) \circ (\delta_H \otimes \delta_H)$$

$$= \delta_H \circ \mu_{H^\sigma}.$$

□

Proposition 3.4 *Let H be a non-associative bimonoid with left division l_H, put $\lambda_H = l_H \circ (H \otimes \eta_H)$, and let σ be a two-cocycle. Define the morphism $f : H \to K$ as $f = \sigma \circ (H \otimes \lambda_H) \circ \delta_H$. If equality (28) holds, then f is convolution invertible with inverse $f^{-1} = \sigma^{-1} \circ (\lambda_H \otimes H) \circ \delta_H$. Moreover, the following identities hold:*

$$f \circ \eta_H = f^{-1} \circ \eta_H = id_K. \tag{55}$$

If r_H is a right division for H, put $\varrho_H = r_H \circ (\eta_H \otimes H)$. Let σ be a two-cocycle. Define the morphism $g : H \to K$ as $g = \sigma^{-1} \circ (\varrho_H \otimes H) \circ \delta_H$. If equality (30) holds, then g is convolution invertible with inverse $g^{-1} = \sigma \circ (H \otimes \varrho_H) \circ \delta_H$. Moreover,

$$g \circ \eta_H = g^{-1} \circ \eta_H = id_K. \tag{56}$$

Proof Indeed,

$$\begin{aligned}
& f * f^{-1} \\
& = (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \otimes H) \\
& \quad \circ (\delta_H \otimes H) \circ \delta_H \; (\text{by } (3) \text{ and naturality of } c) \\
& = (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H \circ \lambda_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \; (\text{by } (24)) \\
& = (\partial^1(\sigma^{-1}) * \partial^4(\sigma)) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes H) \\
& \quad \circ \delta_H \; (\text{by } (23), \text{ naturality of } c, \text{ and counit properties}) \\
& = (\partial^3(\sigma) * \partial^2(\sigma^{-1})) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ \delta_H \; (\text{by } (49)) \\
& = (\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_{H\otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \\
& \quad \circ (((\delta_H \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H) \otimes \delta_H) \circ \delta_H \\
& \quad (\text{by } (3), \; (23) \text{ and naturality of } c) \\
& = (\sigma \otimes \sigma^{-1}) \circ (H \otimes ((\mu_H \otimes (id_H * \lambda_H)) \circ (\lambda_H \otimes c_{H,H}) \\
& \quad \circ (\delta_H \otimes H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \; (\text{by naturality of } c, \text{ and coassociativity}) \\
& = (((\sigma \circ (H \otimes (\lambda_H * id_H)) \circ \delta_H)) \otimes (\sigma^{-1} \circ (\eta_H \otimes H))) \\
& \quad \circ \delta_H \; (\text{by } (13), \text{ and counit properties})) \\
& = \sigma \circ (H \otimes \eta_H) \; (\text{ by } (28), \text{ the normal condition for } \sigma^{-1} \text{ and counit properties}) \\
& = \varepsilon_H \; (\text{by } (48)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& f^{-1} * f \\
& = (\partial^4(\sigma^{-1}) * \partial^1(\sigma)) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \\
& \quad \circ \delta_H \; (\text{by coassociativity, naturality of } c, \text{ and counit properties}) \\
& = (\partial^2(\sigma) * \partial^3(\sigma^{-1})) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H \; (\text{by } (50))
\end{aligned}$$

$$= (\sigma \otimes \sigma^{-1}) \circ (\mu_H \otimes c_{H,H} \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)$$
$$\circ (((((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \otimes \delta_H) \circ \delta_H)$$
$$\otimes (((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H))) \circ \delta_H \text{ (by (3) and (24))}$$
$$= \sigma^{-1} \circ (H \otimes \sigma \otimes (id_H * \lambda_H)) \circ (((\lambda_H \otimes (\lambda_H * id_H)) \circ \delta_H)$$
$$\otimes ((c_{H,H} \circ (H \otimes \lambda_H) \circ \delta_H))) \circ \delta_H \text{ (by coassociativity and naturality of } c)$$
$$= \varepsilon_H \text{ (by (28), (13), the normal condition for } \sigma \text{ and } \sigma^{-1}\text{, naturality of } c\text{, counit properties, and (18))}.$$

Finally, (55) follows from (3), (15), the normal condition for σ and σ^{-1}, and (1). The proof for the right division is similar, and we leave the details to the reader. □

Remark 3.5 Note that equalities (28) and (30) hold for every left Hopf quasigroup. Also, they hold for loop algebras associated with right or left Bol loops. The so-called right Bol identity was introduced by G. Bol in [6] and was also mentioned by Bruck in [7]. Let $(L, \cdot, \diagup, \diagdown)$ be a loop. L is called a right Bol loop if the right Bol identity

$$((x \cdot y) \cdot z) \cdot y = x \cdot ((y \cdot z) \cdot y) \tag{57}$$

holds for all $x, y, z \in L$. If the equality (left Bol identity)

$$y \cdot (z \cdot (y \cdot x)) = (y \cdot (z \cdot y)) \cdot x \tag{58}$$

holds for all $x, y, z \in L$, we say that L is a left Bol loop. As was pointed in [32], Bol loops are more general than Moufang loops because L is Moufang if and only if it satisfies (57) and (58). Also, Bol loops with the automorphic inverse property are Bruck loops.

An interesting example of right Bol loops comes from matrix theory. The set of $n \times n$ positive definite symmetric matrices is a right Bol loop with the operation

$$P \cdot Q = \sqrt{QP^2Q}.$$

Moreover, in the literature we can find other examples of right Bol loops obtained by modifying the operation in a direct product of groups.

The cocommutative non-associative bimonoid RL defined in Example 2.10 satisfies equality (28) if and only if the loop L satisfies

$$(a \diagdown e_L) \cdot a = e_L. \tag{59}$$

If L is a right Bol loop, equality (59) always holds. Indeed, first note that

$$((a \cdot (a \diagdown e_L)) \cdot a) \cdot (a \diagdown e_L) = (e_L \cdot a) \cdot (a \diagdown e_L) = a \cdot (a \diagdown e_L) = e_L.$$

Then,

$$a \cdot (((a \diagdown e_L) \cdot a) \cdot (a \diagdown e_L)) = e_L = a \cdot (a \diagdown e_L).$$

As a consequence,

$$((a\diagdown e_L)\cdot a)\cdot(a\diagdown e_L)=a\diagdown e_L=e_L\cdot(a\diagdown e_L).$$

Therefore, (59) holds. In a similar way, it is easy to see (59) for a left Bol loop.

Proposition 3.6 *Let H be a non-associative bimonoid with left division l_H, put $\lambda_H = l_H\circ(H\otimes\eta_H)$ and assume that (28) holds. Let σ be a two-cocycle and let f, f^{-1} be the morphisms introduced in Proposition (3.4). Define the morphism $l_{H^\sigma}: H\otimes H\rightarrow H$ as*

$$l_{H^\sigma}=\mu_{H^\sigma}\circ(f\otimes\lambda_H\otimes f^{-1}\otimes H)\circ(H\otimes\delta_H\otimes H)\circ(\delta_H\otimes H).$$

Then, the equality

$$\delta_H\circ l_{H^\sigma}=(l_{H^\sigma}\otimes l_{H^\sigma})\circ(H\otimes c_{H,H}\otimes H)\circ((c_{H,H}\circ\delta_H)\otimes\delta_H) \tag{60}$$

holds. Moreover,

$$l_{H^\sigma}\circ(\eta_H\otimes H)=id_H, \tag{61}$$

and

$$l_{H^\sigma}\circ(H\otimes\eta_H)=(f\otimes\lambda_H\otimes f^{-1})\circ(\delta_H\otimes H)\circ\delta_H. \tag{62}$$

Therefore, we have

$$l_{H^\sigma}=\mu_{H^\sigma}\circ((l_{H^\sigma}\circ(H\otimes\eta_H))\otimes H), \tag{63}$$

$$(f^{-1}\otimes l_{H^\sigma})\circ(\delta_H\otimes H)=\mu_{H^\sigma}\circ(((\lambda_H\otimes f^{-1})\circ\delta_H)\otimes H). \tag{64}$$

Finally, if H is cocommutative,

$$l_{H^\sigma}\circ(H\otimes\eta_H)=\lambda_H. \tag{65}$$

Proof Indeed, equality (60) holds because:

$$\begin{aligned}
&(l_{H^\sigma}\otimes l_{H^\sigma})\circ(H\otimes c_{H,H}\otimes H)\circ((c_{H,H}\circ\delta_H)\otimes\delta_H)\\
&\quad=\mu_{H^\sigma\otimes H^\sigma}\circ(((f\otimes\lambda_H\otimes\lambda_H\otimes f^{-1})\circ\delta_{H\otimes H}\circ(H\otimes(f*f^{-1})\otimes H)\\
&\qquad\circ(\delta_H\otimes H)\circ\delta_H)\otimes\delta_H)\ (\text{by coassociativity and naturality of } c)\\
&\quad=\mu_{H^\sigma\otimes H^\sigma}\circ(((f\otimes((\lambda_H\otimes\lambda_H)\circ c_{H,H}\circ\delta_H)\otimes f^{-1})\\
&\qquad\circ(H\otimes\delta_H)\circ\delta_H)\otimes\delta_H)\ (\text{by the invertibility of } f\text{, coassociativity and counit properties})\\
&\quad=(\mu_{H^\sigma}\otimes\mu_{H^\sigma})\circ\delta_{H\otimes H}\circ(f\otimes\lambda_H\otimes f^{-1}\otimes H)\\
&\qquad\circ(H\otimes\delta_H\otimes H)\circ(\delta_H\otimes H)\ (\text{by } (24))\\
&\quad=\delta_H\circ l_{H^\sigma}\ (\text{by } (4) \text{ for } H^\sigma).
\end{aligned}$$

Identity (61) follows trivially because η_H is the unit of H^σ and by (3), (55) and (15). Also, using that η_H is the unit of H^σ, we obtain (62). Equality (63) follows directly from (62), and (64) is a consequence of the coassociativity of δ_H, the invertibility of f and the counit properties.

Finally, if H is cocommutative,

$$
\begin{aligned}
& l_{H^\sigma} \circ (H \otimes \eta_H) \\
& = (f \otimes \lambda_H \otimes f^{-1}) \circ (\delta_H \otimes H) \circ \delta_H \text{ (by (62))} \\
& = (f \otimes f^{-1} \otimes \lambda_H) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \text{ (by coassociativity and naturality of } c) \\
& = ((f * f^{-1}) \otimes \lambda_H) \circ \delta_H \text{ (by coassociativity and cocommutativity of } H) \\
& = \lambda_H \text{ (by the invertibility of } f \text{ and counit properties)}.
\end{aligned}
$$

□

The right division version of Proposition 3.6 is the following:

Proposition 3.7 *Let H be a non-associative bimonoid with right division r_H. Put $\varrho_H = r_H \circ (\eta_H \otimes H)$ and assume that (30) holds. Let σ be a two-cocycle and let g, g^{-1} be the morphisms introduced in Proposition (3.4). Define the morphism $r_{H^\sigma} : H \otimes H \to H$ as*

$$r_{H^\sigma} = \mu_{H^\sigma} \circ (H \otimes g^{-1} \otimes \varrho_H \otimes g) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes \delta_H).$$

Then, the equality

$$\delta_H \circ r_{H^\sigma} = (r_{H^\sigma} \otimes r_{H^\sigma}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H)) \tag{66}$$

holds. Moreover,

$$r_{H^\sigma} \circ (H \otimes \eta_H) = id_H, \tag{67}$$

and

$$r_{H^\sigma} \circ (\eta_H \otimes H) = (g^{-1} \otimes \varrho_H \otimes g) \circ (\delta_H \otimes H) \circ \delta_H. \tag{68}$$

Therefore, we have

$$r_{H^\sigma} = \mu_{H^\sigma} \circ (H \otimes (r_{H^\sigma} \circ (\eta_H \otimes H))), \tag{69}$$

$$(r_{H^\sigma} \otimes g^{-1}) \circ (H \otimes \delta_H) = \mu_{H^\sigma} \circ (H \otimes ((g^{-1} \otimes \varrho_H) \circ \delta_H)). \tag{70}$$

Finally, if H is cocommutative,

$$r_{H^\sigma} \circ (\eta_H \otimes H) = \varrho_H. \tag{71}$$

The following lemmas give two equalities which will be useful to get the main result of this section.

Lemma 3.8 *Let H be a non-associative bimonoid with left division l_H, put $\lambda_H = l_H \circ (H \otimes \eta_H)$ and assume that (28) holds. Let σ be a two-cocycle and let f, f^{-1} be the morphisms introduced in Proposition (3.4). Then, the equalities*

$$\sigma \circ ((l_{H^\sigma} \circ (H \otimes \eta_H)) \otimes H) \circ (H \otimes \mu_{H^\sigma}) \circ (\delta_H \otimes H)$$

$$= (f \otimes \sigma^{-1}) \circ (\delta_H \otimes H), \quad (72)$$

$$\sigma^{-1} \circ (H \otimes (\mu_{H^\sigma} \circ ((l_{H^\sigma} \circ (H \otimes \eta_H)) \otimes H))) \circ (\delta_H \otimes H)$$
$$= \sigma \circ (\lambda_H \otimes f^{-1} \otimes H) \circ (\delta_H \otimes H), \quad (73)$$

hold.

Proof We begin by showing (72):

$$\begin{aligned}
&\sigma \circ ((l_{H^\sigma} \circ (H \otimes \eta_H)) \otimes H) \circ (H \otimes \mu_{H^\sigma}) \circ (\delta_H \otimes H) \\
&= \sigma \circ (((((f \otimes \lambda_H) \circ \delta_H) \otimes f^{-1}) \circ \delta_H) \otimes \mu_{H^\sigma}) \circ (\delta_H \otimes H) \text{ (by (62))} \\
&= \sigma \circ (H \otimes (\partial^4(\sigma^{-1}) * \partial^1(\sigma)) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (((f \otimes \lambda_H) \circ \delta_H) \\
&\quad \otimes ((\lambda_H \otimes H) \circ \delta_H) \otimes H \otimes H \otimes H) \\
&\quad \circ (H \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes H) \text{ (by coassociativity)} \\
&= \sigma \circ (H \otimes (\partial^2(\sigma) * \partial^3(\sigma^{-1})) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (((f \otimes \lambda_H) \circ \delta_H) \\
&\quad \otimes ((\lambda_H \otimes H) \circ \delta_H) \otimes H \otimes H \otimes H) \\
&\quad \circ (H \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes H) \text{ (by (50))} \\
&= \sigma \circ (H \otimes \sigma \otimes \sigma^{-1} \otimes H) \circ (H \otimes ((\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \\
&\quad \circ ((\delta_H \circ \lambda_H) \otimes H \otimes H)) \otimes (((\delta_H \\
&\quad \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (((f \otimes \lambda_H) \circ \delta_H) \otimes H \otimes \delta_{H\otimes H}) \\
&\quad \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \text{ (by coassociativity and naturality of } c) \\
&= \sigma \circ (H \otimes \sigma \otimes \sigma^{-1} \otimes H) \circ (H \otimes ((\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \\
&\quad \circ ((c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes H \otimes H)) \otimes (((\delta_H \\
&\quad \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (((f \otimes \lambda_H) \circ \delta_H) \otimes H \otimes \delta_{H\otimes H}) \\
&\quad \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \text{ (by (24))} \\
&= \sigma \circ (H \otimes \sigma^{-1} \otimes H) \circ (((f \otimes \lambda_H) \circ \delta_H) \otimes ((\lambda_H \otimes \sigma) \\
&\quad \circ (H \otimes (\lambda_H * id_H) \otimes H) \circ (\delta_H \otimes H)) \otimes (((\delta_H \\
&\quad \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (H \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes H) \text{ (by naturality of } c) \\
&= \sigma \circ (H \otimes \sigma^{-1} \otimes H) \circ (((f \otimes \lambda_H) \circ \delta_H) \otimes \lambda_H \\
&\quad \otimes (((\delta_H \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \\
&\quad \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \\
&\quad \text{(by (28), counit properties, naturality of } c\text{, and (48))} \\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \\
&\quad \otimes (((\delta_H \circ \mu_H) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (((f \otimes H) \\
&\quad \circ \delta_H) \otimes H \otimes H) \circ (\delta_H \otimes H) \text{ (by naturality of } c) \\
&= (\sigma * \sigma^{-1}) \circ (((f \otimes \lambda_H) \circ \delta_H) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (\delta_H \otimes H) \text{ (by (24))} \\
&= (f \otimes \sigma^{-1}) \circ (\delta_H \otimes H) \text{ (by invertibility of } \sigma\text{, (18), counit properties, (2), and naturality of } c).
\end{aligned}$$

To get (73), we firstly show the equality

$$(f\otimes\sigma^{-1})\circ(H\otimes\lambda_H\otimes H)\circ(\delta_H\otimes H) = \sigma\circ(H\otimes\mu_H)\circ(H\otimes\lambda_H\otimes H)\circ(\delta_H\otimes H). \quad (74)$$

Indeed,

$$\begin{aligned}
&(f \otimes \sigma^{-1}) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes H)\\
&\quad \circ (\delta_H \otimes H) \text{ (by definition of } f \text{ and coassociativity)}\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes H)\\
&\quad \circ (\delta_H \otimes H) \text{ (by symmetry of } c)\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H \circ \lambda_H) \otimes H) \circ (\delta_H \otimes H) \text{ (by (24))}\\
&= (\partial^3(\sigma) * \partial^2(\sigma^{-1})) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes H) \text{ (by (49) and (52))}\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_{H\otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)\\
&\quad \circ (\delta_H \otimes (c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes \delta_H) \circ (\delta_H \otimes H) \text{ (by (24))}\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ (\delta_H\\
&\quad \otimes((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes \delta_H) \circ (\delta_H \otimes H) \text{ (by naturality of } c)\\
&= (\sigma \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes (id_H * \lambda_H) \otimes \mu_H \otimes H)\\
&\quad \circ (\delta_H \otimes \lambda_H \otimes \delta_H) \circ (\delta_H \otimes H) \text{ (by coassociativity)}\\
&= \sigma \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)\\
&\quad \text{(by (13), counit properties, naturality of } c \text{ and normality for } \sigma^{-1}).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\sigma^{-1} \circ (H \otimes (\mu_{H^\sigma} \circ ((l_{H^\sigma} \circ (H \otimes \eta_H)) \otimes H))) \circ (\delta_H \otimes H)\\
&= \sigma^{-1} \circ (H \otimes \mu_{H^\sigma}) \circ (H \otimes ((f \otimes \lambda_H \otimes f^{-1}) \circ (\delta_H \otimes H) \circ \delta_H) \otimes H)\\
&\quad \circ(\delta_H \otimes H) \text{ (by (62))}\\
&= \sigma^{-1} \circ (H \otimes ((\mu_H \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H)\\
&\quad \circ ((c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes \delta_H)))\\
&\quad \circ (H \otimes ((f \otimes ((\sigma \otimes H) \circ (H \otimes c_{H,H})\\
&\quad \circ(((\lambda_H \otimes H) \circ c_{H,H} \circ \delta_H) \otimes f^{-1} \otimes H)\\
&\quad \circ (\delta_H \otimes H))) \circ (\delta_H \otimes H)) \otimes H)\\
&\quad \circ (\delta_H \otimes \delta_H) \text{ (by naturality of } c \text{ and (24))}\\
&= (\sigma^{-1} \otimes ((f \otimes \sigma^{-1}) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)))\\
&\quad \circ (H \otimes (c_{H,H} \circ (H \otimes (\mu_H \circ (H \otimes \sigma \otimes H)\\
&\quad \circ (((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes f^{-1} \otimes \delta_H)\\
&\quad \circ(\delta_H \otimes H))) \circ (\delta_H \otimes H)) \otimes H)))\\
&\quad \circ (\delta_H \otimes \delta_H) \text{ (by naturality of } c)\\
&= (\sigma^{-1} \otimes (\sigma \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)))
\end{aligned}$$

$$
\begin{aligned}
&\circ (H \otimes (c_{H,H} \circ (H \otimes (\mu_H \circ (H \otimes \sigma \otimes H)\\
&\circ (((\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes f^{-1} \otimes \delta_H)\\
&\circ (\delta_H \otimes H))) \circ (\delta_H \otimes H)) \otimes H)))\\
&\circ (\delta_H \otimes \delta_H) \text{ (by (74))}\\
= &((\sigma^{-1} \circ (H \otimes \mu_H)) \otimes (\sigma \circ (H \otimes \mu_H))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H)\\
&\circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)\\
&\circ (\delta_H \otimes (c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes (\sigma \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H)\\
&\otimes H)) \otimes \delta_H) \circ (\delta_H \otimes H \otimes \delta_H) \circ (\delta_H \otimes H)\\
&\text{(by naturality of } c \text{ and coassociativity)}\\
= &(\partial_3(\sigma^{-1}) * \partial_3(\sigma)) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes (\sigma \circ (\lambda_H \otimes f^{-1} \otimes H)\\
&\circ (\delta_H \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by (24))}\\
= &\sigma \circ (\lambda_H \otimes f^{-1} \otimes H) \circ (\delta_H \otimes H)\\
&\text{(by invertibility of } \partial_3(\sigma) \text{ (see Remark 3.2), counit properties and (18)),}
\end{aligned}
$$

and the proof is complete. □

Lemma 3.9 *Let H be a non-associative bimonoid with right division r_H, put $\varrho_H = r_H \circ (\eta_H \otimes H)$ and assume that (30) holds. Let σ be a two-cocycle and let g, g^{-1} be the morphisms introduced in Proposition (3.4). Then, the equalities*

$$
\begin{aligned}
&\sigma^{-1} \circ (H \otimes (r_{H^\sigma} \circ (\eta_H \otimes H))) \circ (\mu_{H^\sigma} \otimes H) \circ (H \otimes \delta_H)\\
&\quad = (\sigma \otimes g) \circ (H \otimes \delta_H), \qquad (75)\\
&\sigma \circ ((\mu_{H^\sigma} \circ (H \otimes (r_{H^\sigma} \circ (\eta_H \otimes H)))) \otimes H) \circ (H \otimes \delta_H)\\
&\quad = \sigma^{-1} \circ (H \otimes g^{-1} \otimes \varrho_H) \circ (H \otimes \delta_H), \qquad (76)
\end{aligned}
$$

hold.

Proof The proof is similar to the one performed in the previous lemma but using

$$
(\sigma \otimes g) \circ (H \otimes \varrho_H \otimes H) \circ (H \otimes \delta_H) = \sigma^{-1} \circ (\mu_H \otimes H) \circ (H \otimes \varrho_H \otimes H) \circ (H \otimes \delta_H) \quad (77)
$$

instead of (74). □

The following theorem is the main result of this section. We will show that, under suitable conditions, H^σ is a non-associative bimonoid with (right) left division (r_{H^σ}) l_{H^σ}.

Theorem 3.10 *The following assertions hold:*

(i) Let H be a left Hopf quasigroup with left antipode λ_H. Let σ be a two-cocycle. Then, the non-associative bimonoid H^σ defined in Proposition 3.3 is a left Hopf quasigroup with left antipode $\lambda_{H^\sigma} = l_{H^\sigma} \circ (H \otimes \eta_H)$, where l_{H^σ} is the morphism introduced in Proposition 3.6.

(ii) *Let H be a right Hopf quasigroup with right antipode ϱ_H. Let σ be a two-cocycle. Then, the non-associative bimonoid H^{σ} defined in Proposition 3.3 is a right Hopf quasigroup with right antipode $\varrho_{H^{\sigma}} = r_{H^{\sigma}} \circ (\eta_H \otimes H)$, where $r_{H^{\sigma}}$ is the morphism introduced in Proposition 3.7.*

(iii) *Let H be a Hopf quasigroup with antipode λ_H. Let σ be a two-cocycle. Then, the non-associative bimonoid H^{σ} defined in Proposition 3.3 is a Hopf quasigroup with antipode $\lambda_{H^{\sigma}}$.*

Proof We prove (i). The proof for (ii) is similar using Lemma 3.9 instead of Lemma 3.8. The assertion (iii) follows from Remark 2.9. Note that if H is a Hopf quasigroup λ_H is a left and right antipode, then

$$\lambda_{H^{\sigma}} = l_{H^{\sigma}} \circ (H \otimes \eta_H) = r_{H^{\sigma}} \circ (\eta_H \otimes H) = \varrho_{H^{\sigma}},$$

because $f = g^{-1}$ and $f^{-1} = g$.

First, note that by Proposition 3.3, H^{σ} is a non-associative bimonoid. Therefore, to complete the proof we only need to show (5) for $l_{H^{\sigma}}$ and $\mu_{H^{\sigma}}$, because (63) holds (see Proposition 3.6), and then, by Proposition 2.11, we obtain that H^{σ} is a left Hopf quasigroup where $\lambda_{H^{\sigma}} = l_{H^{\sigma}} \circ (H \otimes \eta_H)$ is the left antipode. Indeed, on the one hand we have

$$\begin{aligned}
& l_{H^{\sigma}} \circ (H \otimes \mu_{H^{\sigma}}) \circ (\delta_H \otimes H) \\
&= (\sigma \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (H \otimes c_{H,H} \otimes H) \circ (f \\
&\quad \otimes (((\delta_H \circ \lambda_H) \otimes f^{-1}) \circ \delta_H) \otimes (\delta_H \circ \mu_{H^{\sigma}})) \\
&\quad \circ (\delta_H \otimes H \otimes H) \circ (\delta_H \otimes H) \; (\text{by definition of } \mu_{H^{\sigma}}) \\
&= (\sigma \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (H \otimes c_{H,H} \otimes H) \circ (f \\
&\quad \otimes (((c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H) \otimes f^{-1}) \circ \delta_H) \\
&\quad \otimes ((\mu_{H^{\sigma}} \otimes \mu_{H^{\sigma}}) \circ \delta_{H\otimes H})) \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \; (\text{by (24) and (4) for } H^{\sigma}) \\
&= ((\sigma \circ (\lambda_H \otimes f^{-1} \otimes \mu_{H^{\sigma}}) \circ (((H \otimes \delta_H) \circ \delta_H) \otimes H)) \\
&\quad \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (H \otimes c_{H,H} \otimes \mu_{H^{\sigma}}) \\
&\quad \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (((f \otimes \lambda_H) \\
&\quad \circ \delta_H) \otimes \delta_H \otimes \delta_H) \circ (\delta_H \otimes H) \; (\text{by naturality of } c \text{ and coassociativity}) \\
&= (((\sigma \circ (\lambda_H \otimes f^{-1} \otimes \mu_{H^{\sigma}}) \circ (((H \otimes \delta_H) \circ \delta_H) \otimes H)) \\
&\quad \circ ((((f^{-1} * f) \otimes H) \circ \delta_H) \otimes H)) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \\
&\quad \circ (H \otimes c_{H,H} \otimes \mu_{H^{\sigma}}) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (((f \otimes \lambda_H) \circ \delta_H) \\
&\quad \otimes \delta_H \otimes \delta_H) \circ (\delta_H \otimes H) \; (\text{by invertibility of } f \text{ and counit properties}) \\
&= (((\sigma \circ ((l_{H^{\sigma}} \circ (H \otimes \eta_H)) \otimes \mu_{H^{\sigma}}) \circ (\delta_H \otimes H))) \\
&\quad \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \circ (H \otimes c_{H,H} \otimes \mu_{H^{\sigma}}) \\
&\quad \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (((f \otimes ((\lambda_H \otimes f^{-1}) \circ \delta_H)) \circ \delta_H) \\
&\quad \otimes \delta_H \otimes \delta_H) \circ (\delta_H \otimes H) \; (\text{by naturality of } c\text{, coassociativity of } \delta_H \text{ and (62)}) \\
&= (((f \otimes \sigma^{-1}) \circ (\delta_H \otimes H)) \otimes ((\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H}))
\end{aligned}$$

$$
\begin{aligned}
&\circ (H\otimes c_{H,H}\otimes \mu_{H^{\sigma}})\circ (c_{H,H}\otimes c_{H,H}\otimes H)\\
&\circ (((f\otimes ((\lambda_H\otimes f^{-1})\circ \delta_H))\circ \delta_H)\otimes \delta_H\otimes \delta_H)\circ (\delta_H\otimes H)\ \ (\text{by } (72))\\
&= (\sigma^{-1}\otimes ((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H}))\circ (H\otimes c_{H,H}\otimes \mu_{H^{\sigma}})\\
&\circ (c_{H,H}\otimes c_{H,H}\otimes H)\circ (((f\otimes ((\lambda_H\otimes (f^{-1}*f))\circ \delta_H))\\
&\circ \delta_H)\otimes \delta_H\otimes \delta_H)\circ (\delta_H\otimes H)\ (\text{by naturality of } c \text{ and coassociativity})\\
&= (\sigma^{-1}\otimes ((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H}))\circ (H\otimes c_{H,H}\otimes \mu_{H^{\sigma}})\circ (c_{H,H}\otimes c_{H,H}\otimes H)\\
&\circ (((f\otimes \lambda_H)\circ \delta_H)\otimes \delta_H\otimes \delta_H)\\
&\circ (\delta_H\otimes H)\ (\text{by invertibility of } f \text{ and counit properties})\\
&= (\mu_H\otimes \sigma^{-1})\circ (H\otimes c_{H,H}\otimes H)\circ (\delta_H\otimes (\delta_H\circ \mu_H)\otimes \sigma^{-1})\\
&\circ (((f\otimes \lambda_H)\circ \delta_H)\otimes (\sigma^{-1}*\sigma)\otimes \delta_{H\otimes H})\\
&\circ (H\otimes \delta_{H\otimes H})\circ (\delta_H\otimes H)\ (\text{by naturality of } c\text{, coassociativity of } \delta_H \text{ and definition of } \mu_{H^{\sigma}})\\
&= (\mu_H\otimes \sigma^{-1})\circ (H\otimes c_{H,H}\otimes H)\circ (\delta_H\otimes ((\mu_H\otimes \mu_H)\circ \delta_{H\otimes H})\otimes \sigma^{-1})\\
&\circ (((f\otimes \lambda_H)\circ \delta_H)\otimes (\sigma^{-1}*\sigma)\otimes \delta_{H\otimes H})\\
&\circ (H\otimes \delta_{H\otimes H})\circ (\delta_H\otimes H)\ (\text{by } (4) \text{ and counit properties})\\
&= (\mu_H\otimes (\sigma^{-1}\circ (H\otimes ((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H}))))\circ (H\otimes c_{H,H}\otimes H\otimes H)\\
&\circ (((f\otimes (\delta_H\circ \lambda_H))\circ \delta_H)\otimes \mu_H\otimes H\otimes H)\\
&\circ (H\otimes \delta_{H\otimes H})\circ (\delta_H\otimes H)\ (\text{by naturality of } c \text{ and coassociativity})\\
&= (\mu_H\otimes (\partial^3(\sigma^{-1})*\partial^1(\sigma^{-1})))\circ (H\otimes c_{H,H}\otimes H\otimes H)\\
&\circ (((f\otimes (\delta_H\circ \lambda_H))\circ \delta_H)\otimes \mu_H\otimes H\otimes H)\circ (H\otimes \delta_{H\otimes H})\\
&\circ (\delta_H\otimes H)\ (\text{by } (54))\\
&= (\mu_H\otimes (\partial^2(\sigma^{-1})*\partial^4(\sigma^{-1})))\circ (H\otimes c_{H,H}\otimes H\otimes H)\\
&\circ (((f\otimes (\delta_H\circ \lambda_H))\circ \delta_H)\otimes \mu_H\otimes H\otimes H)\circ (H\otimes \delta_{H\otimes H})\\
&\circ (\delta_H\otimes H)\ (\text{by } (51))\\
&= (\mu_H\otimes (\sigma^{-1}\circ (((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H})\otimes H)))\circ (H\otimes c_{H,H}\otimes H\otimes H)\\
&\circ (((f\otimes (\delta_H\circ \lambda_H))\circ \delta_H)\otimes \mu_H\otimes H\otimes H)\\
&\circ (H\otimes \delta_{H\otimes H})\circ (\delta_H\otimes H)\ (\text{by } (54))\\
&= (\mu_H\otimes (\sigma^{-1}\circ (((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H})\otimes H)))\circ (H\otimes c_{H,H}\otimes H\otimes H)\\
&\circ (((f\otimes (c_{H,H}\circ (\lambda_H\otimes \lambda_H)\\
&\circ \delta_H))\circ \delta_H)\otimes \mu_H\otimes H\otimes H)\circ (H\otimes \delta_{H\otimes H})\circ (\delta_H\otimes H)\ (\text{by } (24))\\
&= (H\otimes (\sigma^{-1}\circ ((\mu_H\otimes \sigma^{-1})\circ \delta_{H\otimes H})\otimes H))\circ (c_{H,H}\otimes H\otimes H)\\
&\circ (f\otimes \lambda_H\otimes (\mu_H\circ (\lambda_H\otimes \mu_H)\circ (\delta_H\otimes H))\\
&\otimes H\otimes H)\circ (H\otimes H\otimes \delta_{H\otimes H})\circ (H\otimes \delta_H\otimes H)\circ (\delta_H\otimes H)\\
&(\text{by naturality of } c \text{ and coassociativity})\\
&= (H\otimes \sigma^{-1})\circ (c_{H,H}\otimes H)\circ (f\otimes ((\mu_H\otimes \sigma^{-1})\circ (H\otimes c_{H,H}\otimes H)\\
&\circ ((\delta_H\circ \lambda_H)\otimes \delta_H)\circ \delta_H)\otimes \delta_H)\circ (\delta_H\otimes H)\\
&(\text{by } (5)\text{, counit properties and naturality of } c)\\
&= (H\otimes \sigma^{-1})\circ (\sigma^{-1}\otimes c_{H,H}\otimes H)\circ (f\otimes ((H\otimes c_{H,H})\circ (H\otimes \mu_H\otimes H)
\end{aligned}
$$

$$\circ\,(((\lambda_H\otimes\lambda_H)\circ\delta_H)\otimes\delta_H)\circ\delta_H)\otimes\delta_H)$$
$$\circ\,(\delta_H\otimes H)\ (\text{by naturality of } c \text{ and } (24))$$
$$=(H\otimes\sigma^{-1})\circ(\sigma^{-1}\otimes c_{H,H}\otimes H)\circ(f\otimes((\lambda_H\otimes(c_{H,H}\circ((\lambda_H*id_H)$$
$$\otimes H)\circ\delta_H))\circ\delta_H)\otimes\delta_H)\circ(\delta_H\otimes H)$$
$$(\text{by coassociativity of } \delta_H)$$
$$=(f*f^{-1})\otimes H\ (\text{by } (28),\ \text{naturality of } c,\ \text{normality for } \sigma^{-1} \text{ and counit properties})$$
$$=\varepsilon_H\otimes H\ (\text{by invertibility of } f).$$

Finally, on the other hand,

$$\mu_{H^\sigma}\circ(H\otimes l_{H^\sigma})\circ(\delta_H\otimes H)$$
$$=(\mu_H\otimes\sigma^{-1})\circ(H\otimes c_{H,H}\otimes H)\circ(\sigma\otimes\delta_H\otimes((l_{H^\sigma}\otimes l_{H^\sigma})$$
$$\circ(H\otimes c_{H,H}\otimes H)\circ((c_{H,H}\circ\delta_H)\otimes\delta_H)))$$
$$\circ\,(H\otimes c_{H,H}\otimes H\otimes H)\circ(\delta_H\otimes((l_{H^\sigma}\otimes H)\circ(H\otimes c_{H,H})$$
$$\circ\,((c_{H,H}\circ\delta_H)\otimes H))\otimes\delta_H)\circ(\delta_H\otimes H)\ (\text{by } (60))$$
$$=(\mu_H\otimes(\sigma^{-1}\circ(H\otimes l_{H^\sigma})\circ(\delta_H\otimes H)))\circ(\sigma\otimes((H\otimes c_{H,H})$$
$$\circ\,(\delta_H\otimes l_{H^\sigma})\circ(\delta_H\otimes H))\otimes H)\circ(H\otimes c_{H,H}\otimes\delta_H)$$
$$\circ\,(\delta_H\otimes l_{H^\sigma}\otimes H)\circ(\delta_H\otimes\delta_H)\ (\text{by naturality of } c \text{ and coassociativity})$$
$$=(\mu_H\otimes(\sigma^{-1}\circ(H\otimes(\mu_{H^\sigma}\circ((l_{H^\sigma}\circ(H\otimes\eta_H))\otimes H)))$$
$$\circ\,(\delta_H\otimes H)))\circ(\sigma\otimes((H\otimes c_{H,H})\circ(\delta_H\otimes l_{H^\sigma})\circ$$
$$(\delta_H\otimes H))\otimes H)\circ(H\otimes c_{H,H}\otimes\delta_H)\circ(\delta_H\otimes l_{H^\sigma}\otimes H)\circ(\delta_H\otimes\delta_H)\ (\text{by } (63))$$
$$=(\mu_H\otimes(\sigma\circ(\lambda_H\otimes f^{-1}\otimes H)\circ(\delta_H\otimes H)))\circ(\sigma$$
$$\otimes((H\otimes c_{H,H})\circ(\delta_H\otimes l_{H^\sigma})\circ(\delta_H\otimes H))\otimes H)$$
$$\circ\,(H\otimes c_{H,H}\otimes\delta_H)\circ(\delta_H\otimes l_{H^\sigma}\otimes H)\circ(\delta_H\otimes\delta_H)\ (\text{by } (73))$$
$$=(\mu_H\otimes\sigma)\circ(H\otimes c_{H,H}\otimes H)\circ(\sigma\otimes((((H\otimes\lambda_H)\circ\delta_H)$$
$$\otimes((f^{-1}\otimes l_{H^\sigma})\circ(\delta_H\otimes H))\otimes H)\circ(\delta_H\otimes\delta_H)))$$
$$\circ\,(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes l_{H^\sigma}\otimes H)\circ(\delta_H\otimes\delta_H)\ (\text{by naturality of } c \text{ and coassociativity})$$
$$=(\mu_H\otimes\sigma)\circ(H\otimes c_{H,H}\otimes H)\circ(\sigma\otimes((((H\otimes\lambda_H)\circ\delta_H)\otimes(\mu_{H^\sigma}$$
$$\circ\,(((\lambda_H\otimes f^{-1})\circ\delta_H)\otimes H))\otimes H)\circ(\delta_H\otimes\delta_H)))$$
$$\circ\,(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes l_{H^\sigma}\otimes H)\circ(\delta_H\otimes\delta_H)\ (\text{by } (64))$$
$$=\mu_H\circ(H\otimes\mu_{H^\sigma}\otimes\sigma)\circ(H\otimes H\otimes c_{H,H}\otimes H)\circ(H\otimes(c_{H,H}$$
$$\circ\,(\lambda_H\otimes\lambda_H)\circ\delta_H)\otimes H\otimes H)\circ(\sigma\otimes\delta_H\otimes\delta_H)$$
$$\circ\,(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes((f^{-1}\otimes l_{H^\sigma})\circ(\delta_H\otimes H))\otimes H)$$
$$\circ(\delta_H\otimes\delta_H)\ (\text{by naturality of } c \text{ and coassociativity})$$
$$=\mu_H\circ(H\otimes\mu_{H^\sigma}\otimes\sigma)\circ(H\otimes H\otimes c_{H,H}\otimes H)\circ(H\otimes(\delta_H\circ\lambda_H)$$
$$\otimes H\otimes H)\circ(\sigma\otimes\delta_H\otimes\delta_H)\circ(H\otimes c_{H,H}\otimes H)$$
$$\circ\,(\delta_H\otimes(\mu_{H^\sigma}\circ(((\lambda_H\otimes f^{-1})\circ\delta_H)\otimes H))\otimes H)$$

$$\circ(\delta_H \otimes \delta_H) \text{ (by (64) and (24))}$$
$$= \mu_H \circ (H \otimes ((((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes (\sigma^{-1} * \sigma)) \circ \delta_{H\otimes H} \circ (\lambda_H \otimes H)))$$
$$\circ (\sigma \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H)$$
$$\circ(\delta_H \otimes \delta_H) \text{ (by naturality of } c \text{ and coassociativity)}$$
$$= \mu_H \circ (H \otimes ((\sigma \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \lambda_H) \otimes \delta_H)))$$
$$\circ (\sigma \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H)$$
$$\text{(by invertibility of } \sigma \text{, naturality of } c \text{ and counit properties)}$$
$$= \mu_H \circ (H \otimes ((\sigma \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H)$$
$$\otimes \delta_H))) \circ (\sigma \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by (24))}$$
$$= \mu_H \circ (H \otimes (\mu_H \circ (H \otimes \sigma \otimes H) \circ (((\lambda_H \otimes \lambda_H) \circ \delta_H)$$
$$\otimes \delta_H))) \circ (\sigma \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by naturality of } c)$$
$$= \mu_H \circ (H \otimes (\mu_H \circ (\lambda_H \otimes H)) \circ (\delta_H \otimes (\sigma \circ (\lambda_H \otimes H))$$
$$\otimes H) \circ (\sigma \otimes \delta_H \otimes \delta_H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by coassociativity)}$$
$$= ((\sigma \circ (\lambda_H \otimes H)) \otimes H) \circ (\sigma \otimes H \otimes \delta_H) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (\delta_H \otimes (\mu_{H^\sigma} \circ (((\lambda_H \otimes f^{-1}) \circ \delta_H) \otimes H)) \otimes H)$$
$$\circ (\delta_H \otimes \delta_H) \text{ (by (5) and counit properties)}$$
$$= (\sigma \otimes H) \circ (H \otimes ((\mu_{H^\sigma} \otimes \sigma) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ (\lambda_H \otimes \lambda_H)$$
$$\circ \delta_H) \otimes f^{-1} \otimes \delta_H)) \otimes H) \circ (\delta_H \otimes H \otimes \delta_H)$$
$$\circ (\delta_H \otimes H) \text{ (by naturality of } c \text{ and coassociativity)}$$
$$= (\sigma \otimes H) \circ (H \otimes ((\mu_{H^\sigma} \otimes \sigma) \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \lambda_H)$$
$$\otimes f^{-1} \otimes \delta_H)) \otimes H) \circ (\delta_H \otimes H \otimes \delta_H) \circ (\delta_H \otimes H) \text{ (by (24))}$$
$$= ((\sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}))) \otimes (\sigma^{-1} * \sigma) \otimes H) \circ (H \otimes H$$
$$\otimes c_{H,H} \otimes H \otimes H) \circ (H \otimes (\delta_H \circ \lambda_H) \otimes f^{-1}$$
$$\otimes \delta_H \otimes H) \circ (\delta_H \otimes H \otimes \delta_H) \circ (\delta_H \otimes H) \text{ (by naturality of } c \text{ and coassociativity)}$$
$$= ((\partial^1(\sigma) * \partial^3(\sigma)) \otimes H) \circ (H \otimes \lambda_H \otimes f^{-1} \otimes \delta_H)$$
$$\circ (\delta_H \otimes H \otimes H) \circ (\delta_H \otimes H) \text{ (by (45))}$$
$$= ((\partial^4(\sigma) * \partial^2(\sigma)) \otimes H) \circ (H \otimes \lambda_H \otimes f^{-1} \otimes \delta_H)$$
$$\circ (\delta_H \otimes H \otimes H) \circ (\delta_H \otimes H) \text{ (by (44))}$$
$$= (\sigma \otimes H) \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H \otimes H) \circ (H \otimes \lambda_H$$
$$\otimes f^{-1} \otimes \delta_H) \circ (\delta_H \otimes H \otimes H) \circ (\delta_H \otimes H) \text{ (by (45))}$$
$$= (\sigma \otimes H) \circ (((\sigma \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ (\lambda_H$$
$$\otimes \lambda_H) \circ \delta_H))) \otimes f^{-1} \otimes \delta_H) \circ (\delta_H \otimes H \otimes H)$$

$$\circ\, (\delta_H \otimes H) \text{ (by (24))}$$
$$= (\sigma \otimes (\sigma \circ ((id_H * \lambda_H) \otimes H))) \circ (H \otimes c_{H,H} \otimes \delta_H) \circ (\delta_H$$
$$\otimes \lambda_H \otimes f^{-1} \otimes H) \circ (\delta_H \otimes H \otimes H) \circ (\delta_H \otimes H)$$
$$\text{(by naturality of } c \text{ and coassociativity)}$$
$$= (f * f^{-1}) \otimes H \text{ (by (13), naturality of } c\text{, normality for } \sigma \text{ and counit properties)}$$
$$= \varepsilon_H \otimes H \text{ (by invertibility of } f\text{)},$$

and the proof is complete. □

4 Two-Cocycles and Skew Pairings

In this section, we will see that, in a similar way that for the Hopf algebra case, a class of two-cocycles is provided by invertible skew pairings for non-associative bimonoids. The following definition is inspired by the corresponding one for Hopf algebras introduced by Doi and Takeuchi in [12] (see also [1] for the monoidal setting).

Definition 4.1 Let A and H be non-associative bimonoids in $\mathcal{C}$. A pairing between A and H over K is a morphism $\tau : A \otimes H \to K$ such that the equalities

(a1) $\tau \circ (\mu_A \otimes H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H)$,
(a2) $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H)$,
(a3) $\tau \circ (A \otimes \eta_H) = \varepsilon_A$,
(a4) $\tau \circ (\eta_A \otimes H) = \varepsilon_H$,

hold.

A skew pairing between A and H is a pairing between A^{cop} and H, i.e., a morphism $\tau : A \otimes H \to K$ satisfying (a1), (a3), (a4) and

(a2$'$) $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)$.

It is easy to see that, by naturality of c, equality (a2$'$) is equivalent to

$$\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes c_{H,H}). \quad (78)$$

Remark 4.2 Note that, if A and H are Hopf quasigroups, a pairing between A and H^{cop} corresponds with the definition of Hopf pairing introduced in [14].

Proposition 4.3 *Let A, H be non-associative bimonoids with left division l_A and l_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then, τ is convolution invertible. Moreover, if τ^{-1} is the inverse of τ, the equalities*

$$\tau^{-1} \circ (\eta_A \otimes H) = \varepsilon_H, \quad (79)$$
$$\tau^{-1} \circ (A \otimes \eta_H) = \varepsilon_A, \quad (80)$$

and

$$\tau^{-1} \circ (A \otimes \mu_H) = (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H), \quad (81)$$

hold.

Proof Define $\tau^{-1} = \tau \circ (\lambda_A \otimes H)$, where $\lambda_A = l_A \circ (A \otimes \eta_A)$. Then,

$$\begin{aligned}
&\tau * \tau^{-1} \\
&= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (((A \otimes \lambda_A) \circ \delta_A) \otimes \delta_H) \text{ (by naturality of } c\text{)} \\
&= \tau \circ ((id_A * \lambda_A) \otimes H) \text{ (by (a1) of Definition 4.1)} \\
&= \varepsilon_A \otimes (\tau \circ (\eta_A \otimes H)) \text{ (by (13) for } A\text{)} \\
&= \varepsilon_A \otimes \varepsilon_H \text{ (by (a4) of Definition 4.1)}.
\end{aligned}$$

Moreover, if $\lambda_H = l_H \circ (H \otimes \eta_H)$, the morphism $\overline{\tau} = \tau \circ (\lambda_A \otimes \lambda_H)$ satisfies $\tau^{-1} * \overline{\tau} = \varepsilon_A \otimes \varepsilon_H$. Indeed,

$$\begin{aligned}
&\tau^{-1} * \overline{\tau} \\
&= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ (\lambda_A \otimes \lambda_A) \circ c_{A,A} \circ \delta_A) \\
&\quad \otimes ((H \otimes \lambda_H) \circ \delta_H)) \text{ (by naturality of } c\text{)} \\
&= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A \circ \lambda_A) \\
&\quad \otimes ((H \otimes \lambda_H) \circ \delta_H)) \text{ (by (24) for } A\text{)} \\
&= \tau \circ (\lambda_A \otimes (id_H * \lambda_H)) \text{ (by (a2}'\text{) of Definition 4.1)} \\
&= \tau \circ (\lambda_A \otimes \varepsilon_H \otimes \eta_H) \text{ (by (13) for } H\text{)} \\
&= (\varepsilon_A \circ \lambda_A) \otimes \varepsilon_H \text{ (by (a3) of Definition 4.1)} \\
&= \varepsilon_A \otimes \varepsilon_H \text{ (by (18) for } A\text{)}.
\end{aligned}$$

As a consequence, $\tau^{-1} = \tau \circ (\lambda_A \otimes H)$ is the convolution inverse of τ because

$$\tau = \tau * (\tau^{-1} * \overline{\tau}) = (\tau * \tau^{-1}) * \overline{\tau} = \overline{\tau}.$$

Thus

$$\tau = \tau^{-1} \circ (A \otimes \lambda_H). \tag{82}$$

It is not difficult to obtain the equalities (79) and (80) because

$$\begin{aligned}
&\tau^{-1} \circ (\eta_A \otimes H) \\
&= \tau \circ ((\lambda_A \circ \eta_A) \otimes H)) \text{ (by definition of } \tau^{-1}\text{)} \\
&= \tau \circ (\eta_A \otimes H)) \text{ (by (19) for } A\text{)} \\
&= \varepsilon_H \text{ (by (a4) of Definition 4.1)},
\end{aligned}$$

and

$$\begin{aligned}
&\tau^{-1} \circ (A \otimes \eta_H) \\
&= \tau \circ (\lambda_A \otimes \eta_H)) \text{ (by definition of } \tau^{-1}\text{)} \\
&= \varepsilon_A \circ \lambda_A \text{ (by (a4) of Definition 4.1)}
\end{aligned}$$

$$= \varepsilon_A \text{ (by (18) for } A).$$

Finally, the proof for (81) is the following:

$$\begin{aligned}
&\tau^{-1} \circ (A \otimes \mu_H) \\
&\quad = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A \circ \lambda_A) \otimes H \otimes H) \text{ (by (a2}') \text{ of Definition 4.1)} \\
&\quad = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ (\lambda_A \otimes \lambda_A) \\
&\qquad \circ c_{A,A} \circ \delta_A) \otimes H \otimes H) \text{ (by (24) for } A) \\
&\quad = (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H) \text{ (by naturality of } c \text{ and } c^2 = id).
\end{aligned}$$

□

Remark 4.4 Note that if A, H are non-associative bimonoids with right division r_A and r_H, respectively, and $\tau : A \otimes H \to K$ is a skew pairing, we can obtain (79), (80) and (81) using a similar proof and defining τ^{-1} as $\tau^{-1} = \tau \circ (\varrho_A \otimes H)$, where $\varrho_A = r_A \circ (\eta_A \otimes A)$.

Remark 4.5 Note that, in the conditions of Proposition 4.3, we obtain that

$$\tau = \tau \circ (\lambda_A \otimes \lambda_H) \tag{83}$$

and

$$\tau^{-1} = \tau^{-1} \circ (\lambda_A \otimes \lambda_H). \tag{84}$$

It should be noted that we can easily obtain the previous equalities for bimonoids with right division.

Proposition 4.6 *Let A, H be non-associative bimonoids with left division l_A and l_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. If $\lambda_H = l_H \circ (H \otimes \eta_H)$ is an isomorphism, the equality*

$$\tau^{-1} \circ (\mu_A \otimes H) = (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes (c_{H,H} \circ \delta_H)) \tag{85}$$

holds, where τ^{-1} is the morphism defined in Proposition 4.3. Moreover, if A is a Hopf quasigroup, equality (85) holds for any non-associative bimonoid H with left division.

Proof By composing with the isomorphism $A \otimes A \otimes \lambda_H$ in the left side of (85), we have

$$\begin{aligned}
&\tau^{-1} \circ (\mu_A \otimes \lambda_H) \\
&\quad = \tau \circ (\mu_A \otimes H) \text{ (by (82))} \\
&\quad = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H) \text{ (by (a1) of Definition 4.1)} \\
&\quad = ((\tau^{-1} \circ (A \otimes \lambda_H)) \otimes (\tau^{-1} \circ (A \otimes \lambda_H))) \circ (A \\
&\qquad \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H) \text{ (by (82))} \\
&\quad = (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes (c_{H,H}
\end{aligned}$$

$$\circ\, (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \text{ (by naturality of } c \text{ and coassociativity)}$$
$$= (\tau^{-1} \otimes \tau^{-1}) \circ (A$$
$$\otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes (c_{H,H} \circ \delta_H \circ \lambda_H)) \text{ (by (24) and } c^2 = id).$$

Therefore, (85) holds.

Finally, if we assume that A is a Hopf quasigroup, the antipode λ_A is antimultiplicative. Then, condition (85) is true without the assumption that λ_H is an isomorphism because

$$\tau^{-1} \circ (\mu_A \otimes H)$$
$$= \tau \circ ((\mu_A \circ (\lambda_A \otimes \lambda_A) \circ c_{A,A}) \otimes H) \text{ (by (31))}$$
$$= (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (c_{A,A} \otimes \delta_H) \text{ (by (a1) of Definition 4.1)}$$
$$= (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A$$
$$\otimes (c_{H,H} \circ \delta_H)) \text{ (by naturality of } c \text{ and } c^2 = id).$$

□

In a similar way, we get the previous result for non-associative bimonoids A and H with right division.

The following proposition gives the connection between skew pairings and two-cocycles for non-associative bimonoids with left division. We leave to the reader the proof of the similar result for non-associative bimonoids with right division.

Proposition 4.7 *Let A, H be non-associative bimonoids with left division l_A and l_H, respectively. Then, $A \otimes H = (A \otimes H, \eta_{A\otimes H}, \mu_{A\otimes H}, \varepsilon_{A\otimes H}, \delta_{A\otimes H})$ is a non-associative bimonoid with left division $l_{A\otimes H} = (l_A \otimes l_H) \circ (A \otimes c_{H,A} \otimes H)$. If A and H are left Hopf quasigroups with left antipodes λ_A, λ_H, respectively, $A \otimes H$ is a left Hopf quasigroup with left antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$.*

Moreover, let $\tau : A \otimes H \to K$ be a skew pairing. The morphism $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a normal two-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$, where τ^{-1} is defined as in Proposition 4.3.

Proof Trivially, $A \otimes H$ is a non-associative bimonoid. The morphism $l_{A\otimes H} = (l_A \otimes l_H) \circ (A \otimes c_{H,A} \otimes H)$ is a left division for $A \otimes H$ because

$$l_{A\otimes H} \circ (A \otimes H \otimes \mu_{A\otimes H}) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$
$$= ((l_A \circ (A \otimes \mu_A) \circ (\delta_A \otimes A)) \otimes (l_H \circ (H \otimes \mu_H) \circ (\delta_H \otimes H)))$$
$$\circ\, (A \otimes c_{H,A} \otimes H) \text{ (by naturality of } c \text{ and } c^2 = id)$$
$$= (\varepsilon_A \otimes A \otimes \varepsilon_H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \text{ (by (5) for } A \text{ and } H)$$
$$= \varepsilon_{A\otimes H} \otimes A \otimes H \text{ (by naturality of } c),$$

and

$$\mu_{A\otimes H} \circ (A \otimes H \otimes l_{A\otimes H}) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

$$= ((\mu_A \circ (A \otimes l_A) \circ (\delta_A \otimes A)) \otimes (\mu_H \circ (H \otimes l_H) \circ (\delta_H \otimes H)))$$
$$\circ (A \otimes c_{H,A} \otimes H) \text{ (by naturality of } c \text{ and } c^2 = id)$$
$$= (\varepsilon_A \otimes A \otimes \varepsilon_H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \text{ (by (5) for } A \text{ and } H)$$
$$= \varepsilon_{A \otimes H} \otimes A \otimes H \text{ (by naturality of } c).$$

If A, H are left Hopf quasigroups with left antipodes λ_A, λ_H, by Proposition 2.11, we have that

$$\lambda_{A\otimes H} = l_{A\otimes H} \circ (A \otimes H \otimes \eta_{A\otimes H}) = \lambda_A \otimes \lambda_H.$$

Then, $A \otimes H$ is a left Hopf quasigroup because

$$\mu_{A\otimes H} \circ ((\lambda_A \otimes \lambda_H) \otimes \mu_{A\otimes H}) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$
$$= ((\mu_A \circ (\lambda_A \otimes \mu_A) \circ (\delta_A \otimes A)) \otimes (\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H)))$$
$$\circ (A \otimes c_{H,A} \otimes H) \text{ (by naturality of } c \text{ and } c^2 = id)$$
$$= (\varepsilon_A \otimes A \otimes \varepsilon_H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \text{ (by (38) for } A \text{ and } H)$$
$$= \varepsilon_{A\otimes H} \otimes A \otimes H \text{ (by naturality of } c).$$

Let $\tau : A \otimes H \to K$ be a skew pairing. Then, $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a two-cocycle. Indeed, on the one hand we have

$$\partial^1(\omega) * \partial^3(\omega)$$
$$= \varepsilon_A \otimes (\tau \circ c_{H,A} \circ (H \otimes (\mu_A \circ (A \otimes (\tau \circ c_{H,A}) \otimes A)$$
$$\circ (A \otimes H \otimes \delta_A)) \otimes \varepsilon_H \text{ (by naturality of } c, \text{ counit properties and (2))}$$
$$= \varepsilon_A \otimes (\tau \circ (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (c_{H,A} \otimes ((\tau \circ c_{H,A})$$
$$\otimes A) \circ (H \otimes \delta_A))) \otimes \varepsilon_H \text{ (by naturality of } c))$$
$$= \varepsilon_A \otimes ((\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H) \circ (A \otimes c_{H,A})$$
$$\circ (c_{H,A} \otimes ((\tau \circ c_{H,A}) \otimes A) \circ (A \otimes \delta_A))) \otimes \varepsilon_H$$
$$\text{(by (a1) of Definition 4.1)}$$
$$= \varepsilon_A \otimes ((\tau \otimes \tau) \circ (A \otimes (c_{A,H} \circ c_{H,A}) \otimes H) \circ (A \otimes H \otimes c_{H,A})$$
$$\circ (A \otimes \delta_H \otimes A) \circ (c_{H,A} \otimes ((\tau \circ c_{H,A}) \otimes A)$$
$$\circ (A \otimes \delta_A))) \otimes \varepsilon_H \text{ (by naturality of } c)$$
$$= \varepsilon_A \otimes ((\tau \otimes \tau) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A) \circ (c_{H,A}$$
$$\otimes ((\tau \circ c_{H,A}) \otimes A) \circ (H \otimes \delta_A))) \otimes \varepsilon_H \text{ (by naturality of } c \text{ and } c^2 = id),$$

and, on the other hand,

$$\partial^4(\omega) * \partial^2(\omega)$$
$$= \varepsilon_A \otimes (((\tau \circ c_{H,A}) \otimes (\tau \circ c_{H,A} \circ (\mu_H \otimes A))) \circ (H \otimes c_{H,A} \otimes H \otimes A)$$
$$\circ (\delta_H \otimes A \otimes H \otimes A)) \otimes \varepsilon_H \text{ (by naturality of } c\text{, counit properties, and (2))}$$

$$= \varepsilon_A \otimes ((\tau \otimes (\tau \circ (A \otimes \mu_H) \circ (c_{H,A} \otimes H))) \circ (A \otimes \delta_H \otimes A \otimes H)$$
$$\circ\, (c_{H,A} \otimes c_{H,A})) \otimes \varepsilon_H \text{ (by naturality of } c)$$
$$= \varepsilon_A \otimes ((\tau \otimes ((\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A)$$
$$\otimes H \otimes H) \circ (c_{H,A} \otimes H))) \circ (A \otimes \delta_H \otimes A \otimes H)$$
$$\circ\, (c_{H,A} \otimes c_{H,A})) \otimes \varepsilon_H \text{ (by (a2}') \text{ of Definition 4.1)}$$
$$= \varepsilon_A \otimes ((\tau \otimes \tau) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A) \circ (c_{H,A} \otimes ((\tau \circ c_{H,A})$$
$$\otimes A) \circ (H \otimes \delta_A))) \otimes \varepsilon_H \text{ (by naturality of } c \text{ and } c^2 = id).$$

Finally, ω is convolution invertible because

$$\omega * \omega^{-1}$$
$$= \varepsilon_A \otimes ((\tau * \tau^{-1}) \circ c_{H,A}) \otimes \varepsilon_H \text{ (by naturality of } c \text{ and counit properties)}$$
$$= \varepsilon_{A\otimes H} \otimes \varepsilon_{A\otimes H} \text{ (by invertibility of } \tau),$$

and similarly,

$$\omega^{-1} * \omega$$
$$= \varepsilon_A \otimes ((\tau^{-1} * \tau) \circ c_{H,A}) \otimes \varepsilon_H \text{ (by naturality of } c \text{ and counit properties)}$$
$$= \varepsilon_{A\otimes H} \otimes \varepsilon_{A\otimes H} \text{ (by invertibility of } \tau).$$

□

As a consequence of Proposition 4.7 and its right division version, we have the following corollary.

Corollary 4.8 *Let A, H be Hopf quasigroups with antipodes λ_A, λ_H, respectively. Then, $A \otimes H = (A \otimes H, \eta_{A\otimes H}, \mu_{A\otimes H}, \varepsilon_{A\otimes H}, \delta_{A\otimes H})$ is a Hopf quasigroup with antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$.*

Moreover, let $\tau : A \otimes H \to K$ be a skew pairing. The morphism $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a two-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$, where τ^{-1} is defined as in Proposition 4.3.

Also, we get the following result which is a generalization of the one given in [14], Proposition 2.2. (There is a slightly difference because the definition of Hopf pairing in [14] corresponds with our notion of pairing between A and H^{cop}.)

Corollary 4.9 *Let A, H be left Hopf quasigroups with left antipodes λ_A, λ_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then, $A \bowtie_\tau H = (A \otimes H, \eta_{A\bowtie_\tau H}, \mu_{A\bowtie_\tau H}, \varepsilon_{A\bowtie_\tau H}, \delta_{A\bowtie_\tau H}, \lambda_{A\bowtie_\tau H})$ has a structure of left Hopf quasigroup, where*

$$\eta_{A\bowtie_\tau H} = \eta_{A\otimes H}, \quad \varepsilon_{A\bowtie_\tau H} = \varepsilon_{A\otimes H}, \quad \delta_{A\bowtie_\tau H} = \delta_{A\otimes H},$$
$$\mu_{A\bowtie_\tau H} = (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H) \circ (A \otimes \delta_{A\otimes H}$$
$$\otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A\otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H) \tag{86}$$

and

$$\lambda_{A\bowtie_\tau H} = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H}. \tag{87}$$

Proof The result follows by application of Theorem 3.10 to the left Hopf quasigroup $A \otimes H$ and the two-cocycle $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$. Using the naturality of c, the counit properties, the coassociativity of the coproducts and (18), it is easy to check that

$$\mu_{A\bowtie_\tau H} = \mu_{(A\otimes H)^\omega}, \quad \lambda_{A\bowtie_\tau H} = \lambda_{(A\otimes H)^\omega}.$$

□

For right Hopf quasigroups, we have a similar corollary and, as a consequence, we obtain the following result:

Corollary 4.10 *Let A, H be Hopf quasigroups with antipodes λ_A, λ_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then $A \bowtie_\tau H = (A \otimes H, \eta_{A\bowtie_\tau H}, \mu_{A\bowtie_\tau H}, \varepsilon_{A\bowtie_\tau H}, \delta_{A\bowtie_\tau H}, \lambda_{A\bowtie_\tau H})$ has a structure of Hopf quasigroup, where*

$$\begin{aligned} \eta_{A\bowtie_\tau H} &= \eta_{A\otimes H}, \quad \varepsilon_{A\bowtie_\tau H} = \varepsilon_{A\otimes H}, \quad \delta_{A\bowtie_\tau H} = \delta_{A\otimes H}, \\ \mu_{A\bowtie_\tau H} &= (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H) \circ (A \otimes \delta_{A\otimes H} \\ &\quad \otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A\otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H) \end{aligned} \tag{88}$$

and

$$\lambda_{A\bowtie_\tau H} = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H}. \tag{89}$$

Remark 4.11 When particularizing to the Hopf algebra setting, it is a well-known fact that the Drinfeld double of a Hopf algebra H (roughly speaking, a product involving H and the opposite comonoid of its dual Hopf algebra H^*) is an example of a deformation of a Hopf algebra by the two-cocycle associated with a skew pairing. We want to point out that in our context we cannot describe the Drinfeld double in this way because the dual of a Hopf quasigroup H is not a Hopf quasigroup but a Hopf coquasigroup.

Example 4.12 Let $\mathbb{F}$ be a field such that Char($\mathbb{F}$)$\neq 2$ and denote the tensor product over $\mathbb{F}$ as $\otimes$. Consider the non-abelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ where σ_0 is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let u be an additional element such that $u^2 = 1$. By Theorem 1 of [10], the set

$$L = M(S_3, 2) = \{\sigma_i u^\alpha \ ; \ \alpha = 0, 1\}$$

is a Moufang loop where the product is defined by

$$\begin{aligned} \sigma_i u^\alpha . \, \sigma_j u^\beta &= (\sigma_i^\nu \sigma_j^\mu)^\nu u^{\alpha+\beta}, \\ &\qquad \nu = (-1)^\beta, \ \mu = (-1)^{\alpha+\beta}. \end{aligned}$$

Then, L is an IP loop and by Example 2.13, $A = \mathbb{F}L$ is a cocommutative Hopf quasigroup.

On the other hand, let H_4 be the four-dimensional Taft Hopf algebra. This Hopf algebra is the smallest non-commutative, non-cocommutative Hopf algebra. The basis of H_4 is $\{1, x, y, w = xy\}$, and the multiplication table is defined by

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The costructure of H_4 is given by

$$\delta_{H_4}(x) = x \otimes x,\ \delta_{H_4}(y) = y \otimes x + 1 \otimes y,\ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$
$$\varepsilon_{H_4}(x) = 1_{\mathbb{F}},\ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0,$$

and the antipode λ_{H_4} is described by

$$\lambda_{H_4}(x) = x,\ \lambda_{H_4}(y) = w,\ \lambda_{H_4}(w) = -y.$$

By Proposition 4.7, $A \otimes H_4$ is a non-commutative, non-cocommutative Hopf quasigroup and the morphism $\tau : A \otimes H_4 \to \mathbb{F}$ defined by

$$\tau(\sigma_i u^\alpha \otimes z) = \begin{cases} 1 & \text{if} \quad z = 1 \\ (-1)^\alpha & \text{if} \quad z = x \\ 0 & \text{if } z = y, w \end{cases}$$

is a skew pairing such that $\tau = \tau^{-1}$. Then, by Proposition 4.7,

$$\omega = \varepsilon_A \otimes (\tau \circ c_{H_4,A}) \otimes \varepsilon_{H_4}$$

is a two-cocycle with convolution inverse $\omega^{-1} = \omega$. Finally, $A \bowtie_\tau H_4$ is a Hopf quasigroup isomorphic to $(A \otimes H_4)^\omega$.

5 Double Cross Products and Skew Pairings

In this section, we will show that the construction of $A \bowtie_\tau H$ introduced in the previous section is also a special case of the double cross product defined in [25]. First of all, we need to recall some definitions, following [8,9] and [20], to state a characterization of double cross products in the quasigroup setting.

Definition 5.1 Let H be a left Hopf quasigroup. We say that (M, φ_M) is a left H-quasimodule if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ (called the action) satisfying

$$\varphi_M \circ (\eta_H \otimes M) = id_M, \tag{90}$$

$$\begin{aligned}&\varphi_M \circ (H \otimes \varphi_M) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes M)\\ &\quad = \varepsilon_H \otimes M = \varphi_M \circ (\lambda_H \otimes \varphi_M) \circ (\delta_H \otimes M).\end{aligned} \tag{91}$$

Given two left H-quasimodules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-quasimodules if

$$\varphi_N \circ (H \otimes f) = f \circ \varphi_M. \tag{92}$$

We denote the category of left H-quasimodules by ${}_H\mathcal{QC}$.

If (M, φ_M) and (N, φ_N) are left H-quasimodules, the tensor product $M \otimes N$ is a left H-quasimodule with the diagonal action

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

This makes the category of left H-quasimodules into a strict monoidal category $({}_H\mathcal{QC}, \otimes, K)$ (see Remark 3.3 of [9]).

We will say that a unital magma A is a left H-quasimodule magma if it is a left H-quasimodule with action $\varphi_A : H \otimes A \to A$ and the following equalities

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \tag{93}$$

$$\mu_A \circ \varphi_{A\otimes A} = \varphi_A \circ (H \otimes \mu_A), \tag{94}$$

hold, i.e., φ_A is a morphism of unital magmas.

A comonoid A is a left H-quasimodule comonoid if it is a left H-quasimodule with action φ_A and

$$\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A, \tag{95}$$

$$\delta_A \circ \varphi_A = \varphi_{A\otimes A} \circ (\delta_H \otimes \delta_A), \tag{96}$$

hold, i.e., φ_A is a comonoid morphism.

Replacing (91) by the equality

$$\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M), \tag{97}$$

we have the definition of left H-module and the ones of left H-module magma and comonoid [because (91) follows trivially from (97)]. Note that the pair (H, μ_H) is not an H-module but it is an H-quasimodule. Morphisms between left H-modules are defined as for H-quasimodules and we denote the category of left H-modules by ${}_H\mathcal{C}$. Obviously, we have similar definitions for the right side.

Proposition 5.2 *Let A, H be (right) left Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Define $\varphi_A : H \otimes A \to A$ and $\phi_H : H \otimes A \to H$ as*

$$\varphi_A = (\tau \otimes A \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_A \otimes H) \circ \delta_{A\otimes H} \circ c_{H,A}$$

and

$$\phi_H = (\tau \otimes H \otimes \tau^{-1}) \circ (A \otimes H \otimes c_{A,H} \otimes H) \\ \circ (A \otimes H \otimes A \otimes \delta_H) \circ \delta_{A\otimes H} \circ c_{H,A}.$$

Then,

(i) The pair (A, φ_A) is a left H-module comonoid.
(ii) If the (right) left antipode of H is an isomorphism, the pair (H, ϕ_H) is a right A-module comonoid.
(iii) If H is a Hopf quasigroup, the pair (H, ϕ_H) is a right A-module comonoid.

Proof We prove the result for left Hopf quasigroups. The proof for right Hopf quasigroups is similar and is left to the reader. Trivially, by (3) for H, (a3) of Definition 4.1, (80), and the counit properties we obtain that $\varphi_A \circ (\eta_H \otimes A) = id_A$. The equality $\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A$ follows by the counit properties, the invertibility of τ and the naturality of c. Moreover,

$$\begin{aligned} &\varphi_A \circ (\mu_H \otimes A) \\ &= ((\tau \circ (A \otimes \mu_H)) \otimes A \otimes (\tau^{-1} \circ (A \otimes \mu_H))) \circ (A \otimes H \otimes H \\ &\quad \otimes \delta_A \otimes H \otimes H) \circ \delta_{A\otimes H\otimes H} \circ (c_{H,A} \otimes H) \\ &\quad \circ (H \otimes c_{H,A}) \text{ (by (a2}') \text{ of Definition 4.1 and (81))} \\ &= (((\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)) \otimes A \\ &\quad \otimes ((\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H))) \\ &\quad \circ (A \otimes H \otimes H \otimes \delta_A \otimes H \otimes H) \circ \delta_{A\otimes H\otimes H} \circ (c_{H,A} \otimes H) \\ &\quad \circ (H \otimes c_{H,A}) \text{ (by naturality of } c \text{ and coassociativity)} \\ &= \varphi_A \circ (H \otimes \varphi_A) \text{ (by naturality of } c \text{ and (4))}. \end{aligned}$$

Finally,

$$\begin{aligned} &(\varphi_A \otimes \varphi_A) \circ \delta_{H\otimes A} \\ &= (A \otimes (\tau^{-1} * \tau) \otimes A) \circ (A \otimes A \otimes c_{A,H}) \circ (A \otimes \delta_A \otimes H) \\ &\quad \circ (\delta_A \otimes H \otimes \tau^{-1}) \circ (\tau \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H} \circ c_{H,A} \\ &\quad \text{(by naturality of } c \text{ and coassociativity)} \\ &= \delta_A \circ \varphi_A \text{ (by naturality of } c\text{, invertibility of } \tau\text{, and counit properties)}. \end{aligned}$$

The proof for (ii) follows a similar pattern but using (a1) of Definition 4.1 and (85) instead of (a2′) and (81). By Proposition 4.6, we obtain (iii) because in the quasigroup setting condition (85) is true without the assumption of λ_H be an isomorphism. □

The following result is a version of [25], Theorem 7.2.2 for left Hopf quasigroups (see also [19], Theorem IX.2.3).

Theorem 5.3 *Let A, H be left Hopf quasigroups with left antipodes λ_A, λ_H, respectively. Assume that (A, φ_A) is a left H-module comonoid and (H, ϕ_H) is a right A-module comonoid. Then, the following assertions are equivalent:*

(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A\bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \varphi_A \otimes \phi_H \otimes H) \circ (A \otimes \delta_{H\otimes A} \otimes H)$$

and tensor product unit, counit and coproduct, is a left Hopf quasigroup with left antipode

$$\lambda_{A\bowtie H} = (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

(ii) The equalities

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \tag{98}$$

$$\phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A, \tag{99}$$

$$(\phi_H \otimes \varphi_A) \circ \delta_{H\otimes A} = c_{A,H} \circ (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}, \tag{100}$$

$$\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ ((\lambda_{A\bowtie H} \circ c_{H,A}) \otimes A), \tag{101}$$

$$\mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes H) \circ (\delta_H \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, \tag{102}$$

$$\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, \tag{103}$$

hold.

Proof $(i) \Rightarrow (ii)$ First of all, we have

$$id_{A\otimes H} = ((\mu_A \circ (A \otimes (\varphi_A \circ (H \otimes \eta_A)))) \otimes H) \circ (A \otimes \delta_H) \tag{104}$$

because

$$\begin{aligned} &id_{A\otimes H} \\ &= \mu_{A\bowtie H} \circ (A \otimes H \otimes \eta_{A\otimes H}) \text{ (by unit properties)} \\ &= ((\mu_A \circ (A \otimes (\varphi_A \circ (H \otimes \eta_A)))) \otimes H) \circ (A \otimes \delta_H) \\ &\quad \text{(by (3) for } A\text{, (90) for } \phi_H\text{, and the properties of } \eta_A). \end{aligned}$$

Therefore, composing with $A \otimes \varepsilon_H$ on the left side and with $\eta_A \otimes H$ on the right side of equality (104), we get (98). In a similar way, the identity $\mu_{A\bowtie H} \circ (\eta_{A\otimes H} \otimes A \otimes H) = id_{A\otimes H}$ leads to (99). As far as (100), it can be obtained by composing with

$\eta_A \otimes H \otimes A \otimes \eta_H$ on the right and with $\varepsilon_A \otimes H \otimes A \otimes \varepsilon_H$ on the left in the two terms of the equality

$$\delta_{A\otimes H} \circ \mu_{A\bowtie H} = (\mu_{A\bowtie H} \otimes \mu_{A\bowtie H}) \circ (A \otimes H \otimes c_{A\otimes H,A\otimes H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes \delta_{A\otimes H}).$$

Indeed:

$$\begin{aligned}
&(\phi_H \otimes \varphi_A) \circ \delta_{H\otimes A} \\
&= ((((\varepsilon_A \circ \varphi_A) \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes ((\varphi_A \otimes (\varepsilon_H \circ \phi_H)) \\
&\quad \circ \delta_{H\otimes A})) \circ \delta_{A\otimes H} \;(\text{by } (95) \text{ for } \varphi_A \text{ and } \phi_H, \text{ naturality of } c, \text{ and counit properties}) \\
&= (\varepsilon_A \otimes H \otimes A \otimes \varepsilon_H) \circ (\mu_{A\bowtie H} \otimes \mu_{A\bowtie H}) \circ (A \otimes H \otimes c_{A\otimes H,A\otimes H} \\
&\quad \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes \delta_{A\otimes H}) \\
&\quad \circ (\eta_A \otimes H \otimes A \otimes \eta_H) \;(\text{by } (3) \text{ for } A \text{ and } H, \text{ naturality of } c, \text{ and the properties of } \eta_A \text{ and } \eta_H) \\
&= (\varepsilon_A \otimes H \otimes A \otimes \varepsilon_H) \circ \delta_{A\otimes H} \circ \mu_{A\bowtie H} \circ (\eta_A \otimes H \otimes A \otimes \eta_H) \;(\text{by } (4) \text{ for } A \bowtie H) \\
&= c_{A,H} \circ (\varphi_A \circ \phi_H) \circ \delta_{H\otimes A} \;(\text{by naturality of } c, \text{ and the properties of } \eta_A, \eta_H, \varepsilon_A \text{ and } \varepsilon_H).
\end{aligned}$$

On the other hand, if $A \bowtie H$ is a left Hopf quasigroup with left antipode $\lambda_{A\bowtie H}$, (27) holds. Then, we have

$$\mu_{A\bowtie H} \circ (\lambda_{A\bowtie H} \otimes \mu_{A\bowtie H}) \circ (\delta_{A\otimes H} \otimes A \otimes H) = \varepsilon_{A\otimes H} \otimes A \otimes H. \tag{105}$$

Composing with $A \otimes \varepsilon_H$ on the left and with $A \otimes H \otimes A \otimes \eta_H$ on the right in the two terms of equality (105), we get

$$\begin{aligned}
&\varepsilon_A \otimes \varepsilon_H \otimes A \\
&\quad = \mu_A \circ (A \otimes \varphi_A) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A)) \otimes \\
&\quad\quad \circ (H \otimes ((A \otimes \mu_A) \circ (\delta_A \otimes A))) \circ (c_{A,H} \otimes A) \\
&\quad\quad \circ (A \otimes ((H \otimes \varphi_A) \circ (\delta_H \otimes A))).
\end{aligned} \tag{106}$$

Indeed:

$$\begin{aligned}
&\varepsilon_A \otimes \varepsilon_H \otimes A \\
&= \varepsilon_A \otimes \varepsilon_H \otimes A \otimes (\varepsilon_H \circ \eta_H) \;(\text{by } (1) \text{ for } H) \\
&= (A \otimes \varepsilon_H) \circ \mu_{A\bowtie H} \circ (\lambda_{A\bowtie H} \otimes \mu_{A\bowtie H}) \circ (\delta_{A\otimes H} \otimes A \otimes \eta_H) \;(\text{by } (105) \text{ for } A \bowtie H) \\
&= \mu_A \circ (A \otimes \varphi_A) \circ (\lambda_{A\bowtie H} \otimes \mu_A) \circ (A \otimes c_{A,H} \otimes A) \circ (\delta_A \otimes ((H \otimes \varphi_A) \\
&\quad \circ (\delta_H \otimes A))) \;(\text{by } (2) \text{ for } H, (95) \text{ for } \phi_H, \text{ the naturality of } c \text{ and counit properties}) \\
&= \mu_A \circ (A \otimes \varphi_A) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A)) \otimes A) \\
&\quad \circ (H \otimes ((A \otimes \mu_A) \circ (\delta_A \otimes A))) \circ (c_{A,H} \otimes A) \\
&\quad \circ (A \otimes ((H \otimes \varphi_A) \circ (\delta_H \otimes A))) \;(\text{by the naturality of } c).
\end{aligned}$$

Having into account that $(H \otimes \varphi_A) \circ (\delta_H \otimes A)$ and $(A \otimes \mu_A) \circ (\delta_A \otimes A)$ are isomorphisms with inverses $(H \otimes \varphi_A) \circ (H \otimes \lambda_H \otimes A) \circ (\delta_H \otimes A)$ and $(A \otimes \mu_A) \circ (A \otimes \lambda_A \otimes A) \circ (\delta_A \otimes A)$, respectively, we have

$$\begin{aligned}
&\mu_A \circ (A \otimes \varphi_A) \circ ((\lambda_{A\bowtie H} \circ c_{H,A}) \otimes A)\\
&\mu_A \circ (A \otimes \varphi_A) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A)) \otimes A) \;(\text{by } c^2 = id)\\
&\quad = \mu_A \circ (A \otimes \varphi_A) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A)) \otimes A)\\
&\qquad \circ (H \otimes ((A \otimes \mu_A) \circ (\delta_A \otimes A))) \circ (c_{A,H} \otimes A)\\
&\qquad \circ (A \otimes ((H \otimes \varphi_A) \circ (\delta_H \otimes A))) \circ (A \otimes ((H \otimes \varphi_A) \circ (H \otimes \lambda_H \otimes A)\\
&\qquad \circ (\delta_H \otimes A))) \circ (c_{H,A} \otimes A) \circ (H \otimes ((A \otimes \mu_A)\\
&\qquad \circ (A \otimes \lambda_A \otimes A) \circ (\delta_A \otimes A)) \;(\text{by composition with the inverses})\\
&\quad = (\varepsilon_A \otimes \varepsilon_H \otimes A) \circ (A \otimes (((H \otimes \varphi_A) \circ (H \otimes \lambda_H \otimes A)\\
&\qquad \circ (\delta_H \otimes A)))) \circ (c_{H,A} \otimes A) \circ (H \otimes ((A \otimes \mu_A)\\
&\qquad \circ (A \otimes \lambda_A \otimes A) \circ (\delta_A \otimes A))) \;(\text{by } (106))\\
&\quad = \varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) \;(\text{by naturality of } c \text{ and counit properties}).
\end{aligned}$$

Therefore, (101) holds. Now, we show (102): Composing with $\varepsilon_A \otimes H$ on the left and with $\eta_A \otimes H \otimes A \otimes H$ on the right in the two terms of equality (105), we get

$$\begin{aligned}
&\varepsilon_H \otimes \varepsilon_A \otimes H\\
&\quad = \mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes H) \circ (\delta_H \otimes A \otimes H)\\
&\qquad (\text{by } (1), (2), (3), (19) \text{ for } A, (93)\ (95) \text{ for } \varphi_A, (90) \text{ for } \phi_H, \text{naturality of } c \text{ and counit properties}).
\end{aligned}$$

Finally, by (27) for $A \bowtie H$, we have

$$\begin{aligned}
&\mu_{A\bowtie H} \circ (A \otimes H \otimes \mu_{A\bowtie H}) \circ (A \otimes H \otimes \lambda_{A\bowtie H} \otimes A \otimes H)\\
&\quad \circ (\delta_{A\otimes H} \otimes A \otimes H) = \varepsilon_{A\otimes H} \otimes A \otimes H.
\end{aligned} \tag{107}$$

Then, composing with $\varepsilon_A \otimes H$ on the left and with $\eta_A \otimes H \otimes A \otimes H$ on the right in the two terms of equality (107),

$$\begin{aligned}
&\varepsilon_H \otimes \varepsilon_A \otimes H\\
&\quad = \mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \varphi_A \circ \phi_H \otimes H) \circ (H \otimes \delta_{H\otimes A} \otimes H)\\
&\qquad \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H)\\
&\qquad (\text{by } (1), (2), (3), (19) \text{ for } A, (93), (95), (98) \text{ for } \varphi_A, (90) \text{ for } \phi_H, \text{naturality of } c \text{ and counit properties}).
\end{aligned}$$

Therefore, (103) holds.

$(ii) \Rightarrow (i)$ We only prove the equalities involving the left antipode. The proof for the other conditions is analogous to the ones given in [25], Theorem 7.2.2.

$$\begin{aligned}
&\mu_{A\bowtie H} \circ (\lambda_{A\bowtie H} \otimes \mu_{A\bowtie H}) \circ (\delta_{A\otimes H} \otimes A \otimes H)\\
&\quad = ((\mu_A \circ (A \otimes \varphi_A)) \otimes \mu_H) \circ (((((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes A
\end{aligned}$$

$$\otimes(\phi_H \circ (\phi_H \otimes A))) \circ \delta_{H\otimes A\otimes A}) \otimes H)$$
$$\circ (((\lambda_H \otimes \lambda_A) \circ c_{A,H}) \otimes \mu_{A\bowtie H}) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$
(by the comonoid morphism condition for ϕ_H, coassociativity, and naturality of c)
$$= ((\mu_A \circ (A \otimes \varphi_A) \circ ((\lambda_{A\bowtie H} \circ c_{H,A}) \otimes A)) \otimes (\mu_H \circ (\phi_H$$
$$\circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A)) \otimes H))$$
$$\circ (H \otimes ((A \otimes c_{H,A} \otimes A) \circ (c_{H,A} \otimes c_{A,A})) \otimes A \otimes H) \circ ((c_{H,H} \circ \delta_H)$$
$$\otimes(c_{A,A} \circ \delta_A) \otimes \delta_A \otimes H) \circ (c_{A,H} \otimes \mu_{A\bowtie H})$$
$$\circ (\delta_{A\otimes H} \otimes A \otimes H)$$
(by (24) for λ_H and λ_A, coassociativity, naturality of c, and condition of A-module for H)
$$= ((\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A)) \otimes (\mu_H \circ (\phi_H$$
$$\circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A)) \otimes H))$$
$$\circ (H \otimes ((A \otimes c_{H,A} \otimes A) \circ (c_{H,A} \otimes c_{A,A})) \otimes A \otimes H) \circ ((c_{H,H} \circ \delta_H)$$
$$\otimes(c_{A,A} \circ \delta_A) \otimes \delta_A \otimes H) \circ (c_{A,H} \otimes \mu_{A\bowtie H})$$
$$\circ (\delta_{A\otimes H} \otimes A \otimes H) \text{ (by (101)))}$$
$$= (A \otimes \mu_H) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes H) \circ (\lambda_H \otimes (\mu_A$$
$$\circ(\lambda_A \otimes \mu_A) \circ (\delta_A \otimes A)) \otimes \mu_H)$$
$$\circ (c_{A,H} \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H)$$
(by (24) for λ_H and λ_A, (4) for A, and naturality of c)
$$= (A \otimes \mu_H) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes \mu_H) \circ (\lambda_H \otimes ((\varphi_A \otimes \phi_H)$$
$$\circ\delta_{H\otimes A}) \otimes H) \circ (\varepsilon_A \otimes \delta_H \otimes A \otimes H)$$
(by (27) for A and naturality of c)
$$= (A \otimes \mu_H) \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes (\phi_H \circ (H \otimes \varphi_A)) \otimes \mu_H)$$
$$\circ (\delta_{H\otimes H\otimes A} \otimes \phi_H \otimes H) \circ (\lambda_H \otimes \delta_{H\otimes A} \otimes H)$$
$$\circ (\varepsilon_A \otimes \delta_H \otimes A \otimes H) \text{ (by the condition of comonoid morphism for } \varphi_A \text{ and naturality of } c)$$
$$= (A \otimes \mu_H) \circ (A \otimes \phi_H \otimes \mu_H) \circ (c_{H,A} \otimes \varphi_A \otimes \phi_H \otimes H)$$
$$\circ ((\lambda_H \otimes (\varphi_A \circ ((\lambda_H * id_H) \otimes A))$$
$$\circ (\delta_H \otimes A)) \otimes H \otimes A \otimes H \otimes A \otimes H) \circ (\delta_{H\otimes A} \otimes$$
$$\otimes A \otimes H) \circ (\varepsilon_A \otimes \delta_{H\otimes A} \otimes H)$$
(by (24) for λ_H, the condition of H-module for A, coassociativity of δ_H, and naturality of c)
$$= (A \otimes (\mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes \varphi_A \otimes \phi_H \otimes H) \circ (H \otimes \delta_{H\otimes A}$$
$$\otimes H) \circ (\delta_H \otimes A \otimes H))) \circ (c_{H,A} \otimes A \otimes H)$$
$$\circ (\varepsilon_A \otimes H \otimes \delta_A \otimes H) \text{ (by (28), the condition of } H\text{-module for } A \text{ and counit properties)}$$
$$= (A \otimes \varepsilon_H \otimes \varepsilon_A \otimes H) \circ (c_{H,A} \otimes A \otimes H) \circ (\varepsilon_A \otimes H \otimes \delta_A \otimes H) \text{ (by (102))}$$
$$= \varepsilon_{A\otimes H} \otimes A \otimes H \text{ (by counit properties and naturality of } c).$$

On the other hand,

$$\mu_{A\bowtie H} \circ (A \otimes H \otimes \mu_{A\bowtie H}) \circ (A \otimes H \otimes \lambda_{A\bowtie H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

$$= \mu_{A\bowtie H} \circ (A \otimes H \otimes (\mu_A \circ (A \otimes \varphi_A) \circ (((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes A))$$
$$\otimes(\mu_H \circ (\phi_H \otimes H) \circ (H \otimes \mu_A \otimes H))) \circ (A \otimes H \otimes \delta_{H\otimes A\otimes A} \otimes H)$$
$$\circ (A \otimes H \otimes ((\lambda_H \otimes \lambda_A) \circ c_{A,H}) \otimes A \otimes H)$$
$$\circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by the condition of comonoid morphism for ϕ_H, coassociativity, the condition of A-module for H, and naturality of c)

$$= \mu_{A\bowtie H} \circ (A \otimes H \otimes (\mu_A \circ (A \otimes \varphi_A) \circ ((\lambda_{A\bowtie H} \circ c_{H,A}) \otimes A))$$
$$\otimes(\mu_H \circ (\phi_H \otimes H) \circ (H \otimes \mu_A \otimes H)$$
$$\circ (\lambda_H \otimes \lambda_A \otimes A \otimes H))) \circ (A \otimes H \otimes H \otimes ((A \otimes c_{H,A}$$
$$\otimes A) \circ (c_{H,A} \otimes c_{A,A})) \otimes A \otimes H)$$
$$\circ (A \otimes H \otimes (c_{H,H} \circ \delta_H) \otimes (c_{A,A} \circ \delta_A) \otimes \delta_A \otimes H)$$
$$\circ (A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by (24) for the antipodes λ_A, λ_H, and naturality of c)

$$= \mu_{A\bowtie H} \circ (A \otimes H \otimes ((\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A))$$
$$\otimes((\mu_H \circ (\phi_H \otimes H) \circ (H \otimes \mu_A \otimes H)$$
$$\circ (\lambda_H \otimes \lambda_A \otimes A \otimes H)))) \circ (A \otimes H \otimes H \otimes ((A \otimes c_{H,A}$$
$$\otimes A) \circ (c_{H,A} \otimes c_{A,A})) \otimes A \otimes H)$$
$$\circ (A \otimes H \otimes (c_{H,H} \circ \delta_H) \otimes (c_{A,A} \circ \delta_A) \otimes \delta_A \otimes H)$$
$$\circ (A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by (101))

$$= \mu_{A\bowtie H} \circ (A \otimes H \otimes A \otimes \mu_H) \circ (A \otimes H \otimes ((\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A)$$
$$\circ ((\delta_H \circ \lambda_H) \otimes ((\mu_A \otimes \mu_A) \circ \delta_{A\otimes A}$$
$$\circ (\lambda_A \otimes A)))) \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes A \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by (24) for λ_A and λ_H, and naturality of c)

$$= (\mu_A \otimes \mu_H) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (H \otimes \varphi_A)) \otimes (\mu_H$$
$$\circ (\phi_H \otimes H))) \circ (A \otimes H \otimes (\delta_{H\otimes A} \circ (\lambda_H \otimes (\mu_A$$
$$\circ (\lambda_A \otimes A))) \circ (c_{A,H} \otimes A)) \otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by the definition of $\mu_{A\bowtie H}$, and (4) for H)

$$= (\mu_A \otimes \mu_H) \circ (A \otimes (\varphi_A \circ (H \otimes \varphi_A)) \otimes (\phi_H \circ (H \otimes \varphi_A))$$
$$\otimes\mu_H) \circ (A \otimes \delta_{H\otimes H\otimes A} \otimes \phi_H \otimes H)$$
$$\circ (A \otimes H \otimes \delta_{H\otimes A} \otimes H) \circ (A \otimes H \otimes H \otimes \mu_A \otimes H)$$
$$\circ (A \otimes H \otimes ((\lambda_H \otimes \lambda_A) \circ c_{A,H}) \otimes A \otimes H)$$
$$\circ (\delta_{A\otimes H} \otimes A \otimes H)$$ (by the condition of comonoid morphism for φ_A)

$$= (\mu_A \otimes ((\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \varphi_A \otimes \phi_H \otimes H) \circ (H \otimes \delta_{H\otimes A}$$
$$\otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H))$$
$$\circ (A \otimes (\varphi_A \circ (\mu_H \otimes A)) \otimes H \otimes A \otimes H) \circ (A \otimes H \otimes \lambda_H \otimes c_{H,A} \otimes A \otimes H)$$
$$\circ (A \otimes H \otimes (c_{H,H} \circ \delta_H) \otimes (\delta_A \circ \mu_A) \otimes H) \circ (A \otimes H$$
$$\otimes H \otimes \lambda_A \otimes A \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes A \otimes H)$$
$$\circ (\delta_{A\otimes H} \otimes A \otimes H)$$

(by (24) for λ_H, the condition of left H-module for A, coassociativity of the coproducts, and naturality of c)

$$= ((\mu_A \circ (A \otimes \varphi_A) \circ (A \otimes \mu_H \otimes A)) \otimes \varepsilon_H \otimes \varepsilon_A \otimes H)$$
$$\circ (A \otimes H \otimes H \otimes c_{H,A} \otimes A \otimes H) \circ (A \otimes H \otimes (c_{H,H}$$
$$\circ (H \otimes \lambda_H) \circ \delta_H) \otimes (\delta_A \circ \mu_A \circ (\lambda_A \otimes A)) \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes A$$
$$\otimes H) \circ (\delta_{A\otimes H} \otimes A \otimes H) \text{ (by (103))}$$
$$= ((\mu_A \circ (A \otimes (\varphi_A \circ ((\mu_H \circ (H \otimes \lambda_H)) \otimes (\mu_A \circ (\lambda_A \otimes A)))$$
$$\circ (H \otimes c_{A,H} \otimes A))) \circ (\delta_{A\otimes H} \otimes A)) \otimes H)$$
$$\text{(by naturality of } c \text{ and counit properties)}$$
$$= ((\mu_A \circ (A \otimes \varphi_A) \circ (A \otimes (id_H * \lambda_H) \otimes \mu_A) \circ (A \otimes H \otimes \lambda_A \otimes A)$$
$$\circ (A \otimes c_{A,H} \otimes A) \circ (\delta_A \otimes H \otimes A)) \otimes H)$$
$$\text{(by naturality of } c)$$
$$= ((\mu_A \circ (A \otimes \mu_A) \circ (A \otimes \lambda_A \otimes A) \circ (\delta_A \otimes A)) \otimes H)$$
$$\circ (A \otimes \varepsilon_H \otimes A \otimes H) \text{ (by (13) and (90) for } \varphi_A)$$
$$= \varepsilon_{A\otimes H} \otimes A \otimes H \text{ (by (27) for } A).$$

□

As in the previous results, we can obtain a similar theorem for right Hopf quasigroups. In this case, the corresponding equalities to (101), (102) and (103) are

$$\phi_H \circ (\mu_H \otimes A) \circ (H \otimes \varrho_H \otimes \varrho_A)$$
$$= \mu_H \circ (\phi_H \otimes H) \circ (H \otimes (\varrho_{A\bowtie H} \circ c_{H,A})), \tag{108}$$
$$\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes \varrho_A)$$
$$\circ (A \otimes H \otimes \delta_A) = A \otimes \varepsilon_H \otimes \varepsilon_A, \tag{109}$$
$$\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}) \otimes A)$$
$$\circ (A \otimes H \otimes ((\varrho_A \otimes A) \circ \delta_A)) = A \otimes \varepsilon_H \otimes \varepsilon_A, \tag{110}$$

where ρ_H and ρ_A denote the right antipodes.

As a consequence of Theorem 5.3 and its right version, we have:

Corollary 5.4 *Let A, H be Hopf quasigroups with antipodes λ_A, λ_H, respectively. Assume that (A, φ_A) is a left H-module comonoid and (H, ϕ_H) a right A-module comonoid. Then, the following assertions are equivalent:*

(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A\bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \varphi_A \otimes \phi_H \otimes H) \circ (A \otimes \delta_{H\otimes A} \otimes H)$$

and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode

$$\lambda_{A\bowtie H} = (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

(ii) The equalities (98), (99), (100), (101), (102), (103), (108), (109), *and* (110) *hold for λ_H and λ_A.*

Now, we show that the construction of $A \bowtie_{\tau} H$ introduced in the previous section is an example of a double cross product. We will prove the left version and leave to the patient reader to get the right one.

Proposition 5.5 *Let A, H be left Hopf quasigroups with left antipodes λ_A, λ_H such that λ_H is an isomorphism, and let $\tau : A \otimes H \to K$ be a skew pairing. Then, the left Hopf quasigroup $A \bowtie_{\tau} H$ introduced in Corollary 4.9 is the double cross product induced by the actions φ_A, ϕ_H defined in Proposition 5.2.*

Proof First, note that

$$(\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} = (\tau \otimes A \otimes H \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H} \circ c_{H,A}. \quad (111)$$

Indeed:

$$\begin{aligned}
&(\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A} \\
&\quad = (A \otimes (\tau^{-1} * \tau) \otimes H) \circ (\tau \otimes \delta_A \otimes \delta_H \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_{A\otimes H}) \\
&\qquad \circ \delta_{A\otimes H} \circ c_{H,A} \; (\text{by naturality of } c, \text{ and coassociativity}) \\
&\quad = (\tau \otimes A \otimes H \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H} \circ c_{H,A} \\
&\qquad (\text{by invertibility of } \tau, \text{ naturality of } c, \text{ and counit properties}).
\end{aligned}$$

As a consequence, it is not difficult to see that $\mu_{A\bowtie_{\tau} H} = \mu_{A\bowtie H}$. On the other hand, $\lambda_{A\bowtie_{\tau} H} = \lambda_{A\bowtie H}$ because

$$\begin{aligned}
&\lambda_{A\bowtie H} \\
&\quad = (\tau \otimes \lambda_A \otimes \lambda_H \otimes \tau^{-1}) \circ (A \otimes H \otimes ((A \otimes c_{A,H} \otimes H) \\
&\qquad \circ ((c_{A,A} \circ \delta_A) \otimes (c_{H,H} \circ \delta_H)))) \circ (A \otimes c_{A,H} \otimes H) \\
&\qquad \circ ((c_{A,A} \circ \delta_A) \otimes (c_{H,H} \circ \delta_H)) \\
&\qquad (\text{by } (24) \text{ for } \lambda_A \text{ and } \lambda_H, \text{ naturality of } c, c^2 = id, (83) \text{ and } (84)) \\
&\quad = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (A \otimes H \otimes c_{A\otimes H, A\otimes H}) \circ (c_{A\otimes H, A\otimes H} \otimes A \\
&\qquad \otimes H) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ c_{A\otimes H, A\otimes H} \circ \delta_{A\otimes H} \\
&\qquad (\text{by naturality of } c, \text{ and } c^2 = id) \\
&\quad = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (\delta_{A\otimes H} \otimes A \otimes H) \circ \delta_{A\otimes H} \\
&\qquad (\text{by naturality of } c, \text{ and } c^2 = id) \\
&\quad = \lambda_{A\bowtie_{\tau} H} \; (\text{by coasociativity}).
\end{aligned}$$

□

Finally, by the previous results, we have the following corollary for quasigroups without conditions over the antipode of H.

Corollary 5.6 *Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Then, the Hopf quasigroup $A \bowtie_{\tau} H$ introduced in Corollary 4.10 is the double cross product induced by the actions φ_A and ϕ_H, defined in Proposition 5.2.*

6 Quasitriangular Hopf Quasigroups, Skew Pairings, Biproducts and Projections

In this section, we will explore the connections between Yetter–Drinfeld modules for Hopf quasigroups, projections of Hopf quasigroups, skew pairings and quasitriangular structures, obtaining the non-associative version of the main results proved in [1].

There is no difference between the notion of left H-comodule for a Hopf algebra and for a Hopf quasigroup since it only depends on the comonoid structure of H. Then, we will denote a left H-comodule by (M, ρ_M) where M is an object in $\mathcal{C}$ and $\rho_M : M \to H \otimes M$ is a morphism in $\mathcal{C}$ (called the coaction) satisfying the comodule conditions:

$$(\varepsilon_H \otimes M) \circ \rho_M = id_M, \tag{112}$$
$$(H \otimes \rho_M) \circ \rho_M = (\delta_H \otimes M) \circ \rho_M. \tag{113}$$

Given two left H-comodules (M, ρ_M) and (N, ρ_N), $f : M \to N$ is a morphism of left H-comodules if $\rho_N \circ f = (H \otimes f) \circ \rho_M$. We denote the category of left H-comodules by ${}^H\mathcal{C}$.

For two left H-comodules (M, ρ_M) and (N, ρ_N), the tensor product $M \otimes N$ is a left H-comodule with the codiagonal coaction

$$\rho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\rho_M \otimes \rho_N).$$

This tensor product endows to the category of left H-comodules with a structure of strict monoidal category $({}^H\mathcal{C}, \otimes, K)$.

Moreover, we will say that a unital magma A is a left H-comodule magma if it is a left H-comodule with coaction ρ_A and the following equalities hold:

$$\rho_A \circ \eta_A = \eta_H \otimes \eta_A, \tag{114}$$
$$\rho_A \circ \mu_A = (H \otimes \mu_A) \circ \rho_{A\otimes A}. \tag{115}$$

Finally, a comonoid A is a left H-comodule comonoid if it is a left H-comodule with coaction ρ_A and

$$(H \otimes \varepsilon_A) \circ \rho_A = \eta_H \otimes \varepsilon_A, \tag{116}$$
$$(H \otimes \delta_A) \circ \rho_A = \rho_{A\otimes A} \circ \delta_A, \tag{117}$$

hold.

Now, following [2], we recall the notion of Yetter–Drinfeld quasimodule for a Hopf quasigroup H.

Definition 6.1 Let H be a Hopf quasigroup. We say that $M = (M, \varphi_M, \rho_M)$ is a left-left Yetter–Drinfeld quasimodule over H if (M, φ_M) is a left H-quasimodule and (M, ρ_M) is a left H-comodule which satisfies the following equalities:

(b1) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M)$.

(b2) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\rho_M \otimes \mu_H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes c_{M,H} \otimes H) \circ (\rho_M \otimes H \otimes H)$.

(b3) $(\mu_H \otimes M) \circ (H \otimes \mu_H \otimes M) \circ (H \otimes H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$.

If M and N are two left–left Yetter–Drinfeld quasimodules over H and $f : M \to N$ is a morphism between them, we will say that f is a morphism of left–left Yetter–Drinfeld quasimodules if it is a morphism of H-quasimodules and H-comodules.

We shall denote by ${}^H_H\mathcal{QYD}$ the category of left–left Yetter–Drinfeld quasimodules over H and by ${}^H_H\mathcal{YD}$ its subcategory of left–left Yetter–Drinfeld modules (the category formed by the objects that are also left H-modules and with the obvious morphisms). Note that if H is a Hopf algebra, conditions (b2) and (b3) trivialize and in this case ${}^H_H\mathcal{YD}$ is the classical category of left–left Yetter–Drinfeld modules over H.

Let (M, φ_M, ρ_M) and (N, φ_N, ρ_N) be two objects in ${}^H_H\mathcal{QYD}$. Then, $M \otimes N$, with the diagonal structure $\varphi_{M\otimes N}$ and the codiagonal costructure $\rho_{M\otimes N}$, is an object in ${}^H_H\mathcal{QYD}$. Therefore, $({}^H_H\mathcal{QYD}, \otimes, K)$ is a strict monoidal category. If moreover λ_H is an isomorphism, $({}^H_H\mathcal{YD}, \otimes, K)$ is a strict braided monoidal category where the braiding t and its inverse are defined by

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\rho_M \otimes N) \tag{118}$$

and

$$t^{-1}_{M,N} = c_{N,M} \circ ((\varphi_N \circ c_{N,H}) \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \rho_M),$$

respectively (see Proposition 1.8 of [2]). As a consequence, we can consider Hopf quasigroups in ${}^H_H\mathcal{YD}$. The definition is the following:

Definition 6.2 Let H be a Hopf quasigroup such that its antipode is an isomorphism. Let (D, u_D, m_D) be a unital magma in $\mathcal{C}$ such that (D, e_D, Δ_D) is a comonoid in $\mathcal{C}$, and let $s_D : D \to D$ be a morphism in $\mathcal{C}$. We say that the triple $(D, \varphi_D, \varrho_D)$ is a Hopf quasigroup in ${}^H_H\mathcal{YD}$ if:

(c1) The triple (D, φ_D, ρ_D) is a left–left Yetter–Drinfeld H-module.
(c2) The triple (D, u_D, m_D) is a unital magma in ${}^H_H\mathcal{YD}$, i.e., (D, u_D, m_D) is a unital magma in $\mathcal{C}$, (D, φ_D) is a left H-module magma and (D, ρ_D) is a left H-comodule magma.
(c3) The triple (D, e_D, Δ_D) is a comonoid in ${}^H_H\mathcal{YD}$, i.e., (D, e_D, Δ_D) is a comonoid in $\mathcal{C}$, (D, φ_D) is a left H-module comonoid and (D, ρ_D) is a left H-comodule comonoid.
(c4) The following identities hold:

(c4-1) $e_D \circ u_D = id_K$,
(c4-2) $e_D \circ m_D = e_D \otimes e_D$,
(c4-3) $\Delta_D \circ e_D = e_D \otimes e_D$,

(c4-4) $\Delta_D \circ m_D = (m_D \otimes m_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\Delta_D \otimes \Delta_D)$,

where $t_{D,D}$ is the braiding of ${}^H_H\mathcal{YD}$ for $M = N = D$.

(c5) The following identities hold:

(c5-1) $m_D \circ (s_D \otimes m_D) \circ (\Delta_D \otimes D) = e_D \otimes D = m_D \circ (D \otimes m_D) \circ (D \otimes s_D \otimes D) \circ (\Delta_D \otimes D)$.

(c5-2) $m_D \circ (m_D \otimes D) \circ (D \otimes s_D \otimes D) \circ (D \otimes \Delta_D) = D \otimes e_D = \mu_D \circ (m_D \otimes s_D) \circ (D \otimes \Delta_D)$.

Note that under these conditions, s_D is a morphism in ${}^H_H\mathcal{YD}$ (see Lemmas 1.11, 1.12 of [2]).

By Theorem 1.14 of [2], we know that if $(D, \varphi_D, \varrho_D)$ is a Hopf quasigroup in ${}^H_H\mathcal{YD}$, then

$$D \rtimes H = (D \otimes H, \eta_{D\rtimes H}, \mu_{D\rtimes H}, \varepsilon_{D\rtimes H}, \delta_{D\rtimes H}, \lambda_{D\rtimes H})$$

is a Hopf quasigroup in $\mathcal{C}$, with the biproduct structure induced by the smash product coproduct, i.e.,

$$\eta_{D\rtimes H} = \eta_D \otimes \eta_H, \quad \mu_{D\rtimes H} = (\mu_D \otimes \mu_H) \circ (D \otimes \Psi_D^H \otimes H),$$
$$\varepsilon_{D\rtimes H} = \varepsilon_D \otimes \varepsilon_H, \quad \delta_{D\rtimes H} = (D \otimes \Gamma_D^H \otimes H) \circ (\delta_D \otimes \delta_H),$$
$$\lambda_{D\rtimes H} = \Psi_D^H \circ (\lambda_H \otimes \lambda_D) \circ \Gamma_D^H,$$

where the morphisms $\Gamma_D^H : D \otimes H \to H \otimes D$, $\Psi_D^H : H \otimes D \to D \otimes H$ are defined by

$$\Gamma_D^H = (\mu_H \otimes D) \circ (H \otimes c_{D,H}) \circ (\varrho_D \otimes H), \quad \Psi_D^H$$
$$= (\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ (\delta_H \otimes D).$$

Let H and B be Hopf quasigroups and let $f : H \to B$ and $g : B \to H$ be morphisms of Hopf quasigroups such that $g \circ f = id_H$. By Proposition 2.1 of [2], we know that $q_H^B = id_B * (f \circ \lambda_H \circ g) : B \to B$ is an idempotent morphism. Moreover, if B_H is the image of q_H^B and $p_H^B : B \to B_H$, $i_H^B : B_H \to B$ a factorization of q_H^B,

$$B_H \xrightarrow{\;i_H^B\;} B \underset{B \otimes \eta_H}{\overset{(B \otimes g) \circ \delta_B}{\rightrightarrows}} B \otimes H$$

is an equalizer diagram. As a consequence, the triple (B_H, u_{B_H}, m_{B_H}) is a unital magma where u_{B_H} and m_{B_H} are the factorizations, through the equalizer i_H^B, of the morphisms η_B and $\mu_B \circ (i_H^B \otimes i_H^B)$, respectively. Therefore, the equalities

$$u_{B_H} = p_H^B \circ \eta_B, \quad m_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B), \tag{119}$$

hold.

Definition 6.3 Let H be a Hopf quasigroup. A Hopf quasigroup projection over H is a triple (B, f, g) where B is a Hopf quasigroup, $f : H \to B$ and $g : B \to H$ are morphisms of Hopf quasigroups such that $g \circ f = id_H$, and the equality

$$q_H^B \circ \mu_B \otimes (B \otimes q_H^B) = q_H^B \circ \mu_B \tag{120}$$

holds.

Let (B, f, g) and (B', f', g') two Hopf quasigroup projections. We will say that a Hopf quasigroup morphism $h : B \to B'$ is a morphism of Hopf quasigroup projections if it satisfies that $h \circ f = f', g' \circ h = g$. The category of Hopf quasigroup projections over H will be denoted by $\mathcal{P}roj(H)$.

If (B, f, g) is a Hopf quasigroup projection over H,

$$B \otimes H \underset{B \otimes \varepsilon_H}{\overset{\mu_B \circ (B \otimes f)}{\rightrightarrows}} B \xrightarrow{\ p_H^B\ } B_H$$

is a coequalizer diagram. Moreover, the triple $(B_H, e_{B_H}, \Delta_{B_H})$ is a comonoid, where e_{B_H} and Δ_{B_H} are the factorizations, through the coequalizer p_H^B, of the morphisms ε_B and $(p_H^B \otimes p_H^B) \circ \delta_B$, respectively. Moreover, the equalities

$$e_{B_H} = \varepsilon_B \circ i_H^B, \quad \Delta_{B_H} = (p_H^B \otimes p_H^B) \circ \delta_B \circ i_H^B \tag{121}$$

hold (see Proposition 2.3 of [2]).

Definition 6.4 Let H be a Hopf quasigroup. We say that a Hopf quasigroup projection (B, f, g) over H is strong if it satisfies

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (i_H^B \otimes f \otimes i_H^B)$$
$$= p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (i_H^B \otimes f \otimes i_H^B), \tag{122}$$
$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B)$$
$$= p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B), \tag{123}$$
$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B)$$
$$= p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B). \tag{124}$$

Note that, by the factorization of q_H^B, we have that (122), (123), and (124) are equivalent to

$$q_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (i_H^B \otimes f \otimes i_H^B)$$
$$= q_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (i_H^B \otimes f \otimes i_H^B), \tag{125}$$
$$q_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B)$$
$$= q_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B), \tag{126}$$

$$q_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B)$$
$$= q_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B). \tag{127}$$

We will denote by $\mathcal{SP}roj(H)$ the category of strong Hopf quasigroup projections over H. The morphisms of $\mathcal{SP}roj(H)$ are the morphisms of $\mathcal{P}roj(H)$.

Let H be a Hopf quasigroup with invertible antipode. By Proposition 2.7 of [2], if D is a Hopf quasigroup in ${}^H_H\mathcal{YD}$, the triple $(D \rtimes H, f = \eta_D \otimes H, g = \varepsilon_D \otimes H)$ is a strong Hopf quasigroup projection over H. In this case $q_H^{D\rtimes H} = D \otimes \eta_H \otimes \varepsilon_H$. As a consequence, we can choose $p_H^{D\rtimes H} = D \otimes \varepsilon_H$ and $i_H^{D\rtimes H} = D \otimes \eta_H$ and then $(D \rtimes H)_H = D$.

On the other hand, by Corollary 2.10 and Proposition 2.5 of [2], we can assure that, if (B, f, g) is a strong Hopf quasigroup projection over H, the triple $(B_H, \varphi_{B_H}, \varrho_{B_H})$ is a Hopf quasigroup in ${}^H_H\mathcal{YD}$, where the magma–comonoid structure is defined by (119) and (121),

$$\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B), \quad \rho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B, \tag{128}$$

and

$$s_{B_H} = p_H^B \circ ((f \circ g) * \lambda_B) \circ i_H^B. \tag{129}$$

Moreover, $w = \mu_B \circ (i_H^B \otimes f) : B_H \rtimes H \to B$ is an isomorphism of Hopf quasigroups in $\mathcal{C}$ with inverse $w^{-1} = (p_H^B \otimes g) \circ \delta_B$ (see Propositions 2.8 and 2.9 of [2]). Therefore, there exists an equivalence between the categories $\mathcal{SP}roj(H)$ and the category of Hopf quasigroups in ${}^H_H\mathcal{YD}$ (see Theorem 2.11 of [2]).

In the final part of the paper, we will prove that we can construct examples of strong projections by working with quasitriangular structures and skew pairings. First, we will introduce the notion of quasitriangular Hopf quasigroup.

Definition 6.5 Let H be a Hopf quasigroup. We will say that H is quasitriangular if there exists a morphism $R : K \to H \otimes H$ such that:

(d1) $(\delta_H \otimes H) \circ R = (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,
(d2) $(H \otimes \delta_H) \circ R = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,
(d3) $\mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes R) = \mu_{H\otimes H} \circ (R \otimes \delta_H)$,
(d4) $(\varepsilon_H \otimes H) \circ R = (H \otimes \varepsilon_H) \circ R = \eta_H$.

In the Hopf algebra setting, the morphism R is convolution invertible with inverse $R^{-1} = (\lambda_H \otimes H) \circ R$ and $R = (\lambda_H \otimes \lambda_H) \circ R$. In our non-associative context, we have that if $S = (\lambda_H \otimes H) \circ R$ and $T = (\lambda_H \otimes \lambda_H) \circ R$, the following identities hold:

$$R * S = S * R = \eta_{H\otimes H}, \tag{130}$$
$$S * T = T * S = \eta_{H\otimes H}. \tag{131}$$

Indeed:

$R * S$

$$= ((\mu_H \circ (H \otimes \lambda_H)) \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R) \text{ (by naturality of } c)$$
$$= ((id_H * \lambda_H) \otimes H) \circ R \text{ (by (d1) of Definition 6.5)}$$
$$= ((\varepsilon_H \otimes \eta_H) \otimes H) \circ R \text{ (by (13))}$$
$$= \eta_{H\otimes H} \text{ (by (d4) of Definition 6.5)}.$$

Similarly, we prove $S * R = \eta_{H\otimes H}$ using (28) instead of (13). On the other hand,

$$S * T$$
$$= ((\mu_H \circ (\lambda_H \otimes \lambda_H)) \otimes (\mu_H \circ (H \otimes \lambda_H)))$$
$$\circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R) \text{ (by naturality of } c)$$
$$= ((\lambda_H \circ \mu_H \circ c_{H,H}) \otimes (\mu_H \circ (H \otimes \lambda_H)))$$
$$\circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R) \text{ (by (31))}$$
$$= ((\lambda_H \circ \mu_H) \otimes (\mu_H \circ (H \otimes \lambda_H) \circ c_{H,H}))$$
$$\circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R) \text{ (by naturality of } c)$$
$$= (\lambda_H \otimes (id_H * \lambda_H)) \circ R \text{ (by (d2) of Definition 6.5)}$$
$$= (\lambda_H \otimes (\varepsilon_H \otimes \eta_H)) \circ R \text{ (by (13))}$$
$$= \eta_{H\otimes H} \text{ (by (d4) of Definition 6.5 and (19))}.$$

The proof for $T * S = \eta_{H\otimes H}$ is similar using (28) instead of (13).

Note that, by the lack of associativity, we cannot assure that S be the unique morphism satisfying (130) and (131).

Finally, the identity

$$(\mu_H \otimes H \otimes (\mu_H \circ c_{H,H})) \circ (H \otimes H \otimes (c_{H,H} \circ (H \otimes \mu_H)) \otimes H)$$
$$\circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (R \otimes R \otimes R)$$
$$= (\mu_H \otimes \mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (R \otimes R \otimes R) \quad (132)$$

holds because

$$(\mu_H \otimes H \otimes (\mu_H \circ c_{H,H})) \circ (H \otimes H \otimes (c_{H,H} \circ (H \otimes \mu_H)) \otimes H)$$
$$\circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (R \otimes R \otimes R)$$
$$= (H \otimes (((\mu_H \circ c_{H,H}) \otimes (\mu_H \circ c_{H,H})) \circ (H \otimes c_{H,H} \otimes H)))$$
$$\circ (((\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H)$$
$$\circ (R \otimes R)) \otimes R) \text{ (by naturality of } c)$$
$$= (H \otimes (((\mu_H \circ c_{H,H}) \otimes (\mu_H \circ c_{H,H})) \circ (H \otimes c_{H,H} \otimes H)))$$
$$\circ (((H \otimes \delta_H) \circ R) \otimes R) \text{ (by (d2) of Definition 6.5)}$$
$$= (H \otimes (\mu_{H\otimes H} \circ c_{H\otimes H, H\otimes H} \circ (\delta_H \otimes R))) \circ R \text{ (by } c^2 = id)$$
$$= (H \otimes (\mu_{H\otimes H} \circ (R \otimes \delta_H))) \circ R \text{ (by naturality of } c)$$
$$= (H \otimes (\mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes R))) \circ R \text{ (by (d3) of Definition 6.5)}$$
$$= (H \otimes \mu_{H\otimes H}) \circ (\mu_H \otimes (c_{H,H} \circ c_{H,H}) \otimes R)$$

$\circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$ (by (d2) of Definition 6.5)

$= (\mu_H \otimes \mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (R \otimes R \otimes R)$ (by naturality of c).

Proposition 6.6 *Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Let $A \bowtie_\tau H$ be the Hopf quasigroup defined in Corollary 4.10. Define the morphism $g : A \bowtie_\tau H \to H$ by*

$$g = (\tau \otimes \mu_H) \circ (A \otimes R \otimes H).$$

Then, g is a morphism of unital magmas if and only if the following equalities hold:

$$\mu_H \circ (g \otimes H) = g \circ (A \otimes \mu_H), \tag{133}$$

$$\mu_H \circ (H \otimes g) = \mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\tau \otimes H) \circ (A \otimes R)) \otimes H). \tag{134}$$

Proof Assume that $g = (\tau \otimes \mu_H) \circ (A \otimes R \otimes H)$ is a magma morphism. Then,

$$g \circ \mu_{A \bowtie_\tau H} = \mu_H \circ (g \otimes g) \tag{135}$$

holds. Moreover,

$g \circ (A \otimes \mu_H)$

$= g \circ \mu_{A \bowtie_\tau H} \circ (A \otimes H \otimes \eta_A \otimes H)$

(by naturality of c, (3), (a3) of Definition 4.1, (79), and unit and counit properties)

$= \mu_H \circ (g \otimes g) \circ (A \otimes H \otimes \eta_A \otimes H)$ (by (135))

$= \mu_H \circ (g \otimes H)$ (by (a3) of Definition 4.1, (d4) of Definition 6.5, and unit properties).

Therefore, (133) holds. On the other hand, the proof for (134) is the following:

$\mu_H \circ (H \otimes g)$

$= \mu_H \circ ((g \circ (\eta_A \otimes H)) \otimes g)$ (by (a3) of Definition 4.1 and unit properties)

$= g \circ \mu_{A \bowtie_\tau H} \circ (\eta_A \otimes H \otimes A \otimes H)$ (by (135))

$= (\tau \otimes (g \circ (A \otimes \mu_H))) \circ (((\delta_{A\otimes H} \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes H)$ (by unit properties)

$= (\tau \otimes (\mu_H \circ (g \otimes H))) \circ (((\delta_{A\otimes H} \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes H)$ (by (133))

$= ((\tau \circ (A \otimes (\mu_H \circ c_{H,H}))) \otimes (\mu_H \circ (\mu_H \otimes H))) \circ (A \otimes ((H \otimes R$

$\otimes H) \circ \delta_H) \otimes \tau^{-1} \otimes H) \circ ((\delta_{A\otimes H} \circ c_{H,A}) \otimes H)$ (by (78))

$= (\tau \otimes \mu_H) \circ (((A \otimes (\mu_{H\otimes H} \circ (R \otimes \delta_H)) \otimes \tau^{-1})$

$\circ \delta_{A\otimes H} \circ c_{H,A}) \otimes H)$ (by naturality of c)

$= (\tau \otimes \mu_H) \circ (((A \otimes (\mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes (R \circ \tau^{-1}))))$

$\circ \delta_{A\otimes H} \circ c_{H,A}) \otimes H)$ (by (d3) of Definition 6.5)

$= \mu_H \circ (((\tau \otimes \tau \otimes \mu_H) \circ (A \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ ((c_{A,A} \circ \delta_A)$

$$\otimes(c_{H,H}\circ\delta_H)\otimes(R\circ\tau^{-1}))\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H)$$
(by (a2$'$) of Definition 4.1)
$$=\mu_H\circ(((\tau\otimes(\mu_H\circ c_{H,H})\otimes(\tau*\tau^{-1}))\circ(A\otimes R\otimes H$$
$$\otimes A\otimes H)\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H)\ \text{(by naturality of } c)$$
$$=(\tau\otimes(\mu_H\circ((\mu_H\circ c_{H,H})\otimes H)))\circ(A\otimes R\otimes H\otimes H)$$
$$\circ\,(c_{H,A}\otimes H)\ \text{(by naturality of } c\text{, invertibility of } \tau \text{ and counit properties)}$$
$$=\mu_H\circ(\mu_H\otimes H)\circ(H\otimes((\tau\otimes H)\circ(A\otimes R))\otimes H)\ \text{(by naturality of } c).$$

Conversely, assume that (133) and (134) hold. Firstly, note that $g\circ\eta_{A\bowtie_\tau H}=\eta_H$ follows by (a4) of Definition 4.1, (d4) of Definition 6.5, and the unit properties. Secondly,

$$g\circ\mu_{A\bowtie_\tau H}$$
$$=\mu_H\circ((g\circ(\mu_A\otimes H)\circ(A\otimes((((\tau\otimes A\otimes H)\circ\delta_{A\otimes H})$$
$$\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})))\otimes H)$$
(by (133))
$$=(((\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ(A\otimes A\otimes\delta_H))\otimes(\mu_H\circ(\mu_H\otimes H)))$$
$$\circ\,(A\otimes\tau\otimes A\otimes R\otimes H\otimes\tau^{-1}\otimes H)$$
$$\circ\,(A\otimes((\delta_{A\otimes H}\otimes A\otimes H)\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H)$$
(by (a1) of Definition 4.1)
$$=\mu_H\circ(((\tau\otimes\tau\otimes(\mu_H\circ(\mu_H\otimes H))\otimes H)\circ(A$$
$$\otimes\tau\otimes c_{A,H}\otimes c_{H,H}\otimes H\otimes H\otimes H)$$
$$\circ\,(A\otimes A\otimes c_{A,H}\otimes\otimes A\otimes R\otimes R\otimes H\otimes H)\circ(A\otimes\delta_A\otimes\delta_H$$
$$\otimes\tau^{-1}\otimes H)\circ(A\otimes(\delta_{A\otimes H}\circ c_{H,A})\otimes H)$$
(by (d1) of Definition 6.5)
$$=\mu_H\circ((\mu_H\circ(((\tau\otimes H)\circ(A\otimes R))\otimes((((\tau\otimes\tau)\circ(A$$
$$\otimes c_{A,H}\otimes H)\circ((c_{A,A}\circ\delta_A)\otimes c_{H,H}))\otimes\mu_H)$$
$$\circ\,(A\otimes((H\otimes R\otimes H)\circ\delta_H)))))\otimes\tau^{-1}\otimes H)\circ(A\otimes(\delta_{A\otimes H}\circ c_{H,A})\otimes H)$$
(by naturality of c, $c^2=id$, and (134))
$$=\mu_H\circ((\mu_H\circ(((\tau\otimes H)\circ(A\otimes R))\otimes((\tau\otimes H)\circ(A$$
$$\otimes((((\mu_H\circ c_{H,H})\otimes\mu_H))\circ(H\otimes R\otimes H)$$
$$\circ\delta_H)))))\otimes\tau^{-1}\otimes H)\circ(A\otimes(\delta_{A\otimes H}\circ c_{H,A})\otimes H)$$
(by (a2$'$) of Definition 4.1)
$$=\mu_H\circ((\mu_H\circ(((\tau\otimes H)\circ(A\otimes R))\otimes((\tau\otimes H)\circ(A$$
$$\otimes((\mu_{H\otimes H}\circ(R\otimes\delta_H)))))))\otimes\tau^{-1}\otimes H)$$
$$\circ\,(A\otimes(\delta_{A\otimes H}\circ c_{H,A})\otimes H)\ \text{(by naturality of } c \text{ and } c^2=id)$$
$$=\mu_H\circ((\mu_H\circ(((\tau\otimes H)\circ(A\otimes R))\otimes((\tau\otimes H)\circ(A$$
$$\otimes((\mu_{H\otimes H}\circ((c_{H,H}\circ\delta_H)\otimes R)))))))\otimes\tau^{-1}\otimes H)$$

$$\circ (A \otimes (\delta_{A\otimes H} \circ c_{H,A}) \otimes H) \text{ (by (d3) of Definition 6.5)}$$
$$= \mu_H \circ ((\mu_H \circ (((\tau \otimes H) \circ (A \otimes R)) \otimes ((\tau \otimes \tau \otimes \mu_H)$$
$$\circ (A \otimes c_{A,H} \otimes c_{H,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes (c_{H,H}$$
$$\circ \delta_H) \otimes R)))) \otimes \tau^{-1} \otimes H) \circ (A \otimes (\delta_{A\otimes H} \circ c_{H,A}) \otimes H)$$
$$\text{(by (a2}'\text{) of Definition 4.1)}$$
$$= \mu_H \circ ((\mu_H \circ (((\tau \otimes H) \circ (A \otimes R)) \otimes ((\tau \otimes (\mu_H \circ c_{H,H}))$$
$$\circ (A \otimes R \otimes H)))) \otimes (\tau * \tau^{-1}) \otimes H)$$
$$\circ (A \otimes (\delta_{A\otimes H} \circ c_{H,A}) \otimes H) \text{ (by naturality of } c)$$
$$= \mu_H \circ ((g \circ (A \otimes \mu_H)) \otimes H) \circ (A \otimes H \otimes ((\tau \otimes H) \circ (A \otimes R)) \otimes H)$$
$$\text{(by invertibility of } \tau \text{ and counit properties)}$$
$$= \mu_H \circ ((\mu_H \circ (g \otimes (((\tau \otimes H) \circ (A \otimes R))))) \otimes H) \text{ (by (133))}$$
$$= \mu_H \circ (g \otimes g) \text{ (by (134))},$$

and then g is morphism of unital magmas. □

Remark 6.7 In the previous Proposition, note that, if H is a Hopf algebra, equalities (133) and (134) always hold.

Theorem 6.8 *Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Let $A \bowtie_\tau H$ be the Hopf quasigroup defined in Corollary 4.10 and let $g : A \bowtie_\tau H \to H$ be the morphism introduced in Proposition 6.6. Define the morphism $f : H \to A \bowtie_\tau H$ by $f = \eta_A \otimes H$. Then, if (133) and (134) hold, the triple $(A \bowtie_\tau H, f, g)$ is a strong Hopf quasigroup projection over H.*

Proof By Proposition 6.6, we know that g is a morphism of unital magmas. Also, by (2), (d4) of Definition 6.5 and (a3) of Definition 4.1, we obtain that $\varepsilon_H \circ g = \varepsilon_{A\bowtie_\tau H}$. Moreover,

$$\delta_H \circ g$$
$$= (\tau \otimes (\mu_{H\otimes H} \circ (\delta_H \otimes \delta_H))) \circ (A \otimes R \otimes H) \text{ (by (4))}$$
$$= (\tau \otimes \tau \otimes \mu_{H\otimes H}) \circ (A \otimes c_{A,H} \otimes H \otimes c_{H,H} \otimes H \otimes H)$$
$$\circ ((c_{A,A} \circ \delta_A) \otimes ((H \otimes c_{H,H} \otimes H) \circ (R \otimes R)) \otimes \delta_H)$$
$$\text{(by (d2) of Definition 6.5, and (a2}'\text{) of Definition 4.1)}$$
$$= (g \otimes g) \circ \delta_{A\bowtie_\tau H} \text{ (by naturality of } c \text{ and } c^2 = id).$$

Therefore, g is a comonoid morphism. On the other hand, trivially $f \circ \eta_H = \eta_{A\bowtie_\tau H}$ and $\mu_{A\bowtie_\tau H} \circ (f \otimes f) = f \circ \mu_H$ follows easily by naturality of c, (3), (a4) of Definition 4.1, (79), and unit and counit properties. By (1), it is clear that $\varepsilon_{A\bowtie_\tau H} \circ f = \varepsilon_H$ and the identity $\delta_{A\bowtie_\tau H} \circ f = (f \otimes f) \circ \delta_H$ can be proved using (3) and the naturality of c. As a consequence, f is a morphism of unital magmas and comonoids. By (a4) of Definition 4.1 and (d4) of Definition 6.5, an easy computation shows that $g \circ f = id_H$.

Our next goal is to obtain a simple expression for the idempotent morphism

$$q_H^{A\bowtie_\tau H} = id_{A\bowtie_\tau H} * (f \circ \lambda_H \circ g) : A \bowtie_\tau H \to A \bowtie_\tau H.$$

Indeed, the equality

$$q_H^{A\bowtie_\tau H} = (A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R \otimes \varepsilon_H) \tag{136}$$

holds because

$$\begin{aligned}
&q_H^{A\bowtie_\tau H}\\
&= (A \otimes (\mu_H \circ (H \otimes (\lambda_H \circ g)))) \circ \delta_{A\otimes H}\\
&\quad (\text{by naturality of } c,\ (3),\ \text{(a4) of Definition 4.1},\ (79),\ \text{and unit and counit properties})\\
&= (A \otimes (\mu_H \circ (H \otimes \tau \otimes (\mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H)))\\
&\quad \circ (H \otimes A \otimes R \otimes H))) \circ \delta_{A\otimes H}\ (\text{by } (31))\\
&= (A \otimes (\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) \circ c_{H,H}\\
&\quad \circ (\tau \otimes \lambda_H \otimes H))) \circ (\delta_A \otimes R \otimes H)\ (\text{by naturality of } c)\\
&= (A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R \otimes \varepsilon_H)\ (\text{by } (27)).
\end{aligned}$$

Then, using (136), we can prove that $(A \bowtie_\tau H, f, g)$ is a Hopf quasigroup projection over H. Indeed:

$$\begin{aligned}
&q_H^{A\bowtie_\tau H} \circ \mu_{A\bowtie_\tau H} \circ (A \otimes H \otimes q_H^{A\bowtie_\tau H})\\
&= (A \otimes \tau \otimes H) \circ ((\delta_A \circ \mu_A) \otimes ((H \otimes \lambda_H) \circ R)) \circ (A \otimes ((\tau \otimes A)\\
&\quad \circ (A \otimes c_{A,H} \otimes \tau^{-1}) \circ (\delta_A \otimes H \otimes A \otimes H)\\
&\quad \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes \varepsilon_H)\\
&\quad (\text{by } (2),\ (18),\ \text{(d4) of Definition 6.5, (a3) of Definition 4.1, and counit properties})\\
&= q_H^{A\bowtie_\tau H} \circ \mu_{A\bowtie_\tau H}\\
&\quad (\text{by } (2),\ \text{and counit properties}).
\end{aligned}$$

Note that, by (3), the naturality of c, (79), (a4) of Definition 4.1, and unit and counit properties, we have the equality

$$\mu_{A\bowtie_\tau H} \circ (A \otimes H \otimes \eta_A \otimes H) = \mu_H, \tag{137}$$

and, by unit and counit properties, and (2), the identity

$$q_H^{A\bowtie_\tau H} \circ \mu_{A\bowtie_\tau H} \circ (A \otimes H \otimes A \otimes (\eta_H \circ \varepsilon_H)) = q_H^{A\bowtie_\tau H} \circ \mu_{A\bowtie_\tau H} \tag{138}$$

holds.

To finish the proof, it is sufficient to show that the projection is strong, i.e., (125), (126) and (127) hold. Let us first prove (125):

$$
\begin{aligned}
& q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(A\otimes H\otimes\mu_{A\bowtie_\tau H})\circ(i_H^{A\bowtie_\tau H}\otimes f\otimes i_H^{A\bowtie_\tau H})\\
&\quad=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes(R\circ\tau^{-1}))\circ(A\otimes((((\tau\otimes H)\\
&\qquad\circ(A\otimes c_{A,H})\circ(\delta_A\otimes H))\otimes A\otimes H)\circ\delta_{A\otimes H}))\\
&\qquad\circ(A\otimes c_{H,A})\circ(i_H^{A\bowtie_\tau H}\otimes((((\tau\otimes H)\circ(A\otimes c_{A,H})\\
&\qquad\circ(\delta_A\otimes H))\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ(c_{H,A}\otimes\varepsilon_H)\\
&\qquad\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by } (2),\ (138) \text{ and unit and counit properties})\\
&\quad=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes(R\circ((\tau\otimes\tau)\circ(A\otimes c_{A,H}\\
&\qquad\otimes H)\circ((c_{A,A}\circ\delta_A)\otimes H\otimes H))))\\
&\qquad\circ(A\otimes(((c_{A,A}\circ\delta_A)\otimes H\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H\\
&\qquad\otimes\tau^{-1})\circ(i_H^{A\bowtie_\tau H}\otimes(((\delta_{A\otimes H}\circ c_{H,A})\otimes\varepsilon_H)\\
&\qquad\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by naturality of } c\text{, coassociativity, and } c^2=id)\\
&\quad=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes(R\circ\tau\circ(A\otimes\mu_H)))\circ(A\otimes(((c_{A,A}\circ\delta_A)\\
&\qquad\otimes H\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H\otimes\tau^{-1})\\
&\qquad\circ(i_H^{A\bowtie_\tau H}\otimes(((\delta_{A\otimes H}\circ c_{H,A})\otimes\varepsilon_H)\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by (a2}'\text{) of Definition 4.1})\\
&\quad=(A\otimes((\tau\otimes\lambda_H)\circ(A\otimes R))\circ((\delta_A\circ\mu_A)\otimes((\tau^{-1}\otimes\tau^{-1})\\
&\qquad\circ(A\otimes c_{A,H}\otimes H)\circ(\delta_A\otimes H\otimes H)))\circ(A\otimes((((\tau\otimes\delta_A)\\
&\qquad\circ(A\otimes c_{A,H})\circ(\delta_A\otimes\mu_H))\otimes H\otimes H)\circ(A\otimes\delta_{H\otimes H})\\
&\qquad\circ(c_{H,A}\otimes H)))\circ(i_H^{A\bowtie_\tau H}\otimes((c_{H,A}\otimes\varepsilon_H)\\
&\qquad\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by naturality of } c\text{, coassociativity, and } c^2=id)\\
&\quad=(A\otimes((\tau\otimes\lambda_H)\circ(A\otimes R))\circ((\delta_A\circ\mu_A)\otimes\tau^{-1})\\
&\qquad\circ(A\otimes((((\tau\otimes\delta_A)\circ(A\otimes c_{A,H}))\otimes H)\\
&\qquad\circ(\delta_A\otimes((\mu_H\otimes\mu_H)\circ\delta_{H\otimes H}))\circ(c_{H,A}\otimes H)))\\
&\qquad\circ(i_H^{A\bowtie_\tau H}\otimes((c_{H,A}\otimes\varepsilon_H)\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by } (81))\\
&\quad=(A\otimes((\tau\otimes\lambda_H)\circ(A\otimes R))\circ((\delta_A\circ\mu_A)\otimes\tau^{-1})\\
&\qquad\circ(A\otimes((((\tau\otimes\delta_A)\circ(A\otimes c_{A,H}))\otimes H)\\
&\qquad\circ(\delta_A\otimes(\delta_H\circ\mu_H))\circ(c_{H,A}\otimes H)))\circ(i_H^{A\bowtie_\tau H}\\
&\qquad\otimes((c_{H,A}\otimes\varepsilon_H)\circ(H\otimes i_H^{A\bowtie_\tau H}))) \ (\text{by } (4))\\
&\quad=(A\otimes((\tau\otimes\lambda_H)\circ(A\otimes R))\circ((\delta_A\circ\mu_A)\otimes(\varepsilon_H\circ\mu_H))\\
&\qquad\circ(A\otimes((((\tau\otimes A\otimes H)\circ\delta_{A\otimes H})\otimes\tau^{-1})\\
&\qquad\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H)\circ(((A\otimes\mu_H)\circ(i_H^{A\bowtie_\tau H}\otimes H))\\
&\qquad\otimes i_H^{A\bowtie_\tau H}) \ (\text{by } (2)\text{, and counit properties})\\
&\quad=q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(\mu_{A\bowtie_\tau H}\otimes A\otimes H)\circ(i_H^{A\bowtie_\tau H}\otimes f\otimes i_H^{A\bowtie_\tau H}) \ (\text{by } (137)).
\end{aligned}
$$

Secondly, we will prove (126):

$$
\begin{aligned}
& q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(\mu_{A\bowtie_\tau H}\otimes A\otimes H)\circ(f\otimes i_H^{A\bowtie_\tau H}\otimes i_H^{A\bowtie_\tau H})\\
&=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes R)\circ(A\otimes((((\tau\otimes A)\circ(A\otimes c_{A,H})\\
&\quad\circ(\delta_A\otimes H))\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A}))\\
&\quad\circ(((\tau\otimes A\otimes\mu_H)\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H))\\
&\quad\otimes A)\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\\
&\quad\text{(by (2), and unit and counit properties)}\\
&=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes R)\circ(A\otimes((((\tau\otimes A)\circ(A\otimes c_{A,H})\\
&\quad\circ(\delta_A\otimes H))\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\\
&\quad\circ(\delta_A\otimes(\delta_H\circ\mu_H))\circ(c_{H,A}\otimes H)))\circ(((\tau\otimes A\otimes H)\\
&\quad\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})))\otimes c_{H,A})\\
&\quad\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ \text{(by naturality of } c)\\
&=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes R)\circ(A\otimes((((\tau\otimes A)\circ(A\otimes c_{A,H})\\
&\quad\circ(\delta_A\otimes H))\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\\
&\quad\circ(\delta_A\otimes((\mu_H\otimes\mu_H)\circ\delta_{H\otimes H}))\circ(c_{H,A}\otimes H)))\circ(((\tau\otimes A\otimes H)\\
&\quad\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})))\otimes c_{H,A})\\
&\quad\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ \text{(by (4))}\\
&=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes(R\circ\tau^{-1}\circ(A\otimes\mu_H)))\\
&\quad\circ(A\otimes(((\tau\circ(A\otimes\mu_H))\otimes A\otimes A\otimes H\otimes H)\\
&\quad\circ(A\otimes c_{A\otimes A,H\otimes H}\otimes H\otimes H)\circ(((\delta_A\otimes A)\circ\delta_A)\otimes\delta_{H\otimes H})\circ(c_{H,A}\\
&\quad\otimes H)))\circ(((\tau\otimes A\otimes H)\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\\
&\quad\circ\delta_{A\otimes H}\circ c_{H,A})))\otimes c_{H,A})\circ(H\otimes i_H^{A\bowtie_\tau H}\\
&\quad\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ \text{(by naturality of } c)\\
&=(A\otimes\tau\otimes\lambda_H)\circ((\delta_A\circ\mu_A)\otimes(R\circ\tau^{-1}\circ(A\otimes\mu_H)))\circ(A\\
&\quad\otimes((((\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ((c_{A,A}\circ\delta_A)\\
&\quad\otimes H\otimes H))\otimes A\otimes A\otimes H\otimes H)\circ(A\otimes c_{A\otimes A,H\otimes H}\otimes H\otimes H)\\
&\quad\circ(((\delta_A\otimes A)\circ\delta_A)\otimes\delta_{H\otimes H})\circ(c_{H,A}\otimes H)))\\
&\quad\circ(((\tau\otimes A\otimes H)\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})))\otimes c_{H,A})\\
&\quad\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\\
&\quad\text{(by (a2}'\text{) of Definition 4.1)}\\
&=(((\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ(A\otimes A\otimes\delta_H))\otimes((A\\
&\quad\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes R)))\circ(A\otimes((A\otimes c_{A,H})\\
&\quad\circ(((A\otimes\mu_A)\circ(c_{A,A}\otimes A)\circ(A\otimes\delta_A))\otimes H)\circ(A\\
&\quad\otimes c_{H,A}))\otimes\tau\otimes(\tau^{-1}\circ(A\otimes\mu_H)))\\
&\quad\circ(\delta_A\otimes H\otimes(c_{A,A}\circ\delta_A)\otimes c_{A,H}\otimes H\otimes H)\circ(A\otimes H\otimes\delta_A
\end{aligned}
$$

$$
\begin{aligned}
&\otimes c_{H,H} \otimes H) \circ (A \otimes H \otimes c_{H,A} \otimes \delta_H) \\
&\circ(((A \otimes \delta_H \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes c_{H,A}) \circ (H \otimes i_H^{A\bowtie_\tau H} \otimes ((A \otimes \varepsilon_H) \\
&\circ i_H^{A\bowtie_\tau H})) \text{ (by naturality of } c\text{, coassociativity, and } c^2 = id) \\
&= (\tau \otimes ((A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R))) \circ (A \otimes c_{A,H}) \\
&\circ (((\mu_A \otimes \mu_A) \circ \delta_{A\otimes A}) \otimes H) \\
&\circ(A \otimes c_{H,A} \otimes \tau \otimes (\tau^{-1} \circ (A \otimes \mu_H))) \circ (A \otimes H \otimes (c_{A,A} \circ \delta_A) \otimes c_{A,H} \\
&\otimes H \otimes H) \circ (A \otimes H \otimes \delta_A \otimes c_{H,H} \otimes H) \\
&\circ (A \otimes H \otimes c_{H,A} \otimes \delta_H) \circ (((A \otimes \delta_H \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes c_{H,A}) \\
&\circ (H \otimes i_H^{A\bowtie_\tau H} \otimes ((A \otimes \varepsilon_H) \circ i_H^{A\bowtie_\tau H})) \text{ (by (a1) of Definition 4.1)} \\
&= (\tau \otimes ((A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R))) \circ (A \otimes c_{A,H}) \circ (((\delta_A \circ \mu_A) \otimes H) \\
&\circ (A \otimes c_{H,A} \otimes \tau \otimes (\tau^{-1} \circ (A \otimes \mu_H))) \circ (A \otimes H \otimes (c_{A,A} \circ \delta_A) \otimes c_{A,H} \\
&\otimes H \otimes H) \circ (A \otimes H \otimes \delta_A \otimes c_{H,H} \otimes H) \\
&\circ (A \otimes H \otimes c_{H,A} \otimes \delta_H) \circ (((A \otimes \delta_H \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \\
&\otimes c_{H,A}) \circ (H \otimes i_H^{A\bowtie_\tau H} \otimes ((A \otimes \varepsilon_H) \circ i_H^{A\bowtie_\tau H})) \\
&\text{(by (4))} \\
&= (\tau \otimes ((A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R))) \circ (A \otimes c_{A,H}) \circ (((\delta_A \circ \mu_A) \otimes H) \\
&\circ (A \otimes c_{H,A} \otimes \tau \otimes ((\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H))) \\
&\circ(A \otimes H \otimes (c_{A,A} \circ \delta_A) \otimes c_{A,H} \otimes H \otimes H) \\
&\circ(A \otimes H \otimes \delta_A \otimes c_{H,H} \otimes H) \circ (A \otimes H \otimes c_{H,A} \otimes \delta_H) \\
&\circ (((A \otimes \delta_H \otimes \tau^{-1}) \circ \delta_{A\otimes H} \circ c_{H,A}) \otimes c_{H,A}) \\
&\circ (H \otimes i_H^{A\bowtie_\tau H} \otimes ((A \otimes \varepsilon_H) \circ i_H^{A\bowtie_\tau H})) \text{ (by (81))} \\
&= (A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R) \circ (((\tau \otimes A) \circ (A \otimes c_{A,H}) \circ ((\delta_A \circ \mu_A) \\
&\otimes H)) \otimes ((\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \\
&\circ (A \otimes A \otimes (c_{H,H} \circ \delta_H)))) \circ \delta_{A\otimes A\otimes H} \circ (A \otimes c_{H,A}) \\
&\circ (c_{H,A} \otimes A) \circ (H \otimes A \otimes ((A \otimes \tau \otimes \tau^{-1}) \\
&\circ ((c_{A,A} \circ \delta_A) \otimes H \otimes A \otimes H) \circ \delta_{A\otimes H} \circ c_{H,A})) \circ (H \otimes i_H^{A\bowtie_\tau H} \\
&\otimes((A \otimes \varepsilon_H) \circ i_H^{A\bowtie_\tau H})) \text{ (by naturality of } c\text{, coassociativity, and } c^2 = id) \\
&= (A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R) \circ (((\tau \otimes A) \circ (A \otimes c_{A,H}) \\
&\circ ((\delta_A \circ \mu_A) \otimes H)) \otimes (\tau^{-1} \circ (\mu_A \otimes H)) \\
&\circ \delta_{A\otimes A\otimes H} \circ (A \otimes c_{H,A}) \circ (c_{H,A} \otimes A) \circ (H \otimes A \otimes ((A \otimes \tau \otimes \tau^{-1}) \\
&\circ ((c_{A,A} \circ \delta_A) \otimes H \otimes A \otimes H) \circ \delta_{A\otimes H} \circ c_{H,A})) \\
&\circ (H \otimes i_H^{A\bowtie_\tau H} \otimes ((A \otimes \varepsilon_H) \circ i_H^{A\bowtie_\tau H})) \text{ (by (85))} \\
&= (A \otimes \tau \otimes \lambda_H) \circ (\delta_A \otimes R) \circ (((\tau \otimes A) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes H)) \\
&\otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (((\mu_A \otimes \mu_A) \\
&\circ \delta_{A\otimes A}) \otimes \delta_H) \circ (A \otimes c_{H,A}) \circ (c_{H,A} \otimes A) \circ \circ(H \otimes A \\
&\otimes((A \otimes \tau \otimes \tau^{-1}) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes A \otimes H)
\end{aligned}
$$

$$
\begin{aligned}
&\circ\delta_{A\otimes H}\circ c_{H,A}))\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ (\text{by naturality of } c)\\
&=(A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes R)\circ(((\tau\otimes A)\circ(A\otimes c_{A,H})\circ(\delta_A\\
&\otimes H))\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\circ((\delta_A\\
&\circ\mu_A)\otimes\delta_H)\circ(A\otimes c_{H,A})\circ(c_{H,A}\otimes A)\circ(H\otimes A\\
&\otimes((A\otimes\tau\otimes\tau^{-1})\circ((c_{A,A}\circ\delta_A)\otimes H\otimes A\otimes H)\\
&\circ\delta_{A\otimes H}\circ c_{H,A}))\circ(H\otimes i_H^{A\bowtie_\tau H}\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ (\text{by } (4))\\
&=q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(A\otimes A\otimes(\eta_H\circ\varepsilon_H))\circ(A\otimes H\otimes\mu_{A\bowtie_\tau H})\circ(f\otimes i_H^{A\bowtie_\tau H}\\
&\otimes i_H^{A\bowtie_\tau H})\ (\text{by } (2), (1), \text{naturality of } c, \text{ and unit and counit properties})\\
&=q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(A\otimes H\otimes\mu_{A\bowtie_\tau H})\circ(f\otimes i_H^{A\bowtie_\tau H}\otimes i_H^{A\bowtie_\tau H})\ (\text{by } (138)).
\end{aligned}
$$

Finally, we prove (127):

$$
\begin{aligned}
&q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(A\otimes H\otimes\mu_{A\bowtie_\tau H})\circ(f\otimes f\otimes i_H^{A\bowtie_\tau H})\\
&\quad=(A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes R)\circ(((\tau\otimes A)\circ(A\otimes c_{A,H})\circ(\delta_A\otimes H))\\
&\quad\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A}\circ(H\otimes((((\tau\otimes A)\\
&\quad\circ(A\otimes c_{A,H})\circ(\delta_A\otimes H))\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A}))\circ(H\otimes H\otimes((A\\
&\quad\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\ (\text{by } (2), \text{ and unit and counit properties})\\
&\quad=(((\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ((c_{A,A}\circ\delta_A)\otimes H\otimes H))\\
&\quad\otimes((A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ\tau^{-1}))))\\
&\quad\circ(A\otimes H\otimes c_{A,H}\otimes A\otimes H)\circ(((((A\otimes c_{A,H})\circ(\delta_A\otimes H))\\
&\quad\otimes H\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H\otimes A\otimes H)\\
&\quad\circ(H\otimes(\delta_{A\otimes H}\circ c_{H,A}))\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\\
&\quad(\text{by naturality of } c, \text{ coassociativity, and } c^2=id)\\
&\quad=((\tau\circ(A\otimes\mu_H))\otimes((A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ\tau^{-1}))))\\
&\quad\circ(A\otimes H\otimes c_{A,H}\otimes A\otimes H)\circ(((((A\otimes c_{A,H})\circ(\delta_A\otimes H))\\
&\quad\otimes H\otimes\tau^{-1})\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H\otimes A\otimes H)\\
&\quad\circ(H\otimes(\delta_{A\otimes H}\circ c_{H,A}))\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\\
&\quad(\text{by (a2}') \text{ of Definition } 4.1)\\
&\quad=((\tau\circ(A\otimes\mu_H))\otimes((A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ((\tau^{-1}\\
&\quad\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\circ(\delta_A\otimes H\otimes H))))))\\
&\quad\circ(A\otimes c_{A\otimes A,H\otimes H}\otimes H\otimes H)\circ(((\delta_A\otimes A)\circ\delta_A)\\
&\quad\otimes\delta_{H\otimes H})\circ(c_{H,A}\otimes H)\circ(H\otimes c_{H,A})\\
&\quad\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\\
&\quad(\text{by naturality of } c, \text{ coassociativity, and } c^2=id)\\
&\quad=((\tau\circ(A\otimes\mu_H))\otimes((A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ(\tau^{-1}\circ(A\otimes\mu_H))))))\\
&\quad\circ(A\otimes c_{A\otimes A,H\otimes H}\otimes H\otimes H)\circ(((\delta_A\otimes A)
\end{aligned}
$$

$$
\begin{aligned}
&\circ\delta_A)\otimes\delta_{H\otimes H})\circ(c_{H,A}\otimes H)\circ(H\otimes c_{H,A})\\
&\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\;(\text{by }(81))\\
&=(A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ\tau^{-1}))\circ(((\tau\otimes A)\circ(A\otimes c_{A,H})\\
&\circ(\delta_A\otimes H))\otimes A\otimes H)\circ(A\otimes c_{A,H}\otimes H)\\
&\circ(\delta_A\otimes((\mu_H\otimes\mu_H)\circ\delta_{H\otimes H}))\circ(c_{H,A}\otimes H)\circ(H\otimes c_{H,A})\\
&\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\;(\text{by naturality of } c)\\
&=(A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ\tau^{-1}))\circ(((\tau\otimes A)\circ(A\otimes c_{A,H})\\
&\circ(\delta_A\otimes H))\otimes A\otimes H)\circ(A\otimes c_{A,H}\otimes H)\\
&\circ(\delta_A\otimes((\delta_H\circ\mu_H))\circ(c_{H,A}\otimes H)\circ(H\otimes c_{H,A})\\
&\circ(H\otimes H\otimes((A\otimes\varepsilon_H)\circ i_H^{A\bowtie_\tau H}))\;(\text{by }(4))\\
&=(\tau\otimes((A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes(R\circ\varepsilon_H\circ\mu_H))))\circ(((\delta_{A\otimes H}\otimes\tau^{-1})\\
&\circ\delta_{A\otimes H}\circ c_{H,A})\otimes H)\circ(\mu_H\otimes i_H^{A\bowtie_\tau H})\\
&\quad(\text{by naturality of } c,\ (2),\ \text{and counit properties})\\
&=q_H^{A\bowtie_\tau H}\circ\mu_{A\bowtie_\tau H}\circ(\mu_{A\bowtie_\tau H}\otimes A\otimes H)\circ(f\otimes f\otimes i_H^{A\bowtie_\tau H})\\
&\quad(\text{by }(2),\ (137),\ (138)\text{ and unit and counit properties}).
\end{aligned}
$$

□

Corollary 6.9 *Let A, H be Hopf quasigroups and let $\tau : A\otimes H\to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Then, if (133) and (134) hold, there exist an action φ_A and a coaction ρ_A such that (A,φ_A,ρ_A) is a Hopf quasigroup in ${}^H_H\mathcal{YD}$. Moreover, $A\rtimes H$ and $A\bowtie_\tau H$ are isomorphic Hopf quasigroups in $\mathcal{C}$.*

Proof By the proof of the previous theorem, we know that $(A\bowtie_\tau H, f=\eta_A\otimes H, g=(\tau\otimes\mu_H)\circ(A\otimes R\otimes H))$ is a strong projection over H and (136) holds. Put

$$p_H^{A\bowtie_\tau H}=A\otimes\varepsilon_H,\quad i_H^{A\bowtie_\tau H}=(A\otimes\tau\otimes\lambda_H)\circ(\delta_A\otimes R).$$

Then, $q_H^{A\bowtie_\tau H}=i_H^{A\bowtie_\tau H}\circ p_H^{A\bowtie_\tau H}$ and $p_H^{A\bowtie_\tau H}\circ i_H^{A\bowtie_\tau H}=id_A$ because

$$
\begin{aligned}
&p_H^{A\bowtie_\tau H}\circ i_H^{A\bowtie_\tau H}\\
&=(A\otimes\tau\otimes\varepsilon_H)\circ(\delta_A\otimes R)\;(\text{by }(18))\\
&=(A\otimes\tau)\circ(\delta_A\otimes\eta_H)\;(\text{by (d4) of Definition 6.5})\\
&=id_A\;(\text{by (a3) of Definition 4.1, and counit properties}).
\end{aligned}
$$

Therefore, we can choose $A=(A\bowtie_\tau H)_H$,

$$A\xrightarrow{\;i_H^{A\bowtie_\tau H}\;}A\bowtie_\tau H\;\underset{A\bowtie_\tau H\otimes\eta_H}{\overset{(A\bowtie_\tau H\otimes g)\circ\delta_{A\bowtie_\tau H}}{\rightrightarrows}}\;A\bowtie_\tau H\otimes H$$

is an equalizer diagram and

$$A \bowtie_\tau H \otimes H \underset{A \bowtie_\tau H \otimes \varepsilon_H}{\overset{\mu_{A \bowtie_\tau H} \circ (A \bowtie_\tau H \otimes f)}{\rightrightarrows}} A \bowtie_\tau H \xrightarrow{p_H^{A \bowtie_\tau H}} A$$

is a coequalizer diagram. Moreover, by the general theory developed in [2] [see (128)], we know that $(A, \varphi_A, \varrho_A)$ is a Hopf quasigroup in ${}^H_H\mathcal{YD}$, where φ_A is the action defined in Proposition 5.2, and

$$\rho_A = (\tau \otimes c_{A,H}) \circ (A \otimes c_{A,H} \otimes \mu_H) \circ (\delta_A \otimes (R \circ \tau) \otimes \lambda_H) \circ (\delta_A \otimes R).$$

By (119) and (121), the new magma–comonoid structure of A is

$$u_A = \eta_A, \quad m_A = \mu_A \circ (A \otimes \varphi_A) \circ (i_H^{A \bowtie_\tau H} \otimes A), \quad e_A = \varepsilon_A, \quad \Delta_A = \delta_A,$$

and the antipode s_A [see (129)] admits the following expression $s_A = (\tau \otimes \varphi_A) \circ (A \otimes R \otimes \lambda_A) \circ \delta_A$.

Finally, the isomorphism of Hopf quasigroups $w = \mu_{A \bowtie_\tau H} \circ (i_H^{A \bowtie_\tau H} \otimes f) : A \rtimes H \to A \bowtie_\tau H$ is $w = (A \otimes \mu_H) \circ (i_H^{A \bowtie_\tau H} \otimes H)$. □

Example 6.10 Let H_4 be the four-dimensional Taft Hopf algebra and consider the Hopf quasigroup $A \bowtie_\tau H_4$ constructed in Example 4.12. By [31], we know that H_4 has a one-parameter family of quasitriangular structures R_α defined by

$$R_\alpha = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \frac{\alpha}{2}(y \otimes y - y \otimes w + w \otimes y + w \otimes w).$$

Therefore, we are in the conditions of the previous corollary and, as a consequence, A admits a structure of Hopf quasigroup in the category ${}^{H_4}_{H_4}\mathcal{YD}$. Moreover, $A \bowtie_\tau H_4 \simeq A \rtimes H_4$.

Note that in this case the action on A trivializes because A is cocommutative. As a consequence, this example does not lead to new solutions of the Yang–Baxter equation because the associated braiding with A in ${}^{H_4}_{H_4}\mathcal{YD}$ is the usual twist in the category of vector spaces.

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