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Some Properties of the Schur Multiplier and Stem Covers of Leibniz Crossed Modules

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Abstract

In this article, we investigate the interplay between stem covers, the Schur multiplier of Leibniz crossed modules and the non-abelian exterior product of Leibniz algebras. Explicitly, we obtain a six-term exact sequence associated with a central extension of Leibniz crossed modules, which is useful to characterize stem covers. We show the existence of stem covers and determine the structure of all stem covers of Leibniz crossed modules. Also, we give the connection between the stem cover of a Lie crossed module in the categories of Lie and Leibniz crossed modules, respectively.

Keywords Leibniz algebra · Leibniz crossed module · Schur multiplier · Stem cover · Stem extension

Mathematics Subject Classification 17A32 · 17B55 · 18G05

1 Introduction

Leibniz algebras are algebraic structures introduced by Bloh in [1,2] as a non-skew symmetric generalization of Lie algebras. In the 1990s, Loday rediscovered and developed them [22,23] when he handled periodicity phenomena in algebraic K-theory [24].

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This structure is not only important by algebraic reasons, but also for its applications in other branches such as geometry or physics (see, for instance, [11,18,20,25]).

A Leibniz algebra is a $\mathbb{K}$-vector space $\mathfrak{q}$ equipped with a bilinear map $[-,-] : \mathfrak{q} \times \mathfrak{q} \longrightarrow \mathfrak{q}$ satisfying the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, for all $x, y, z \in \mathfrak{q}$. If we assume $[x,x] = 0$ for all $x \in \mathfrak{q}$, then $\mathfrak{q}$ is a Lie algebra.

An active line of research consists in the extension of properties from Lie algebras to Leibniz algebras. As an example of these generalizations, stem covers and stem extensions of a Leibniz algebra were studied in [10]; in [19] was extended to Leibniz algebras the notion of non-abelian tensor product of Lie algebras introduced by Ellis in [14]; in [12], authors investigated the interplay between the non-abelian tensor and exterior products of Leibniz algebras with the low-dimensional Leibniz homology of Leibniz algebras.

Crossed modules of groups were described for the first time by Whitehead in the late 1940s [32] as an algebraic model for path-connected CW spaces whose homotopy groups are trivial in dimensions greater than 2. Crossed modules of different algebraic objects can be regarded as algebraic structures that generalize simultaneously the notions of normal subobject and module. They were used in many branches of mathematics such as category theory, cohomology of algebraic structures, differential geometry or physics. Also crossed modules were defined in different categories such as Lie algebras and commutative algebras [21,29], either as tools or as algebraic structures in their own right. Leibniz crossed modules were introduced in [24] to study the cohomology of Leibniz algebras. They were also used as coefficients for non-abelian (co)homology of Leibniz algebras in [8]. Since Leibniz crossed modules are generalizations of Lie crossed modules and Leibniz algebras, it is of interest to extend results from Leibniz algebra and Lie crossed modules to Leibniz crossed modules.

Accordingly, in this paper we show that Leibniz crossed modules constitute a semi-abelian category with enough projective objects; hence, the Baer invariant $\frac{(\mathfrak{u},\mathfrak{r},\mu)\cap[(\mathfrak{m},\mathfrak{f},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}{[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}$ associated with the projective presentation $0 \longrightarrow (\mathfrak{u},\mathfrak{r},\mu) \longrightarrow (\mathfrak{m},\mathfrak{f},\mu) \stackrel{(\pi_1,\pi_2)}{\longrightarrow} (\mathfrak{n},\mathfrak{q},\delta) \longrightarrow 0$, called the Schur multiplier of the Leibniz crossed module $(\mathfrak{n},\mathfrak{q},\delta)$ and denoted by $\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)$, plays a central role in the study of connections with the non-abelian exterior product of Leibniz algebras, in the study of stem covers of Leibniz crossed modules and in the study of connections with stem covers of Lie crossed modules.

The paper is organized as follows: in Sect. 2, we recall some basic categorical concepts such as the commutator of two ideals, the center and central extensions of Leibniz crossed modules. Moreover, we show that the category of Leibniz crossed modules has enough projective objects. In Sect. 3, we describe the Schur multiplier of a Leibniz crossed module and analyze its interplay with the non-abelian exterior product of Leibniz algebras given in [12]. Explicitly, we show that $\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta) \cong \mathsf{Ker}\,((\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) \longrightarrow (\mathfrak{n},\mathfrak{q},\delta))$ and we construct the six-term exact sequence

$$(I, \mathfrak{b} \curlywedge \mathfrak{p}, \sigma \curlywedge \mathsf{id}) \longrightarrow \mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathrm{ab}} \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)_{\mathrm{ab}} \longrightarrow 0.$$

associated with the central extension of Leibniz crossed modules $(e) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \xrightarrow{\varphi} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$.

In Sect. 4, we characterize and study properties of stem covers of Leibniz crossed modules through the Schur multiplier; in particular, we show the existence of stem covers for an arbitrary Leibniz crossed module and similarly to a result of Schur in the group case [31], we determine the structure of all stem covers of a Leibniz crossed module whose Schur multiplier is finite dimensional. Finally, in the last section, we study the connections between the stem covers of a Lie crossed module in the categories of Lie and Leibniz crossed modules, respectively.

2 Preliminaries on Leibniz Crossed Modules

This section is devoted to recall some basic definitions on Leibniz crossed modules, which will be needed in the sequel.

Definition 1 [24] Let $\mathfrak{m}$ and $\mathfrak{n}$ be Leibniz algebras. A Leibniz action of $\mathfrak{m}$ on $\mathfrak{n}$ is a couple of bilinear maps $\mathfrak{m} \times \mathfrak{n} \longrightarrow \mathfrak{n}$, $(m, n) \longmapsto {}^m n$, and $\mathfrak{n} \times \mathfrak{m} \longrightarrow \mathfrak{n}$, $(n, m) \longmapsto n^m$, satisfying the following axioms:

$$\begin{aligned} {}^{[m,m']}n &= {}^m({}^{m'}n) + ({}^m n)^{m'}, & {}^m[n, n'] &= [{}^m n, n'] - [{}^m n', n], \\ n^{[m,m']} &= (n^m)^{m'} - (n^{m'})^m, & [n, n']^m &= [n^m, n'] + [n, n'^m], \\ {}^m({}^{m'}n) &= -{}^m(n^{m'}), & [n, {}^m n'] &= -[n, n'^m], \end{aligned}$$

for each $m, m' \in \mathfrak{m}$, $n, n' \in \mathfrak{n}$.

Definition 2 [24] A Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is a homomorphism of Leibniz algebras $\delta : \mathfrak{n} \longrightarrow \mathfrak{q}$ together with a Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$ such that

(i) $\delta({}^q n) = [q, \delta(n)], \qquad \delta(n^q) = [\delta(n), q]$,
(ii) ${}^{\delta(n_1)}n_2 = [n_1, n_2] = {n_1}^{\delta(n_2)}$,

for all $q \in \mathfrak{q}$ and $n_1, n_2 \in \mathfrak{n}$.

Example 1 (i) Let $\mathfrak{n}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$, then $(\mathfrak{n}, \mathfrak{q}, i)$ is a Leibniz crossed module, where i is the inclusion map and the Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$ is given by the Leibniz bracket. In this way, every Leibniz algebra $\mathfrak{q}$ can be regarded as crossed module in the two obvious ways $(0, \mathfrak{q}, i)$ and $(\mathfrak{q}, \mathfrak{q}, \mathsf{id})$.
(ii) For any $\mathfrak{q}$-module $\mathfrak{m}$ the trivial map $0 : \mathfrak{m} \to \mathfrak{q}$ is a crossed module with the trivial action of $\mathfrak{q}$ on the abelian Leibniz algebra $\mathfrak{m}$.
(iii) Any homomorphism of Leibniz algebras $\delta : \mathfrak{n} \to \mathfrak{q}$, with $\mathfrak{n}$ abelian and $\mathsf{Im}(\delta)$ in the center of $\mathfrak{q}$, provides a crossed module with $\mathfrak{q}$ acting trivially on $\mathfrak{n}$.

Definition 3 [6] A homomorphism of Leibniz crossed modules, $(\varphi, \psi) : (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow (\mathfrak{n}', \mathfrak{q}', \delta')$, is a pair of Leibniz algebra homomorphisms $\varphi : \mathfrak{n} \longrightarrow \mathfrak{n}'$ and $\psi : \mathfrak{q} \longrightarrow \mathfrak{q}'$ such that $\psi \circ \delta = \delta' \circ \varphi$ and φ preserves the Leibniz action of $\mathfrak{q}$ via ψ, i.e., $\varphi({}^{q}n) = {}^{\psi(q)}\varphi(n)$ and $\varphi(n^{q}) = \varphi(n)^{\psi(q)}$, for all $n \in \mathfrak{n}$ and $q \in \mathfrak{q}$.

A homomorphism of crossed modules (φ, ψ) is called injective if both φ and ψ are injective homomorphisms of Leibniz algebras. Also, (φ, ψ) is called surjective if φ and ψ are onto maps.

It is clear that Leibniz crossed modules constitute a category, which is denoted by **XLb**. Theorem 10 in [8], in the particular case $n = 2$, provides the equivalence between the categories **XLb** and **Cat**^{1}**-Lb** of cat^1-Leibniz algebras (see also [6]). Moreover, [4] shows that **Cat**^{1}**-Lb** is a modified category of interest, which is a semi-abelian category. Hence, **XLb** is a semi-abelian category (see also [6]). Subobjects and normal subobjects in **XLb** are the crossed submodules and ideals of a crossed module, that is, $(\mathfrak{m}, \mathfrak{g}, \partial)$ is a crossed submodule of a crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ if $\mathfrak{m}$ is a subalgebra of $\mathfrak{n}$, $\mathfrak{g}$ is a subalgebra of $\mathfrak{q}$, $\partial = \delta_{|\mathfrak{m}}$ and the Leibniz action of $\mathfrak{g}$ on $\mathfrak{m}$ is the restriction of the Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$. A crossed submodule $(\mathfrak{m}, \mathfrak{g}, \partial)$ is said to be an ideal of $(\mathfrak{n}, \mathfrak{q}, \delta)$ when ${}^{g}n, n^{g} \in \mathfrak{m}$, for all $g \in \mathfrak{g}, n \in \mathfrak{n}$, and ${}^{q}m, m^{q} \in \mathfrak{m}$, for all $m \in \mathfrak{m}, q \in \mathfrak{q}$.

According to [17] (see also [15,16]), we have the following notions corresponding to the category **XLb**:

- The *commutator* of two ideals $(\mathfrak{s}, \mathfrak{h}, \delta)$ and $(\mathfrak{t}, \mathfrak{j}, \delta)$ of a Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is the ideal

$$[(\mathfrak{s}, \mathfrak{h}, \delta), (\mathfrak{t}, \mathfrak{j}, \delta)] = \big(< D_{\mathfrak{h}}(\mathfrak{t}), D_{\mathfrak{j}}(\mathfrak{s}) >, [\mathfrak{h}, \mathfrak{j}], \delta_{|}\big)$$

  where $D_{\mathfrak{h}}(\mathfrak{t}) = \{{}^{h}t, t^{h} \mid h \in \mathfrak{h}, t \in \mathfrak{t}\}$ and $D_{\mathfrak{j}}(\mathfrak{s}) = \{{}^{j}s, s^{j} \mid j \in \mathfrak{j}, s \in \mathfrak{s}\}$.
- In particular, the *derived crossed module* of a crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is

$$(\mathfrak{n}, \mathfrak{q}, \delta)' = [(\mathfrak{n}, \mathfrak{q}, \delta), (\mathfrak{n}, \mathfrak{q}, \delta)] = \big(D_{\mathfrak{q}}(\mathfrak{n}), [\mathfrak{q}, \mathfrak{q}], \delta_{|}\big)$$

- Following [5], the ideal $(\mathfrak{n}, \mathfrak{q}, \delta)^{\text{ann}}$ is the crossed submodule $(\overline{\mathfrak{n}}, \mathfrak{q}_{\mathsf{Lie}}, \bar{\delta})$, where $\overline{\mathfrak{n}}$ is the ideal of $\mathfrak{n}$ generated by all elements $[n, n]$ and $[q, n] + [n, q]$, $q \in \mathfrak{q}, n \in \mathfrak{n}$, and $\mathfrak{q}^{\text{ann}}$ is the ideal of $\mathfrak{q}$ generated by all elements $[q, q]$ for $q \in \mathfrak{q}$. Moreover, $(\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{Lie}} = (\mathfrak{n}, \mathfrak{q}, \delta)/(\mathfrak{n}, \mathfrak{q}, \delta)^{\text{ann}}$ is a Lie crossed module.
- Following [6], the ideal $Z(\mathfrak{n}, \mathfrak{q}, \delta) = \big(\mathfrak{n}^{\mathfrak{q}}, st_{\mathfrak{q}}(\mathfrak{n}) \cap Z(\mathfrak{q}), \delta_{|}\big)$ is the *center* of the crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$, where $Z(\mathfrak{q})$ is center of $\mathfrak{q}$, $\mathfrak{n}^{\mathfrak{q}} = \{n \in \mathfrak{n} \mid {}^{q}n = n^{q} = 0$, for all $q \in q\}$ and $st_{\mathfrak{q}}(\mathfrak{n}) = \{q \in \mathfrak{q} \mid {}^{q}n = n^{q} = 0$, for all $n \in \mathfrak{n}\}$.
- An extension of Leibniz crossed modules $(e) : 0 \to (\mathfrak{a}, \mathfrak{b}, \sigma) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \to (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ is said to be central if $(\mathfrak{a}, \mathfrak{b}, \sigma) \subseteq Z(\mathfrak{h}, \mathfrak{p}, \sigma)$, equivalently $[(\mathfrak{a}, \mathfrak{b}, \sigma), (\mathfrak{h}, \mathfrak{p}, \sigma)] = 0$.
- A Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is said to be **finite dimensional** if the Leibniz algebras $\mathfrak{n}$ and $\mathfrak{q}$ are both finite dimensional.
- A Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is **perfect** if it coincides with its commutator crossed submodule and it is **abelian** if it coincides with its center. It is easy to show

that $(\mathfrak{n}, \mathfrak{q}, \delta)$ is abelian if and only if $\mathfrak{n}$ and $\mathfrak{q}$ are abelian Leibniz algebras and the Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$ is trivial. We will denote the category of abelian crossed modules by **AbXmod**. Obviously, $(\mathfrak{n}, \mathfrak{q}, \delta)_{\mathrm{ab}} = (\mathfrak{n}, \mathfrak{q}, \delta)/[(\mathfrak{n}, \mathfrak{q}, \delta), (\mathfrak{n}, \mathfrak{q}, \delta)] = \big(\mathfrak{n}/D_{\mathfrak{q}}(\mathfrak{n}), \mathfrak{q}/[\mathfrak{q}, \mathfrak{q}], \overline{\delta}\big)$ is an abelian crossed module called the abelianization of $(\mathfrak{n}, \mathfrak{q}, \delta)$.

Theorem 1 **XLb** *is a category with enough projective objects.*

Proof There is a faithful functor $\mathbf{U}_1 : \mathbf{XLb} \to \mathbf{Lb}$, which assigns to a Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ the direct product of Leibniz algebras $\mathfrak{n} \times \mathfrak{q}$. Now we define the functor $\mathbf{F}_1 : \mathbf{Lb} \to \mathbf{XLb}$ that assigns to any Leibniz algebra $\mathfrak{h}$ the inclusion crossed module $(\overline{\mathfrak{h}}, \mathfrak{h} * \mathfrak{h}, inc)$, where $*$ is the coproduct of Leibniz algebras, with the natural inclusions $i_1, i_2 : \mathfrak{h} \to \mathfrak{h} * \mathfrak{h}$, and $\overline{\mathfrak{h}}$ is the kernel of the retraction $p_2 : \mathfrak{h} * \mathfrak{h} \to \mathfrak{h}$ determined by the conditions $p_2 \circ i_1 = 0$ and $p_2 \circ i_2 = \mathrm{id}_{\mathfrak{h}}$. A direct adaptation of the proof of [7, Proposition 2.1.1] to Leibniz algebras case shows that $\mathbf{F_1}$ is left adjoint to $\mathbf{U_1}$.

Now consider the forgetful functor $\mathbf{U_2} : \mathbf{Lb} \to \mathbf{Set}$ that assigns to a Leibniz algebra $\mathfrak{q}$ its underlying set. It is well known (see, for instance, [12]) that $\mathbf{U_2}$ has as left adjoint the free Leibniz algebra functor $\mathbf{F_2} : \mathbf{Set} \to \mathbf{Lb}$.

Hence, the composition $(\mathbf{F}, \mathbf{U}) = (\mathbf{F}_1 \circ \mathbf{F_2}, \mathbf{U_2} \circ \mathbf{U_1})$ is an adjoint pair, so the free Leibniz crossed module $\mathbf{F}(X)$, for $X \in \mathbf{Set}$, is a projective object with respect to regular epimorphisms in **XLb** and any Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ admits a projective presentation by means of the counit of the adjunction $\mathbf{FU}(\mathfrak{n}, \mathfrak{q}, \delta) \twoheadrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)$. □

The following lemma is useful in our investigation.

Lemma 1 (i) *An abelian crossed module (A, B, μ) is projective in the category* **AbXmod** *if and only if μ is injective.*

(ii) *Let $(\mathfrak{m}, \mathfrak{f}, \mu)$ be a projective Leibniz crossed module, then:*

(a) *any crossed submodule of $(\mathfrak{m}, \mathfrak{f}, \mu)_{\mathrm{ab}}$ is projective in* **AbXmod**.

(b) *the homomorphism μ is injective, $\mathfrak{f}$ and $\mathfrak{f}/\mu(\mathfrak{m})$ are projective Leibniz algebras and $HL_i(\mathfrak{f}/\mu(\mathfrak{m})) = 0, i \geq 2$.*

Proof (i) We can consider (A, B, μ) as a Leibniz crossed module. According to Theorem 1, (A, B, μ) admits a projective presentation by means of the counit of the adjunction $\mathbf{FU}(A, B, \mu) \overset{(\pi_1,\pi_2)}{\twoheadrightarrow} (A, B, \mu)$. If (A, B, μ) is projective, then the morphism (π_1, π_2) is split. Thus, (A, B, μ) is isomorphic to a crossed submodule of $\mathbf{FU}(A, B, \mu) = (\overline{F(X)}, F(X) * F(X), inc)$, so μ is injective.

Conversely, let (A, B, μ) be aspherical (that is μ is injective), and X, Y be the basis of A and B, respectively. We may assume that $X \subseteq Y$. Let $(\delta_1, \delta_2) : (A, B, \mu) \longrightarrow (T_2, L_2, \sigma_2)$ and $(\varepsilon_1, \varepsilon_2) : (T_1, L_1, \sigma_1) \twoheadrightarrow (T_2, L_2, \sigma_2)$ be homomorphisms of crossed modules in **AbXmod**, such that $(\varepsilon_1, \varepsilon_2)$ is surjective. There is a homomorphism $\theta_1 : A \longrightarrow T_1$ such that $\varepsilon_1 \circ \theta_1 = \delta_1$. We define the map $h : Y \longrightarrow L_1$, as $h(x) = \sigma_1 \circ \theta_1(x)$ if $x \in X$, otherwise $h(x) = l_x$, where l_x is a preimage of $\delta_2(x)$ via ε_2, that is $\varepsilon_2(l_x) = \delta_2(x)$. Then, h extends to a homomorphism $\theta_2 : B \longrightarrow L_2$. It is readily verified that $(\theta_1, \theta_2) : (A, B, \mu) \longrightarrow (T_1, L_1, \sigma)$ is a homomorphism of crossed modules.

(ii) (a) The abelianization functor **Ab** : **XLb** $\longrightarrow$ **AbXmod**, **Ab**$(\mathfrak{n}, \mathfrak{q}, \delta) = (\mathfrak{n}, \mathfrak{q}, \delta)_{\text{ab}}$ is left adjoint of the inclusion functor inc : **AbXmod** $\longrightarrow$ **XLb**. Since the inclusion functor preserves epimorphisms, **Ab** preserves projective objects. Now, if $(\mathfrak{m}, \mathfrak{f}, \mu)$ is a projective Leibniz crossed module, then $(\mathfrak{m}, \mathfrak{f}, \mu)_{\text{ab}}$ is also projective in the category **AbXmod** and the result follows.

(b) The homomorphism μ is injective by applying an argument similar to the proof of statement (i). Now, assume there exist the homomorphisms of Leibniz algebras $g : \mathfrak{q} \twoheadrightarrow \mathfrak{q}_2$ and $h : \mathfrak{f}/\mu(\mathfrak{m}) \longrightarrow \mathfrak{q}_2$. So, we have the induced morphisms of crossed modules $(0, g) : (0, \mathfrak{q}_1, i) \twoheadrightarrow (0, \mathfrak{q}_2, i)$ and $(0, h \circ \pi) : (\mathfrak{m}, \mathfrak{f}, \mu) \longrightarrow (0, \mathfrak{q}_2, i)$. By the assumptions, there is a morphism of crossed modules $(\beta_1, \beta_2) : (\mathfrak{m}, \mathfrak{f}, \mu) \longrightarrow (0, \mathfrak{q}_1, i)$, such that $g \circ \beta_2 = h \circ \pi$. Since $\beta_2(\mu(\mathfrak{m})) = 0$, $\overline{\beta} : \mathfrak{f}/\mu(\mathfrak{m}) \longrightarrow \mathfrak{q}_1$ is induced by β_2. It is easy to check that $g \circ \overline{\beta} = h$, so $\mathfrak{f}/\mu(\mathfrak{m})$ is projective. Similarly, $\mathfrak{f}$ is projective. Finally, since $HL_*(\mathfrak{f}/\mu(\mathfrak{m})) = TOR_*^{UL(\mathfrak{f}/\mu(\mathfrak{m}))}\left(U\left((\mathfrak{f}/\mu(\mathfrak{m}))_{\text{Lie}}\right)\right)$ [24, Theorem 3.4], and $(\mathfrak{f}/\mu(\mathfrak{m}))_{\text{Lie}}$ is a projective Lie algebra, $U\left((\mathfrak{f}/\mu(\mathfrak{m}))_{\text{Lie}}\right)$ is projective; then, the result follows for any $i \geq 2$. □

Remark 1 Let $\mathfrak{q}$ be a projective Leibniz algebra. Then, Lemma 1 (ii) applied to the Leibniz crossed module $(0, \mathfrak{q}, i)$ implies that $HL_i(\mathfrak{q}) = 0$ for $i \geq 2$.

3 Schur Multiplier of Leibniz Crossed Modules

Due to Theorem 1, any Leibniz crossed module has a projective presentation $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \xrightarrow{(\pi_1,\pi_2)} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ and following [16, Theorem 6.9 and Corollary 6.10] the quotient

$$\frac{(\mathfrak{u}, \mathfrak{r}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]}{[(\mathfrak{u}, \mathfrak{r}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]}$$

is a Baer invariant, which means it does not depend on the chosen projective presentation. By analogy with other algebraic theories, we call this term *Schur multiplier* of the Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ and we denote it by $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$.

Remark 2 Let $\mathfrak{q}$ be any Leibniz algebra, and $0 \to (\mathfrak{m}, \mathfrak{r}, \mu) \to (\mathfrak{m}, \mathfrak{f}, \mu) \to (0, \mathfrak{q}, i) \to 0$ be a projective presentation of the Leibniz crossed module $(0, \mathfrak{q}, i)$. Then,

$$\mathcal{M}(0, \mathfrak{q}, i) \cong \left(0, \frac{\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]}, \overline{\mu}\right)$$

On the other hand, by the proof of Theorem 1, there is a projective presentation $0 \to (\mathfrak{m}_1, \mathfrak{r}_1, \mu_1) \to (\mathfrak{m}_1, \mathfrak{f}_1, \mu_1) \to (0, \mathfrak{q}, i) \to 0$ of $(0, \mathfrak{q}, i)$ such that $(\mathfrak{m}_1, \mathfrak{f}_1, \mu_1)$ and $\mathfrak{f}_1$ are free objects in **XLb** and **Lb**, respectively. So, $0 \to \mathfrak{r}_1 \to \mathfrak{f}_1 \to \mathfrak{q} \to 0$ is a free presentation of $\mathfrak{q}$, implying that $\mathcal{M}(\mathfrak{q}) \cong \left(\frac{\mathfrak{r}_1 \cap [\mathfrak{f}_1, \mathfrak{f}_1]}{[\mathfrak{r}_1, \mathfrak{f}_1]}\right) \cong HL_2(\mathfrak{q})$. We therefore conclude that $\mathcal{M}(0, \mathfrak{q}, i) \cong (0, HL_2(\mathfrak{q}), i)$.

With a similar reasoning, we can conclude that $\mathcal{M}(\mathfrak{q}, \mathfrak{q}, \mathrm{id}) \cong (HL_2(\mathfrak{q}), HL_2(\mathfrak{q}), \mathrm{id})$.

Moreover, associated with a short exact sequence of Leibniz crossed modules (e) : $0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \stackrel{(i_1,i_2)}{\longrightarrow} (\mathfrak{h}, \mathfrak{p}, \sigma) \stackrel{(f_1,f_2)}{\longrightarrow} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ there exists the following five-term exact sequence:

$$\mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) \to \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \stackrel{\theta_*(e)}{\to} \frac{(\mathfrak{a}, \mathfrak{b}, \sigma)}{[(\mathfrak{a}, \mathfrak{b}, \sigma), (\mathfrak{h}, \mathfrak{p}, \sigma)]} \to (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathrm{ab}} \to (\mathfrak{n}, \mathfrak{q}, \mu)_{\mathrm{ab}} \to 0. \quad (1)$$

3.1 Non-abelian Tensor and Exterior Product

Definition 4 [19] Let $\mathfrak{m}$ and $\mathfrak{n}$ be Leibniz algebras with mutual Leibniz actions on each other. The non-abelian tensor product of $\mathfrak{m}$ and $\mathfrak{n}$, denoted by $\mathfrak{m}\star\mathfrak{n}$, is the Leibniz algebra generated by the symbols $m * n$ and $n * m$, for all $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$, subject to the following relations:

$(1a)\ k(m * n) = km * n = m * kn,$
$(1b)\ k(n * m) = kn * m = n * km,$
$(2a)\ (m + m') * n = m * n + m' * n,$
$(2b)\ (n + n') * m = n * m + n' * m,$
$(2c)\ m * (n + n') = m * n + m * n',$
$(2d)\ n * (m + m') = n * m + n * m',$
$(3a)\ m * [n, n'] = m^n * n' - m^{n'} * n,$
$(3b)\ n * [m, m'] = n^m * m' - n^{m'} * m$
$(3c)\ [m, m'] * n =^m n * m' - m * n^{m'},$
$(3d)\ [n, n'] * m = {}^n m * n' - n * m^{n'},$
$(4a)\ m *^{m'} n = -m * n^{m'},$
$(4b)\ n *^{n'} m = -n * m^{n'},$
$(5a)\ m^n *^{m'} n' = [m * n, m' * n'] =^m n * m'^{n'},$
$(5b)\ {}^n m * n'^{m'} = [n * m, n' * m'] = n^m *^{n'} m',$
$(5c)\ m^n * n'^{m'} = [m * n, n' * m'] =^m n *^{n'} m',$
$(5d)\ {}^n m *^{m'} n' = [n * m, m' * n'] = n^m * m'^{n'},$

for all $k \in \mathbb{K}$, $m, m' \in \mathfrak{m}$ and $n, n' \in \mathfrak{n}$.

Let us consider two Leibniz crossed modules $\eta : \mathfrak{m} \longrightarrow \mathfrak{q}$ and $\delta : \mathfrak{n} \longrightarrow \mathfrak{q}$. Then, there are induced Leibniz actions of $\mathfrak{m}$ and $\mathfrak{n}$ on each other via the action of $\mathfrak{q}$. Therefore, we can consider the non-abelian tensor product $\mathfrak{m}\star\mathfrak{n}$. In [12] is defined $\mathfrak{m}\square\mathfrak{n}$ as the vector subspace of $\mathfrak{m}\star\mathfrak{n}$ generated by the elements $m * n' - n * m'$ such that $\eta(m) = \delta(n)$ and $\eta(m') = \delta(n')$. The vector subspace $\mathfrak{m}\square\mathfrak{n}$ is contained in the center of $\mathfrak{m}\star\mathfrak{n}$, so in particular it is an ideal of $\mathfrak{m}\star\mathfrak{n}$ [12, Proposition 1].

Definition 5 [12] The non-abelian exterior product $\mathfrak{m} \curlywedge \mathfrak{n}$ of $\mathfrak{m}$ and $\mathfrak{n}$ is the quotient

$$\mathfrak{m} \curlywedge \mathfrak{n} = \frac{\mathfrak{m}\star\mathfrak{n}}{\mathfrak{m}\square\mathfrak{n}}.$$

The cosets of $m * n$ and $n * m$ will be denoted by $m \curlywedge n$ and $n \curlywedge m$, respectively.

Given a crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$, by the Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$ and the Leibniz action of $\mathfrak{n}$ on $\mathfrak{q}$, given by δ, we can form the non-abelian tensor products $\mathfrak{q}\star\mathfrak{n}$ and $\mathfrak{q}\star\mathfrak{q}$. As explained in [19, Proposition 4.3], the homomorphisms $\lambda_{\mathfrak{q}} : \mathfrak{q}\star\mathfrak{n} \to \mathfrak{q}$, $\lambda_{\mathfrak{q}}(q * n) = {}^q n$, $\lambda_{\mathfrak{q}}(n * q) = n^q$ and $\mu_{\mathfrak{q}} : \mathfrak{q}\star\mathfrak{q} \to \mathfrak{q}$, $\mu_{\mathfrak{q}}(q * q') = [q, q']$ are Leibniz crossed modules, where the Leibniz action of $\mathfrak{q}$ on $\mathfrak{q}\star\mathfrak{n}$ is given by:

$$ {}^q(q' * n') = [q, q'] * n' - {}^q n' * q', \quad {}^q(n' * q') = {}^q n' * q' - [q, q'] * n',$$
$$ (q' * n')^q = [q', q] * n' + q' * n'^q, \quad (n' * q')^q = n'^q * q' + n' * [q', q].$$

The Leibniz action of $\mathfrak{q}$ on $\mathfrak{q}\star\mathfrak{q}$ is defined similarly. It is apparent that $\lambda_{\mathfrak{q}}(\mathfrak{q}\Box\mathfrak{n}) = 0$ and $\mu_{\mathfrak{q}}(\mathfrak{q}\Box\mathfrak{q}) = 0$, so the induced homomorphisms $\overline{\lambda}_{\mathfrak{q}} : \mathfrak{q} \curlywedge \mathfrak{n} \to \mathfrak{q}$ and $\overline{\mu}_{\mathfrak{q}} : \mathfrak{q} \curlywedge \mathfrak{q} \to \mathfrak{q}$ are Leibniz crossed modules.

Remark 3 Given homomorphisms of Leibniz algebras $\varphi_1 : \mathfrak{m} \longrightarrow \mathfrak{n}$ and $\varphi_2 : \mathfrak{p} \longrightarrow \mathfrak{q}$ such that $\mathfrak{m}$ and $\mathfrak{p}$, respectively, $\mathfrak{n}$ and $\mathfrak{q}$, have mutual Leibniz actions on each other, then there is an induced homomorphism $\varphi_1 \curlywedge \varphi_2 : \mathfrak{m} \curlywedge \mathfrak{p} \longrightarrow \mathfrak{n} \curlywedge \mathfrak{q}$ defined by $(\varphi_1 \curlywedge \varphi_2)(m \curlywedge p) = \varphi_1(m) \curlywedge \varphi_2(p)$, $(\varphi_1 \curlywedge \varphi_2)(p \curlywedge m) = \varphi_2(p) \curlywedge \varphi_1(m)$.

Proposition 1 *Let* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *be a Leibniz crossed module. Then, the following statements hold:*

(i) *There is a Leibniz action of* $\mathfrak{q} \curlywedge \mathfrak{q}$ *on* $\mathfrak{q} \curlywedge \mathfrak{n}$ *defined by* ${}^{x}y = {}^{\overline{\mu_{\mathfrak{q}}}(x)}y$ *and* $y^x = y^{\overline{\mu_{\mathfrak{q}}}(x)}$, *for all* $x \in \mathfrak{q} \curlywedge \mathfrak{q}$ *and* $y \in \mathfrak{q} \curlywedge \mathfrak{n}$.
(ii) *The map* $\mathsf{id} \curlywedge \delta : \mathfrak{q} \curlywedge \mathfrak{n} \longrightarrow \mathfrak{q} \curlywedge \mathfrak{q}$ *with the Leibniz action defined in statement (i) is a Leibniz crossed module.*
(iii) *There is a homomorphism* $\phi = (\bar{\lambda}_{\mathfrak{n}}, \bar{\mu}_{\mathfrak{q}}) : (\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)$ *such that* $\mathsf{Ker}(\phi) \subseteq Z(\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta)$.

Proof For statement (i), thanks to the Leibniz action of $\mathfrak{q}$ on $\mathfrak{q} \curlywedge \mathfrak{n}$, everything can be easily checked.

For statement (ii), it immediately follows, by using relations (5a)–(5d) in Definition 4, that $\mathsf{id} \curlywedge \delta$ is a homomorphism of Leibniz algebras. Also, by using the defining conditions of Leibniz crossed module and Leibniz action of $\mathfrak{q}$ on $\mathfrak{q} \curlywedge \mathfrak{n}$, it is readily checked that $(\mathsf{id} \curlywedge \delta)({}^{x}y) = [x, \mathsf{id} \curlywedge \delta(y)]$ and $(\mathsf{id} \curlywedge \delta)(y^x) = [\mathsf{id} \curlywedge \delta(y), x]$ for all $x \in \mathfrak{q} \curlywedge \mathfrak{q}$, $y \in \mathfrak{q} \curlywedge \mathfrak{n}$.

Now we indicate that ${}^{\mathsf{id}\curlywedge\delta(y_1)}y_2 = [y_1, y_2] = y_1^{\mathsf{id}\curlywedge\delta(y_2)}$ for all $y_1, y_2 \in \mathfrak{q} \curlywedge \mathfrak{n}$. Let $y_i = q_i \curlywedge n_i$, for $i = 1, 2$, then we have

$$\begin{aligned}
{}^{q_1\curlywedge\delta(n_1)}(q_2 \curlywedge n_2) &= {}^{[q_1,\delta(n_1)]}(q_2 \curlywedge n_2)\\
&= [[q_1, \delta(n_1)], q_2] \curlywedge n_2 - {}^{[q_1,\delta(n_1)]} n_2 \curlywedge q_2\\
&= -[q_1, \delta(n_1)] \curlywedge n_2^{q_2}\\
&= -{q_1}^{n_1} \curlywedge {n_2}^{q_2}\\
&= [q_1 \curlywedge n_1, q_2 \curlywedge n_2]\\
&= {}^{q_1}n_1 \curlywedge {q_2}^{n_2}\\
&= [q_1, [q_2, \delta(n_2)]] \curlywedge n_1 + q_1 \curlywedge n_1^{[q_2,\delta(n_2)]}\\
&= (q_1 \curlywedge n_1)^{q_2\curlywedge\delta(n_2)}.
\end{aligned}$$

For the other generators, it can proved in a similar way, so we obtain the result.

For statement (iii), it is easy to check that ϕ is a crossed module homomorphism. To show that $\mathsf{Ker}(\phi) \subseteq Z(\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta)$, let $x \in \mathsf{Ker}(\bar{\lambda}_n)$. We may assume that $x = q \curlywedge n$. So for any $q_1 \curlywedge q_2 \in \mathfrak{q} \curlywedge \mathfrak{q}$, we have:

$$ {}^{q_1\curlywedge q_2}(q \curlywedge n) = {}^{[q_1,q_2]}(q \curlywedge n) = [[q_1, q_2], q] \curlywedge n - {}^{[q_1,q_2]} n \curlywedge q = [q_1, q_2] \curlywedge {}^{q} n = 0, $$

by relations (3c) and (4a) in Definition 4. Other generators can be proved similarly, so we conclude that $\mathsf{Ker}(\bar{\lambda}_n) \subseteq (\mathfrak{q} \curlywedge \mathfrak{n})^{\mathfrak{q} \curlywedge \mathfrak{q}}$.

Also, if $q_1 \curlywedge q_2 \in \mathsf{Ker}(\bar{\mu}_q)$, then $[q_1, q_2] = 0$. So it is an easy task to check that $\mathsf{Ker}(\bar{\mu}_q) \subseteq Z(\mathfrak{q} \curlywedge \mathfrak{q}) \cap st_{\mathfrak{q} \curlywedge \mathfrak{q}}(\mathfrak{q} \curlywedge \mathfrak{n})$, and the result follows. □

Lemma 2 *Let* $(\mathfrak{h}, \mathfrak{p}, \sigma)$ *be a Leibniz crossed module and* $(\mathfrak{a}, \mathfrak{b}, \sigma)$ *be an ideal of* $(\mathfrak{h}, \mathfrak{p}, \sigma)$ *such that* $(\mathfrak{a}, \mathfrak{b}, \sigma) \subseteq Z(\mathfrak{h}, \mathfrak{p}, \sigma)$. *Then, the map*

$$\sigma \curlywedge \mathsf{id} : I \longrightarrow \mathfrak{b} \curlywedge \mathfrak{p}$$

is an abelian Leibniz crossed module, where I *is the ideal of* $\mathfrak{p} \curlywedge \mathfrak{h}$ *generated by all elements* $p \curlywedge a, a \curlywedge p, b \curlywedge h$ *and* $h \curlywedge b$ *for any* $p \in \mathfrak{p}, a \in \mathfrak{a}, b \in \mathfrak{b}$ *and* $h \in \mathfrak{h}$.

Proof By the assumption $\mathfrak{b} \subseteq Z(\mathfrak{p}) \cap st_{\mathfrak{p}}(\mathfrak{h})$ and $\mathfrak{a} \subseteq \mathfrak{h}^{\mathfrak{p}}$, so by relation (5c) in Definition 4, we have

$$\begin{aligned} &[b \curlywedge p, p' \curlywedge b'] = [b, p] \curlywedge [p', b'] = 0, && [a \curlywedge p, p' \curlywedge a'] = a^p \curlywedge p'^{a'} = 0, \\ &[b \curlywedge h, b' \curlywedge h'] = {}^b h \curlywedge b'^{h'} = 0, && [a \curlywedge p, p' \curlywedge h'] = a^p \curlywedge p'^{h'} = 0, \end{aligned}$$

for all $b, b' \in \mathfrak{b}$, $p, p' \in \mathfrak{p}$, $a, a' \in \mathfrak{a}$, and $h, h' \in \mathfrak{h}$. Therefore, I and $\mathfrak{b} \curlywedge \mathfrak{p}$ are abelian Leibniz algebras. Evidently, the canonical homomorphism $\sigma \curlywedge \mathsf{id}$ is an abelian Leibniz crossed module. □

Lemma 3 *Let* $\varphi = (\varphi_1, \varphi_2) : (\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)$ *be a surjective homomorphism of Leibniz crossed modules. Then* $\varphi \curlywedge \varphi = (\varphi_2 \curlywedge \varphi_1, \varphi_2 \curlywedge \varphi_2) : (\mathfrak{p} \curlywedge \mathfrak{h}, \mathfrak{p} \curlywedge \mathfrak{p}, \mathsf{id} \curlywedge \sigma) \longrightarrow (\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta)$ *is also a surjective homomorphism of Leibniz crossed modules.*

Proof Obviously, homomorphism (φ_1, φ_2) induces surjective homomorphisms of Leibniz algebras $\varphi_2 \curlywedge \varphi_1 : \mathfrak{p} \curlywedge \mathfrak{h} \longrightarrow \mathfrak{q} \curlywedge \mathfrak{n}$ and $\varphi_2 \curlywedge \varphi_2 : \mathfrak{p} \curlywedge \mathfrak{p} \longrightarrow \mathfrak{q} \curlywedge \mathfrak{q}$. It is easy to check that $(\varphi_2 \curlywedge \varphi_1) \circ (\mathsf{id} \curlywedge \sigma) = (\mathsf{id} \curlywedge \delta) \circ (\varphi_2 \curlywedge \varphi_2)$ and $\varphi_2 \curlywedge \varphi_1$ preserves the action of crossed module via $\varphi_2 \curlywedge \varphi_2$, for instance

$$\begin{aligned} &\varphi_2 \curlywedge \varphi_1 \left({}^{p_1 \curlywedge p_2} h \curlywedge p\right) \\ &\quad = \varphi_2 \curlywedge \varphi_1 \left({}^{[p_1, p_2]} h \curlywedge p\right) \\ &\quad = \varphi_2 \curlywedge \varphi_1 \left({}^{[p_1, p_2]} h \curlywedge p - [[p_1, p_2], p] \curlywedge h\right) \\ &\quad = {}^{\varphi_2([p_1, p_2])} \varphi_1(h) \curlywedge \varphi_2(p) - [[\varphi_2(p_1), \varphi_2(p_2)], \varphi_2(p)] \curlywedge \varphi_1(h) \\ &\quad = {}^{\varphi_2(p_1) \curlywedge \varphi_2(p_2)} (\varphi_1(h) \curlywedge \varphi_2(p)). \end{aligned}$$

Therefore, it is a homomorphism of crossed modules, as required. □

Remark 4 Under the assumptions of Lemma 3, let $\mathsf{Ker}(\varphi_1, \varphi_2) = (\mathfrak{a}, \mathfrak{b}, \sigma)$, then we have the natural induced map of Leibniz algebras $\psi_1 : \mathfrak{p} \curlywedge \mathfrak{a} + \mathfrak{b} \curlywedge \mathfrak{h} \longrightarrow \mathfrak{p} \curlywedge \mathfrak{h}$ and $\psi_2 : \mathfrak{p} \curlywedge \mathfrak{b} \longrightarrow \mathfrak{p} \curlywedge \mathfrak{p}$, such that $\mathsf{Im}(\psi_1) = \mathsf{Ker}(\varphi_2 \curlywedge \varphi_1)$ and $\mathsf{Im}(\psi_2) = \mathsf{Ker}(\varphi_2 \curlywedge \varphi_2)$ (see [12]). So we may assume that the $\mathsf{Ker}(\varphi_2 \curlywedge \varphi_1)$ is an ideal of $\mathfrak{p} \curlywedge \mathfrak{h}$ generated by all elements $p \curlywedge a, a \curlywedge p, b \curlywedge h$ and $h \curlywedge b$ for any $p \in \mathfrak{p}, a \in \mathfrak{a}, h \in \mathfrak{h}$ and $b \in \mathfrak{b}$. Moreover, $\mathsf{Ker}(\varphi_2 \curlywedge \varphi_2)$ is an ideal of $\mathfrak{p} \curlywedge \mathfrak{p}$ generated by $p \curlywedge b$ and $b \curlywedge p$ for all $p \in \mathfrak{p}$ and $b \in \mathfrak{b}$.

3.2 Connections Between the Schur Multiplier and the Non-abelian Exterior Product

The following result shows the connection between a projective presentation of the given crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ and the non-abelian exterior product of Leibniz algebras $\mathfrak{q}$ and $\mathfrak{n}$.

Theorem 2 *Let* $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \stackrel{(\pi_1,\pi_2)}{\longrightarrow} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *be a projective presentation of the Leibniz crossed module* $(\mathfrak{n}, \mathfrak{q}, \delta)$*. Then, there is an isomorphism*

$$(\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) \cong \left(\frac{[\mathfrak{f}, \mathfrak{m}]}{[\mathfrak{f}, \mathfrak{u}] + [\mathfrak{r}, \mathfrak{m}]}, \frac{[\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]}, \bar{\mu} \right).$$

Proof According to Lemma 1 (ii), $\mathfrak{f}$ and $\mathfrak{f}/\mathfrak{m}$ are projective Leibniz algebras and so $HL_i(\mathfrak{f}) = 0 = HL_i(\mathfrak{f}/\mathfrak{m})$ for $i \geq 2$. So by [12, Proposition 2 and Proposition 7] the surjective homomorphism $\theta_{\mathfrak{f},\mathfrak{m}} : \mathfrak{f} \curlywedge \mathfrak{m} \longrightarrow [\mathfrak{f}, \mathfrak{m}]$ is an isomorphism. It is easy to see that $\theta_{\mathfrak{f},\mathfrak{m}}(\mathsf{Ker}(\pi_2 \curlywedge \pi_1)) = [\mathfrak{f}, \mathfrak{u}] + [\mathfrak{r}, \mathfrak{m}]$ by Remark 4. So, it gives rise to the isomorphism

$$\bar{\theta}_{\mathfrak{f},\mathfrak{m}} : \frac{\mathfrak{f} \curlywedge \mathfrak{m}}{\mathsf{Ker}(\pi_2 \curlywedge \pi_1)} \longrightarrow \frac{[\mathfrak{f}, \mathfrak{m}]}{[\mathfrak{f}, \mathfrak{u}] + [\mathfrak{r}, \mathfrak{m}]}.$$

Also, invoking [12, Theorem 4], $\mathsf{Ker}(\mathfrak{f} \curlywedge \mathfrak{f} \longrightarrow \mathfrak{f}) = HL_2(f) = 0$ so the surjection $\theta_{\mathfrak{f},\mathfrak{f}} : \mathfrak{f} \curlywedge \mathfrak{f} \longrightarrow [\mathfrak{f}, \mathfrak{f}]$ is an isomorphism in which $\theta_{\mathfrak{f},\mathfrak{f}}(\mathsf{Ker}(\pi_2 \curlywedge \pi_2)) = [\mathfrak{f}, \mathfrak{r}]$. So we obtain the induced isomorphism $\bar{\theta}_{\mathfrak{f},\mathfrak{f}} : \mathfrak{f} \curlywedge \mathfrak{f}/\mathsf{Ker}(\pi_2 \curlywedge \pi_2) \longrightarrow [\mathfrak{f}, \mathfrak{f}]/[\mathfrak{f}, \mathfrak{r}]$. Easily, the pair $(\bar{\theta}_{\mathfrak{f},\mathfrak{m}}, \bar{\theta}_{\mathfrak{f},\mathfrak{f}})$ is an isomorphism of crossed modules. Therefore, we conclude from Lemma 3 that

$$(\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) \cong \frac{(\mathfrak{f} \curlywedge \mathfrak{m}, \mathfrak{f} \curlywedge \mathfrak{f}, \mathsf{id} \curlywedge \mu)}{\mathsf{Ker}(\pi_2 \curlywedge \pi_1, \pi_2 \curlywedge \pi_2)} \cong \left(\frac{[\mathfrak{f}, \mathfrak{m}]}{[\mathfrak{f}, \mathfrak{u}] + [\mathfrak{r}, \mathfrak{m}]}, \frac{[\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]}, \bar{\mu} \right).$$

The proof is complete. □

For any Leibniz algebra $\mathfrak{q}$, we have $HL_2(\mathfrak{q}) \cong \mathsf{Ker}(\mathfrak{q} \curlywedge \mathfrak{q} \longrightarrow \mathfrak{q})$ [12, Theorem 4]. As an immediate consequence of the above theorem, we generalize this result for Leibniz crossed modules as follows:

Corollary 1 *Let* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *be a Leibniz crossed module. Then, we have*

$$\begin{aligned} \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) &\cong \mathsf{Ker}\left((\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)\right) \\ &= (\mathsf{Ker}(\mathfrak{q} \curlywedge \mathfrak{n} \longrightarrow \mathfrak{n}), \mathsf{Ker}(\mathfrak{q} \curlywedge \mathfrak{q} \longrightarrow \mathfrak{q}), \mathsf{id} \curlywedge \delta). \end{aligned}$$

Remark 5 Corollary 1 shows that for any abelian Leibniz crossed module $(\mathfrak{a}, \mathfrak{b}, \sigma)$ we have $\mathcal{M}(\mathfrak{a}, \mathfrak{b}, \sigma) = (\mathfrak{b} \curlywedge \mathfrak{a}, \mathfrak{b} \curlywedge \mathfrak{b}, \mathsf{id} \curlywedge \sigma)$.

Now we extend sequence (1) to a six-term natural exact sequence as follows:

Theorem 3 *Let* $(e) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \xrightarrow{\varphi} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *be a central extension of Leibniz crossed modules, then the exact sequence* (1) *can be extended one term further to the following natural exact sequence*

$$\begin{array}{l} (I, \mathfrak{b} \curlywedge \mathfrak{p}, \sigma \curlywedge \mathsf{id}) \longrightarrow \mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \\ \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathsf{ab}} \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{ab}} \longrightarrow 0. \end{array} \tag{2}$$

Proof Considering the inclusion map $\beta : I \longrightarrow \mathfrak{p} \curlywedge \mathfrak{h}$ and the homomorphism $\alpha : \mathfrak{b} \curlywedge \mathfrak{p} \longrightarrow \mathfrak{p} \curlywedge \mathfrak{p}$ one easily sees that $\phi = (\beta, \alpha) : (I, \mathfrak{b} \curlywedge \mathfrak{p}, \sigma \curlywedge \mathsf{id}) \longrightarrow (\mathfrak{p} \curlywedge \mathfrak{h}, \mathfrak{p} \curlywedge \mathfrak{p}, \mathsf{id} \curlywedge \sigma)$ is a homomorphism of crossed modules. Thanks to Remark 4, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & (I, \mathfrak{b} \curlywedge \mathfrak{p}, \sigma \curlywedge \mathsf{id}) & \xrightarrow{\phi} & (\mathfrak{p} \curlywedge \mathfrak{h}, \mathfrak{p} \curlywedge \mathfrak{p}, \mathsf{id} \curlywedge \sigma) & \xrightarrow{\varphi \curlywedge \varphi} & (\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \mathsf{id} \curlywedge \delta) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle (\bar{\lambda}_{\mathfrak{h}}, \bar{\lambda}_{\mathfrak{p}})_{|}} & & \Big\downarrow {\scriptstyle (\bar{\lambda}_{\mathfrak{h}}, \bar{\lambda}_{\mathfrak{p}})} & & \Big\downarrow {\scriptstyle (\bar{\lambda}_{\mathfrak{n}}, \bar{\lambda}_{\mathfrak{q}})} & & \\
0 & \longrightarrow & (\mathfrak{a}, \mathfrak{b}, \sigma) & \longrightarrow & (\mathfrak{h}, \mathfrak{p}, \sigma) & \xrightarrow{\varphi} & (\mathfrak{n}, \mathfrak{q}, \delta) & \longrightarrow & 0.
\end{array}$$

Now the Snake Lemma (which is valid in any semi-abelian category [3]) completes the proof, thanks to Corollary 1 and since $\mathsf{Im}(\bar{\lambda}_{\mathfrak{h}}, \bar{\lambda}_{\mathfrak{p}})_{|} = 0$, and $\mathsf{Coker}(\bar{\lambda}_{\mathfrak{h}}, \bar{\lambda}_{\mathfrak{p}}) \cong (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathsf{ab}}$, and $\mathsf{Coker}(\bar{\lambda}_{\mathfrak{n}}, \bar{\lambda}_{\mathfrak{q}}) \cong (\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{ab}}$. □

Remark 6 If we consider Leibniz algebras as crossed modules in any of the two usual ways [Example 1 (i)], we get the corresponding results for Leibniz algebras in [12, Proposition 7].

4 Stem Extensions and Stem Covers of Leibniz Crossed Modules

This section is devoted to generalizing the notions of stem extension and stem covers of Leibniz algebras to the context of Leibniz crossed modules. For that the homomorphism $\theta_*(e)$ in exact sequence (1) plays a central role. When a Leibniz algebra is regarded as a Leibniz crossed module in the two usual ways [Example 1 (i)], then the subsequent results recover the corresponding ones for stem extensions and stem covers of Leibniz algebras in [10,13].

Definition 6 A central extension of Leibniz crossed modules $(e) : 0 \to (\mathfrak{a}, \mathfrak{b}, \sigma) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \overset{\varphi=(\varphi_1,\varphi_2)}{\to} (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ is said to be a stem extension if $(\mathfrak{a}, \mathfrak{b}, \sigma) \subseteq [(\mathfrak{h}, \mathfrak{p}, \sigma), (\mathfrak{h}, \mathfrak{p}, \sigma)]$.

Also, if $(\mathfrak{a}, \mathfrak{b}, \sigma) \cong \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$, then the stem extension (e) is called a stem cover or covering of $(\mathfrak{n}, \mathfrak{q}, \delta)$.

The following result provides a characterization of stem extensions and stem covers.

Proposition 2 *Let* $(e) : 0 \to (\mathfrak{a}, \mathfrak{b}, \sigma) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \overset{\varphi=(\varphi_1,\varphi_2)}{\to} (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ *be a central extension of Leibniz crossed modules.*

(i) *The following statements are equivalent:*

(a) (e) *is stem extension of* $(\mathfrak{n}, \mathfrak{q}, \delta)$.
(b) *The homomorphism* $\theta_*(e) : \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma)$ *is surjective.*
(c) *The homomorphism* $(\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)/[(\mathfrak{h}, \mathfrak{p}, \sigma), (\mathfrak{h}, \mathfrak{p}, \sigma)]$ *is the zero map.*
(d) *The homomorphism* $\frac{(\mathfrak{h},\mathfrak{p},\sigma)}{[(\mathfrak{h},\mathfrak{p},\sigma),(\mathfrak{h},\mathfrak{p},\sigma)]} \longrightarrow \frac{(\mathfrak{n},\mathfrak{q},\delta)}{[(\mathfrak{n},\mathfrak{q},\delta),(\mathfrak{n},\mathfrak{q},\delta)]}$ *is an isomorphism.*

(ii) *Under the assumption* $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ *is finite dimensional in the central extension* (e)*, the following statements are equivalent:*

(a) (e) *is a stem cover.*
(b) $\theta_*(e)$ *is an isomorphism.*
(c) *The homomorphism* $\frac{(\mathfrak{h},\mathfrak{p},\sigma)}{[(\mathfrak{h},\mathfrak{p},\sigma),(\mathfrak{h},\mathfrak{p},\sigma)]} \longrightarrow \frac{(\mathfrak{n},\mathfrak{q},\delta)}{[(\mathfrak{n},\mathfrak{q},\delta),(\mathfrak{n},\mathfrak{q},\delta)]}$ *is an isomorphism and the induced homomorphism* $\mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ *is the zero map.*

Proof Direct checking from the exact sequence (1). □

Corollary 2 *Let* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *be a perfect Leibniz crossed module. Then the central extension* $(e) : 0 \to (\mathfrak{a}, \mathfrak{b}, \sigma) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \to (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ *is a stem cover if and only if* $(\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathrm{ab}} = \mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) = 0.$

Proof According to Proposition 2 (i) (d), $(\mathfrak{h}, \mathfrak{p}, \sigma)$ is a perfect crossed module. Hence, $(\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathrm{ab}} = 0$. So we have $\mathfrak{p} = [\mathfrak{p}, \mathfrak{p}]$ and $\mathfrak{h} = D_{\mathfrak{p}}(\mathfrak{h})$.

We claim that the crossed module $(I, \mathfrak{b} \curlywedge \mathfrak{p}, \sigma \curlywedge \mathrm{id})$ is trivial. Indeed, thanks to [19, Proposition 4.2], $\mathfrak{b}\star\mathfrak{p} = \mathfrak{b} \otimes \mathfrak{p}_{ab} \oplus \mathfrak{p}_{ab} \otimes \mathfrak{b} = 0$ and so $\mathfrak{b} \curlywedge \mathfrak{p} = 0$. Now, let $b \curlywedge h \in I$, then we can assume $h = {}^{p_0}h_0$, for some $h_0 \in \mathfrak{h}$, $p_0 \in \mathfrak{p}$, then we have

$$b \curlywedge h = b \curlywedge {}^{p_0}h_0 = -b \curlywedge h_0^{p_0} = [b, p_0] \curlywedge h_0 - {}^{b}h_0 \curlywedge p_0 = 0,$$

by relations (4a) and (3c) in Definition 2. Similar computations can be done with the other generators of I. Thus, we can conclude that I is trivial. Then $\mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) = 0$ from sequence (2) and Theorem 2 (ii) (c).

The converse is immediately followed from sequence (2) and Theorem 2. □

The following proposition plays a basic role in the proofs of most of the subsequent results.

Proposition 3 *Let* $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \overset{\pi=(\pi_1,\pi_2)}{\longrightarrow} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *be a projective presentation of* $(\mathfrak{n}, \mathfrak{q}, \delta)$*, then the following statements hold:*

(i) *The following exact sequence is split*

$$0 \longrightarrow \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow (\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) \longrightarrow \frac{(\mathfrak{u}, \mathfrak{r}, \mu)}{(\mathfrak{u}, \mathfrak{r}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]} \longrightarrow 0,$$

where $(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) = \frac{(\mathfrak{u},\mathfrak{r},\mu)}{[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}$.

(ii) *If* $0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \xrightarrow{\gamma=(\gamma_1,\gamma_2)} (\mathfrak{n}_1, \mathfrak{q}_1, \delta_1) \longrightarrow 0$ *is a stem extension of another Leibniz crossed module* $(\mathfrak{n}_1, \mathfrak{q}_1, \delta_1)$ *and* $\alpha = (\alpha_1, \alpha_2) : (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow (\mathfrak{n}_1, \mathfrak{q}_1, \delta_1)$ *is a homomorphism of Leibniz crossed modules, then there exists a homomorphism* $\overline{\beta} = (\overline{\beta_1}, \overline{\beta_2}) : (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)$ *such that* $\bar{\beta}(\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)) \subseteq \bar{\beta}(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) \subseteq (\mathfrak{a}, \mathfrak{b}, \sigma)$*, and the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \overline{\mu}) & \longrightarrow & (\overline{\mathfrak{m}}, \bar{\mathfrak{f}}, \overline{\mu}) & \xrightarrow{\overline{\pi}} & (\mathfrak{n}, \mathfrak{q}, \delta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \overline{\beta}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & (\mathfrak{a}, \mathfrak{b}, \sigma) & \longrightarrow & (\mathfrak{h}, \mathfrak{p}, \sigma) & \xrightarrow{\gamma} & (\mathfrak{n}_1, \mathfrak{q}_1, \delta_1) & \longrightarrow & 0. \end{array}$$

where $(\overline{\mathfrak{m}}, \bar{\mathfrak{f}}, \overline{\mu}) = \frac{(\mathfrak{m},\mathfrak{f},\mu)}{[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}$*. Furthermore, if* α *is surjective, then so is* $\bar{\beta}$*, and* $\bar{\beta}(\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)) = (\mathfrak{a}, \mathfrak{b}, \sigma)$*.*

Proof (i) By the isomorphism theorem.

(ii) It is a straightforward adaptation of the proof of Lemma 3.3 in [26]. □

In the following, we determine the structure of stem covers of Leibniz crossed modules which is analogous to similar results in group and in Lie crossed modules [26,30].

Theorem 4 *Let* $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *be a projective presentation of a Leibniz crossed module* $(\mathfrak{n}, \mathfrak{q}, \delta)$*. Then, the following statements hold:*

(i) *If* $(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) \cong \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \oplus (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})$ *for some ideal* $(\mathfrak{t}, \mathfrak{s}, \mu)$ *of* $(\mathfrak{m}, \mathfrak{f}, \mu)$*, where* $(\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})$ *denotes the quotient* $\frac{(\mathfrak{t},\mathfrak{s},\mu)}{[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}$*, then the extension* $(e) : 0 \longrightarrow (\mathfrak{u}/\mathfrak{t}, \mathfrak{r}/\mathfrak{s}, \bar{\mu}) \longrightarrow (\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu}) \longrightarrow (n, q, \delta) \longrightarrow 0$ *is a stem cover of* $(\mathfrak{n}, \mathfrak{q}, \delta)$*.*

(ii) *If* $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ *is finite dimensional and* $(e_1) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (h, p, \sigma) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *is a stem cover of* $(\mathfrak{n}, \mathfrak{q}, \delta)$*, then there is an ideal* $(\mathfrak{t}, \mathfrak{s}, \mu)$ *of* $(\mathfrak{m}, \mathfrak{f}, \mu)$ *satisfying statement (i) and such that* $(\mathfrak{h}, \mathfrak{p}, \sigma) \cong (\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu})$ *and* $(\mathfrak{a}, \mathfrak{b}, \sigma) \cong (\mathfrak{u}/\mathfrak{t}, \mathfrak{r}/\mathfrak{s}, \bar{\mu})$*.*

Proof (i) By the assumption, we have $(\mathfrak{u}/\mathfrak{t}, \mathfrak{r}/\mathfrak{s}, \bar{\mu}) \cong \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ and $(\mathfrak{u}, \mathfrak{r}, \mu) = (\mathfrak{u}, \mathfrak{r}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] + (\mathfrak{t}, \mathfrak{s}, \mu) \subseteq [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] + (\mathfrak{t}, \mathfrak{s}, \mu)$, then

$$\begin{aligned} (\mathfrak{u}/\mathfrak{t}, \mathfrak{r}/\mathfrak{s}, \bar{\mu}) &\subseteq \frac{[(\mathfrak{m},\mathfrak{f},\mu),(\mathfrak{m},\mathfrak{f},\mu)]+(\mathfrak{t},\mathfrak{s},\mu)}{(\mathfrak{t},\mathfrak{s},\mu)} \cap Z\left(\frac{(\mathfrak{m},\mathfrak{f},\mu)}{(\mathfrak{t},\mathfrak{s},\mu)}\right) \\ &\subseteq [(\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu}), (\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu})] \cap Z\left(\frac{(\mathfrak{m},\mathfrak{f},\mu)}{(\mathfrak{t},\mathfrak{s},\mu)}\right), \end{aligned}$$

so (e) is stem cover of $(\mathfrak{n}, \mathfrak{q}, \delta)$.

(ii) According to Proposition 3 (ii), there is a surjective homomorphism $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2) : (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)$ such that $\bar{\beta}(\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)) = \bar{\beta}(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) = (\mathfrak{a}, \mathfrak{b}, \sigma)$. Setting $\mathsf{Ker}(\bar{\beta}) = (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})$, we have that $(\mathfrak{h}, \mathfrak{p}, \sigma) \cong (\mathfrak{m}, \mathfrak{f}, \mu)/(\mathfrak{t}, \mathfrak{s}, \mu)$ and $(\mathfrak{a}, \mathfrak{b}, \sigma) \cong (\mathfrak{u}, \mathfrak{r}, \mu)/(\mathfrak{t}, \mathfrak{s}, \mu)$. Also the restriction of $\bar{\beta}_|$ from $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ to $(\mathfrak{a}, \mathfrak{b}, \sigma)$ is surjective and since $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ is finite dimensional, $\bar{\beta}_|$ is an isomorphism; therefore,

$\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)\cap \mathsf{Ker}(\bar\beta)=\mathsf{Ker}(\bar\beta_|)=0$. As the kernel of the restriction of $\bar\beta$ to $(\bar{\mathfrak{u}},\bar{\mathfrak{r}},\bar\mu)$ is $\mathsf{Ker}(\bar\beta)$ and the image of this restriction is $(\mathfrak{a},\mathfrak{b},\sigma)$, the result follows. □

Thanks to Theorem 4 and Proposition 3, we provide the following important consequence.

Corollary 3 *Any Leibniz crossed module* $(\mathfrak{n},\mathfrak{q},\delta)$ *admits at least one stem cover. In particular, any Leibniz algebra admits at least one stem cover.*

Proof Let $0\longrightarrow(\mathfrak{u},\mathfrak{r},\mu)\longrightarrow(\mathfrak{m},\mathfrak{f},\mu)\longrightarrow(\mathfrak{n},\mathfrak{q},\delta)\longrightarrow 0$ be a projective presentation of a Leibniz crossed module $(\mathfrak{n},\mathfrak{q},\delta)$. Then, by Proposition 3 (i), there is an ideal $(\mathfrak{t},\mathfrak{s},\mu)$ of $(\mathfrak{m},\mathfrak{f},\mu)$ such that $(\bar{\mathfrak{u}},\bar{\mathfrak{r}},\bar\mu)\cong\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)\oplus(\bar{\mathfrak{t}},\bar{\mathfrak{s}},\bar\mu)$. Now the result follows by Theorem 4 (i). □

It is very interesting to find the relations between two stem covers of given Leibniz crossed module. In the following, we prove that some crossed submodules and factor crossed modules of covering crossed modules are always isomorphic.

Corollary 4 *Let* $(\mathfrak{n},\mathfrak{q},\delta)$ *be a Leibniz crossed module with finite dimensional Schur multiplier and let* $(e_i):0\longrightarrow(\mathfrak{a}_i,\mathfrak{b}_i,\sigma_i)\longrightarrow(\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i)\overset{\varphi_i=(\varphi_{i1},\varphi_{i2})}{\longrightarrow}(\mathfrak{n},\mathfrak{q},\delta)\longrightarrow 0$ *be two stem covers of* $(\mathfrak{n},\mathfrak{q},\delta)$*, for* $i=1,2$*. Then*

(i) $[(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1),(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)]\cong[(\mathfrak{h}_2,\mathfrak{p}_2,\sigma_2),(\mathfrak{h}_2,\mathfrak{p}_2,\sigma_2)]$.
(ii) $(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)/Z(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)\cong(\mathfrak{h}_2,\mathfrak{p}_2,\sigma_2)/Z(\mathfrak{h}_2,\mathfrak{p}_2,\sigma_2)$.
(iii) $Z(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)/(\mathfrak{a}_1,\mathfrak{b}_1,\sigma_1)\cong Z(\mathfrak{h}_2,\mathfrak{p}_2,\sigma_2)/(\mathfrak{a}_2,\mathfrak{b}_2,\sigma_2)$.

Proof (i) Let $(f):0\longrightarrow(\mathfrak{u},\mathfrak{r},\mu)\longrightarrow(\mathfrak{m},\mathfrak{f},\mu)\overset{\pi=(\pi_1,\pi_2)}{\longrightarrow}(\mathfrak{n},\mathfrak{q},\delta)\longrightarrow 0$ be a projective presentation of $(\mathfrak{n},\mathfrak{q},\delta)$. By applying a similar argument to the proof of [26, Theorem 3.7], we can show that the crossed modules

$$[(\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i),(\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i)],\ (\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i)/Z(\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i),\ \text{and } Z(\mathfrak{h}_i,\mathfrak{p}_i,\sigma_i)/(\mathfrak{a}_i,\mathfrak{b}_i,\sigma_i),$$

are uniquely determined by the projective presentation (f).

By virtue of Proposition 3 (ii) and Theorem 4 (ii), there is a surjective homomorphism $\bar\beta:(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)\longrightarrow(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)$ such that $\bar\beta(\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta))=(\mathfrak{a}_1,\mathfrak{b}_1,\sigma_1)$. Since $\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)$ is finite dimensional, then the restriction of $\bar\beta$ from $\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)$ onto $(\mathfrak{a}_1,\mathfrak{b}_1,\sigma_1)$ is an isomorphism. Thus, $0=\mathsf{Ker}(\bar\beta_|)=\mathsf{Ker}(\bar\beta)\cap\mathcal{M}(\mathfrak{n},\mathfrak{q},\delta)$, and it implies that $\mathsf{Ker}(\bar\beta)\cap[(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu),(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)]=0$. It is easy to see that $\bar\beta$ induces the surjective homomorphism $\hat\beta:[(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu),(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)]\longrightarrow[(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1),(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)]$ with $\mathsf{Ker}(\hat\beta)=\mathsf{Ker}(\bar\beta)\cap[(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu),(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)]=0$. Therefore, we have $[(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1),(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)]\cong[(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu),(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)]$.
(ii), (iii) Now, put $\mathsf{Ker}(\bar\beta)=(\bar{\mathfrak{t}},\bar{\mathfrak{s}},\bar\mu)=(\mathfrak{t},\mathfrak{s},\mu)/[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]$ and $Z(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)=(\bar{\mathfrak{k}},\bar{\mathfrak{l}},\bar\mu)=(\mathfrak{k},\mathfrak{l},\mu)/[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]$. We claim that it is sufficient to prove that $Z(\mathfrak{m}/\mathfrak{t},\mathfrak{f}/\mathfrak{s},\bar\mu)=(\mathfrak{k}/\mathfrak{t},\mathfrak{l}/\mathfrak{s}/\bar\mu)$. Hence, we have

$$\frac{(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)}{Z(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)}\cong\frac{(\bar{\mathfrak{m}},\bar{\mathfrak{f}},\bar\mu)/\mathsf{Ker}(\bar\beta)}{(\mathfrak{k}/\mathfrak{t},\mathfrak{l}/\mathfrak{s}/\bar\mu)}\cong\left(\frac{\mathfrak{m}}{\mathfrak{k}},\frac{\mathfrak{f}}{\mathfrak{l}},\bar\mu\right),$$
$$\frac{Z(\mathfrak{h}_1,\mathfrak{p}_1,\sigma_1)}{(\mathfrak{a}_1,\mathfrak{b}_1,\sigma_1)}\cong\frac{(\mathfrak{k}/\mathfrak{t},\mathfrak{l}/\mathfrak{s}/\bar\mu)}{(\bar{\mathfrak{u}},\bar{\mathfrak{r}},\bar\mu)/\mathsf{Ker}(\bar\beta)}\cong\left(\frac{\mathfrak{k}}{\mathfrak{u}},\frac{l}{r},\bar\mu\right),$$

and we get exactly what we want to prove.

To prove our assertion, clearly, $\mathsf{Ker}(\bar{\beta}) \subseteq (\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) \subseteq Z(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})$ and $[(\mathfrak{k}, \mathfrak{l}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] \subseteq (\mathfrak{t}, \mathfrak{s}, \mu)$. So, $(\mathfrak{k}/\mathfrak{t}, \mathfrak{l}/\mathfrak{s}, \mu) \subseteq Z\,(\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu})$. To prove the opposite content, assume $Z\,(\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu}) = (\mathfrak{x}/\mathfrak{t}, \mathfrak{z}/\mathfrak{s}, \bar{\mu})$, then by the assumptions we have

$$[(\mathfrak{x}, \mathfrak{z}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] \subseteq (\mathfrak{t}, \mathfrak{s}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] = [(\mathfrak{u}, \mathfrak{r}, \mu), (m, f, \mu)].$$

Therefore,

$$\frac{(\mathfrak{x}, \mathfrak{z}, \mu)}{[(\mathfrak{u}, \mathfrak{r}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]} \subseteq Z\left(\frac{(\mathfrak{m}, \mathfrak{f}, \mu)}{[(\mathfrak{u}, \mathfrak{r}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]}\right) = \frac{(\mathfrak{k}, \mathfrak{l}, \mu)}{[(\mathfrak{u}, \mathfrak{r}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]}$$

and so $Z\,(\mathfrak{m}/\mathfrak{t}, \mathfrak{f}/\mathfrak{s}, \bar{\mu}) = (\mathfrak{k}/\mathfrak{t}, \mathfrak{l}/\mathfrak{s}/)$. The proof is complete. □

Remark 7 Corollary 4 shows that every perfect crossed module admits only one isomorphism class of stem covers.

Theorem 5 *The central extension* $(e) : 0 \to \mathsf{Ker}(\phi) \to (\mathfrak{q} \curlywedge \mathfrak{n}, \mathfrak{q} \curlywedge \mathfrak{q}, \delta \curlywedge \mathsf{id}) \xrightarrow{\phi} (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ *is a stem cover of* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *if and only if* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *is perfect.*

Proof Let the extension (e) be a stem cover of $(\mathfrak{n}, \mathfrak{q}, \delta)$. Then we have $\mathsf{Im}(\phi) = [(\mathfrak{n}, \mathfrak{q}, \delta), (\mathfrak{n}, \mathfrak{q}, \delta)] = (\mathfrak{n}, \mathfrak{q}, \delta)$, so the crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is perfect.

Conversely, if $(\mathfrak{n}, \mathfrak{q}, \delta)$ is perfect then $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$ and $D_{\mathfrak{q}}(\mathfrak{n}) = \mathfrak{n}$. For every $q \in \mathfrak{q}, n \in \mathfrak{n}$, we can assume $q = [q_1, q_2]$ and $n = {}^{q'}n'$ for some $q_1, q_2, q' \in \mathfrak{q}$ and $n' \in \mathfrak{n}$. Then

$$\begin{aligned} q \curlywedge n &= [q_1, q_2] \curlywedge {}^{q'}n' = [[q_1, q_2], q'] \curlywedge n' - {}^{[q_1,q_2]}n' \curlywedge q' \\ &= {}^{(q_1 \curlywedge q_2)}q' \curlywedge n' \in D_{\mathfrak{q}\curlywedge\mathfrak{q}}(\mathfrak{q} \curlywedge \mathfrak{n}), \end{aligned}$$

by the relations (4a) and (3c) in Definition 4 and the Leibniz action of $\mathfrak{q} \curlywedge \mathfrak{q}$ on $\mathfrak{q} \curlywedge \mathfrak{n}$. Consequently, $\mathfrak{q} \curlywedge \mathfrak{n} \subseteq D_{\mathfrak{q}\curlywedge\mathfrak{q}}(\mathfrak{q} \curlywedge \mathfrak{n})$.

Easily, $[\mathfrak{q} \curlywedge \mathfrak{q}, \mathfrak{q} \curlywedge \mathfrak{q}] = \mathfrak{q} \curlywedge \mathfrak{q}$, so the Leibniz crossed module $(\mathfrak{n} \curlywedge \mathfrak{q}, \mathfrak{q} \curlywedge \mathfrak{q}, \delta \curlywedge \mathsf{id})$ is perfect. Now the result follows by Corollary 1 and Proposition 1 (iii). □

Theorem 6 *Let* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *be a Leibniz crossed module such that* $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$ *is finite dimensional. If* $(e) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \xrightarrow{\varphi=(\varphi_1,\varphi_2)} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *is a stem extension of* $(\mathfrak{n}, \mathfrak{q}, \delta)$*, then there is a stem cover* $(e_1) : 0 \longrightarrow (\mathfrak{a}_1, \mathfrak{b}_1, \sigma_1) \longrightarrow (\mathfrak{h}_1, \mathfrak{p}_1, \sigma_1) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ *such that* (e) *is homomorphic image of* (e_1).

Proof Assume that $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \xrightarrow{\pi=(\pi_1,\pi_2)} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ is a projective presentation of $(\mathfrak{n}, \mathfrak{q}, \delta)$. Thanks to Proposition 3 (ii), there is a surjective homomorphism $\bar{\beta} : (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)$ such that $\gamma \circ \bar{\beta} = \bar{\pi}$ and $\bar{\beta}(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) = (\mathfrak{a}, \mathfrak{b}, \sigma)$. Putting $\mathsf{Ker}(\bar{\beta}) = (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})$, then we have

$$\frac{(\mathfrak{u}, \mathfrak{r}, \mu)}{(\mathfrak{t}, \mathfrak{s}, \mu)} \cong (\mathfrak{a}, \mathfrak{b}, \sigma) \cong \frac{((\mathfrak{u}, \mathfrak{r}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)]) + (\mathfrak{t}, \mathfrak{s}, \mu)}{(\mathfrak{t}, \mathfrak{s}, \mu)},$$

so $(\mathfrak{u}, \mathfrak{r}, \mu) = (\mathfrak{u}, \mathfrak{r}, \mu) \cap [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] + (\mathfrak{t}, \mathfrak{s}, \mu)$, because $(\mathfrak{a}, \mathfrak{b}, \sigma)$ has finite dimension. On the other hand, the following exact sequence splits by Lemma 1 (ii):

$$0 \to [(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}), (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})] \cap (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu}) \to (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu}) \to \frac{(\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})}{[(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}), (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})] \cap (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})} \to 0$$

Thus, $(\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu}) = ([(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}), (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})] \cap (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})) \oplus (\bar{\mathfrak{t}_1}, \bar{\mathfrak{s}_1}, \bar{\mu})$, for some ideal $(\bar{\mathfrak{t}_1}, \bar{\mathfrak{s}_1}, \bar{\mu})$ of $(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})$, where $(\bar{\mathfrak{t}_1}, \bar{\mathfrak{s}_1}, \bar{\mu}) = \frac{(\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})}{[(\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu}), (\bar{\mathfrak{m}}, \bar{\mathfrak{f}}, \bar{\mu})] \cap (\bar{\mathfrak{t}}, \bar{\mathfrak{s}}, \bar{\mu})}$. It yields that $(\bar{\mathfrak{u}}, \bar{\mathfrak{r}}, \bar{\mu}) = \mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) \oplus (\bar{\mathfrak{t}_1}, \bar{\mathfrak{s}_1}, \bar{\mu})$ and so by Theorem 4 (i), the extension (e_1) : $0 \to (\mathfrak{u}/\mathfrak{t}_1, \mathfrak{r}/\mathfrak{s}_1, \bar{\mu}) \to (\mathfrak{m}/\mathfrak{t}_1, \mathfrak{f}/\mathfrak{s}_1, \bar{\mu}) \to (\mathfrak{n}, \mathfrak{q}, \delta) \to 0$ is a stem cover of $(\mathfrak{n}, \mathfrak{q}, \delta)$ and the extension (e) is homomorphic image of (e_1), as required. □

An immediate consequence of the above theorem is a new characterization of stem covers of finite dimensional Leibniz crossed modules.

Corollary 5 *A stem extension (e) of a finite dimensional crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)$ is a stem cover if and only if any surjective homomorphism of other stem extension of $(\mathfrak{n}, \mathfrak{q}, \delta)$ onto (e) is an isomorphism.*

Proof The sufficient condition follows from Theorem 6. For necessary condition, let $(e) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ be a stem cover of $(\mathfrak{n}, \mathfrak{q}, \delta)$ and (e_1) be a stem extension of $(\mathfrak{n}, \mathfrak{q}.\delta)$ such that there is a surjective homomorphism $\alpha = (\alpha_1, \alpha_2) : (e_1) \longrightarrow (e)$. According to Theorem 6, we can find a stem cover $(e_2) : 0 \longrightarrow (\mathfrak{a}_2, \mathfrak{b}_2, \sigma_2) \longrightarrow (\mathfrak{h}_2, \mathfrak{p}_2, \sigma_2) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ of $(\mathfrak{n}, \mathfrak{q}.\delta)$ such that (e_1) is homomorphic image of (e_2) and $\beta = (\beta_1, \beta_2) : (e_2) \longrightarrow (e_1)$ is a surjective homomorphism. So, we have the surjective homomorphism $\alpha \circ \beta$ from (e_2) onto (e). Now, since $(\mathfrak{n}, \mathfrak{q}, \delta)$ is finite dimensional, then we have $(\dim(\mathfrak{h}_2), \dim(\mathfrak{p}_2)) = (\dim(\mathfrak{n}), \dim(\mathfrak{q})) + (\dim(\mathfrak{a}_2), \dim(\mathfrak{b}_2)) = (\dim(\mathfrak{n}), \dim(\mathfrak{q})) + (\dim(\mathfrak{a}), \dim(\mathfrak{b})) = (\dim(\mathfrak{h}), \dim(\mathfrak{p}))$. Therefore, $\alpha \circ \beta$ is an isomorphism and then α is an isomorphism, as required. □

5 Connection with Stem Cover of Lie Crossed Modules

In this section, we investigate the interplay between the notions of stem cover and the Schur multiplier of Leibniz crossed modules with the same notations for Lie crossed modules in [9].

Theorem 7 *Let (T, L, τ) be a Lie crossed module with finite dimensional Schur multiplier $\mathcal{M}(T, L, \tau)$ as Leibniz crossed module and let $(e) : 0 \longrightarrow (A, B, \vartheta) \longrightarrow (M, P, \vartheta) \longrightarrow (T, L, \tau) \longrightarrow 0$ be a stem cover of (T, L, τ) in* **XLie** *(the category of Lie crossed modules). Then, there exists a stem cover $(e_1) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow (T, L, \tau) \longrightarrow 0$ of Leibniz crossed modules, such that (e) is homomorphic image of (e_1).*

Moreover, if (T, L, τ) is finite dimensional Lie crossed module, then $(\mathfrak{h}, \mathfrak{p}, \sigma)_{Lie} \cong (M, P, \vartheta)$.

Proof Clearly, (e) is a stem extension of (T, L, τ) in **XLb**. According to Theorem 6, there exists a stem cover $(e_1) : 0 \to (\mathfrak{a}, \mathfrak{b}, \sigma) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \to (T, L, \tau) \to 0$ and a surjective homomorphism $\beta = (\beta_1, \beta_2) : (\mathfrak{h}, \mathfrak{p}, \sigma) \twoheadrightarrow (M, P, \vartheta)$. Obviously, β induces a surjective homomorphism $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2) : (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathsf{Lie}} \longrightarrow (M, P, \vartheta)$.

On the other hand, $0 \longrightarrow ((\mathfrak{a}, \mathfrak{b}, \sigma) + (\mathfrak{h}, \mathfrak{p}, \sigma)^{\text{ann}})/(\mathfrak{h}, \mathfrak{p}, \sigma)^{\text{ann}} \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathsf{Lie}} \longrightarrow (T, L, \tau) \longrightarrow 0$ is a stem extension of (T, L, τ) in the category **XLie** of crossed modules in Lie algebras, so by the proof of Theorem 3.6 in [30], there is a surjective homomorphism $\alpha = (\alpha_1, \alpha_2)$ from (M_1, P_1, ϑ_1) to $(\mathfrak{h}, \mathfrak{p}, \sigma)_{\mathsf{Lie}}$, where (M_1, P_1, ϑ_1) is a stem cover of (T, L, τ) in **XLie**. Now, combining [30, Corollary 3.5] with [28, Proposition 22] and [27, Theorem 13], we deduce that $M \cong M_1$ and $P \cong P_1$. So, the homomorphisms $M \longrightarrow M_1 \xrightarrow{\alpha_1} \bar{\mathfrak{h}} \xrightarrow{\bar{\beta}_1} M$ and $P \longrightarrow P_1 \xrightarrow{\alpha_2} \mathfrak{p}_{\mathsf{Lie}} \xrightarrow{\bar{\beta}_2} P$ are surjective. Thus, they are isomorphism, in finite dimensional case.

It is easy to check that $\bar{\beta}_1$ and $\bar{\beta}_2$ are isomorphisms and so $\bar{\beta}$ is an isomorphism of crossed modules, as required. □

Remark 8 Theorem 7 applied to the particular Lie crossed modules (T, T, id) or $(0, T, inc)$ recovers the corresponding results for Lie algebras provided in [13, Theorem 3.4].

Note that in the proof of Theorem 7 we have $\mathsf{Ker}(\beta) \subseteq (\mathfrak{a}, \mathfrak{b}, \sigma)$. This fact and exact sequence (1) provide the following consequence:

Corollary 6 *Under the assumptions of Theorem 7, the exact sequence* $(\hat{e}) : 0 \to \mathsf{Ker}(\beta) \to (\mathfrak{h}, \mathfrak{p}, \sigma) \to (M, P, \vartheta) \to 0$ *is a stem extension of* (M, P, ϑ) *in the category* **XLb**.

Moreover, if (T, L, τ) *is a perfect Lie crossed module, then* $(\hat{e})$ *is a stem cover of* (M, P, ϑ) *in* **XLb**.

Proof By the assumptions of Theorem 7 we have $\mathsf{Ker}(\beta) \subseteq (\mathfrak{a}, \mathfrak{b}, \sigma) \subseteq Z(\mathfrak{h}, \mathfrak{p}, \sigma) \cap [(\mathfrak{h}, \mathfrak{p}, \sigma), (\mathfrak{h}, \mathfrak{p}, \sigma)]$. So $(\hat{e})$ is a stem extension. By application of sequence (1), we obtain the following exact sequence:

$$\mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow \mathcal{M}(M, P, \vartheta) \longrightarrow \mathsf{Ker}(\beta) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma)_{\text{ab}} \longrightarrow (M, P, \vartheta)_{\text{ab}} \longrightarrow 0$$

Now if (T, L, τ) is a perfect Lie crossed module, it also is a perfect Leibniz crossed module. Since $(\hat{e})$ is a stem cover of (T, L, τ), then thanks to Corollary 2 we have $(\mathfrak{h}, \mathfrak{p}, \sigma)_{\text{ab}} = \mathcal{M}(\mathfrak{h}, \mathfrak{p}, \sigma) = 0$. Therefore, $\mathsf{Ker}(\beta) \cong (\mathcal{M}, \mathcal{P}, \vartheta)$. The proof is complete. □

Corollary 7 *Let* (T, L, τ) *be a Lie crossed module. Then* $\mathcal{M}^{\mathsf{Lie}}(T, L, \tau)$ *(the Schur multiplier in* **XLie** [9]*) is homomorphic image of* $\mathcal{M}(T, L, \tau)$.

Corollary 8 *Let* (T, L, τ) *be a perfect Lie crossed module with finite dimensional Schur multiplier* $\mathcal{M}(T, L, \tau)$ *as a Leibniz crossed module. Then, the stem cover* $(e) : 0 \longrightarrow (\mathfrak{a}, \mathfrak{b}, \sigma) \longrightarrow (\mathfrak{h}, \mathfrak{p}, \sigma) \xrightarrow{\varphi} (T, L, \tau) \longrightarrow 0$ *of* (T, L, τ) *in* **XLb** *is a stem cover of* (T, L, τ) *in* **XLie**, *if and only if* $\mathcal{M}(T, L, \tau) \cong \mathcal{M}^{\mathsf{Lie}}(T, L, \tau)$.

Proof Let $(e_1) : 0 \longrightarrow (A, B, \sigma_1) \longrightarrow (H, P, \sigma_1) \stackrel{\bar{\varphi}}{\longrightarrow} (T, L, \tau) \longrightarrow 0$ be a stem cover of (T, L, τ) in **XLie**. Then (e_1) is a stem extension of (T, L, τ) in **XLb**, so by Theorem 6 and Remark 7, there is a surjective homomorphism $\beta : (\mathfrak{h}, \mathfrak{p}, \sigma) \longrightarrow (H, P, \sigma_1)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathfrak{a}, \mathfrak{b}, \sigma) & \longrightarrow & (\mathfrak{h}, \mathfrak{p}, \sigma) & \stackrel{\varphi}{\longrightarrow} & (T, L, \tau) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_|} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & (A, B, \sigma_1) & \longrightarrow & (H, P, \sigma_1) & \stackrel{\bar{\varphi}}{\longrightarrow} & (T, L, \tau) & \longrightarrow & 0.
\end{array}$$

Note that, $\mathsf{Ker}(\beta) \subseteq (\mathfrak{a}, \mathfrak{b}, \sigma)$, so the restriction $\beta_|$ from $(\mathfrak{a}, \mathfrak{b}, \sigma)$ onto (A, B, σ_1) is an isomorphism if and only if β is an isomorphism, Thus, by finiteness of $\mathcal{M}(\mathfrak{a}, \mathfrak{b}, \sigma)$, we conclude $\mathcal{M}(T, L, \tau) \cong \mathcal{M}^{\mathsf{Lie}}(T, L, \tau)$ if and only if $(H, P, \sigma_1) \cong (\mathfrak{h}, \mathfrak{p}, \sigma)$. The result follows. □

Corollary 7 is extended to Leibniz crossed modules as follows:

Theorem 8 *Let* $(\mathfrak{n}, \mathfrak{q}, \delta)$ *be a Leibniz crossed module. Then* $\mathcal{M}^{\mathsf{Lie}}((\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{Lie}})$ *is homomorphic image of* $\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta)$.

Proof Let $0 \longrightarrow (\mathfrak{u}, \mathfrak{r}, \mu) \longrightarrow (\mathfrak{m}, \mathfrak{f}, \mu) \stackrel{\pi}{\longrightarrow} (\mathfrak{n}, \mathfrak{q}, \delta) \longrightarrow 0$ be a projective presentation of $(\mathfrak{n}, \mathfrak{q}, \delta)$. Since $(\mathfrak{m}, \mathfrak{f}, \mu)_{\mathsf{Lie}}$ is a projective crossed module in **XLie**, we have the projective presentation

$$0 \to \frac{(\mathfrak{u}, \mathfrak{r}, \mu) + (\mathfrak{m}, \mathfrak{f}, \mu)^{\mathrm{ann}}}{(\mathfrak{m}, \mathfrak{f}, \mu)^{\mathrm{ann}}} \to (\mathfrak{m}, \mathfrak{f}, \mu)_{\mathsf{Lie}} \to (\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{Lie}} \to 0$$

of the Lie crossed module $(\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{Lie}}$. Clearly, the projection homomorphism $pr : (\mathfrak{m}, \mathfrak{f}, \mu) \to (\mathfrak{m}, \mathfrak{f}, \mu)_{\mathsf{Lie}}$ induces the surjective homomorphism $\bar{pr} : [(\mathfrak{m}, \mathfrak{f}, \mu), (\mathfrak{m}, \mathfrak{f}, \mu)] \to [(\mathfrak{m}, \mathfrak{f}, \mu)_{\mathsf{Lie}}, (\mathfrak{m}, \mathfrak{f}, \mu)_{\mathsf{Lie}}]$, which gives rise to a surjective homomorphism

$$\begin{array}{rcl}
\mathcal{M}(\mathfrak{n}, \mathfrak{q}, \delta) & = & \frac{(\mathfrak{u},\mathfrak{r},\mu) \cap [(\mathfrak{m},\mathfrak{f},\mu),(\mathfrak{m},\mathfrak{f},\mu)]}{[(\mathfrak{u},\mathfrak{r},\mu),(\mathfrak{m},\mathfrak{f},\mu)]} \\
 & & \downarrow{\scriptstyle \tilde{pr}} \\
\mathcal{M}^{\mathsf{Lie}}((\mathfrak{n}, \mathfrak{q}, \delta)_{\mathsf{Lie}}) & = & \frac{\overline{(\mathfrak{u},\mathfrak{r},\mu)} \cap [(\mathfrak{m},\mathfrak{f},\mu)_{\mathsf{Lie}},(\mathfrak{m},\mathfrak{f},\mu)_{\mathsf{Lie}}]}{[\overline{(\mathfrak{u},\mathfrak{r},\mu)},(\mathfrak{m},\mathfrak{f},\mu)_{\mathsf{Lie}}]},
\end{array}$$

where $\overline{(\mathfrak{u}, \mathfrak{r}, \mu)} = ((\mathfrak{u}, \mathfrak{r}, \mu) + (\mathfrak{m}, \mathfrak{f}, \mu)^{\mathrm{ann}})/(\mathfrak{m}, \mathfrak{f}, \mu)^{\mathrm{ann}}$, and the result follows. □

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