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# Regularity Criteria for a Ginzburg–Landau–Navier–Stokes in a Bounded Domain

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**Abstract**
In this work, we prove some regularity criteria for a Ginzburg–Landau–Navier–Stokes system with the Coulomb gauge in a bounded domain $\Omega \subset \mathbb{R}^3$.

**Keywords** Ginzburg–Landau · Navier–Stokes · Regularity criterion

**Mathematics Subject Classifications** 82D55 · 35Q30 · 35Q56 · 76D03

## 1 Introduction

In this work, we consider the following Ginzburg–Landau–Navier–Stokes system with the Coulomb gauge:

$$\operatorname{div} u = 0, \tag{1.1}$$
$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = |\psi|^2 \nabla h, \tag{1.2}$$

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$$\eta \partial_t \psi + i\eta k \phi \psi + u \cdot \nabla \psi + \left( \frac{i}{k} \nabla + A \right)^2 \psi + (|\psi|^2 - 1)\psi = 0, \tag{1.3}$$

$$\partial_t A + \nabla \phi - \Delta A + \mathrm{Re} \left\{ \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} \right\} = 0, \tag{1.4}$$

$$\mathrm{div}\, A = 0 \ \text{ in } \ \Omega \times (0, \infty), \tag{1.5}$$

$$u = 0, \frac{\partial \psi}{\partial n} = 0, A \cdot n = 0, \mathrm{rot}\, A \times n = 0 \ \text{ on } \ \partial\Omega \times (0, \infty), \tag{1.6}$$

$$(u, \psi, A)(\cdot, 0) = (u_0, \psi_0, A_0)(\cdot) \ \text{ in } \ \Omega, \tag{1.7}$$

where $u$ is the velocity, $\pi$ is the pressure, $\psi$ is complex the order parameter, $A$ is the vector potential, and $\phi$ is the electric potential, respectively. $\eta$ and $k$ are the positive Ginzburg–Landau constants. $\overline{\psi}$ is the complex conjugate of $\psi$, $\mathrm{Re}\psi := \frac{\psi + \overline{\psi}}{2}$ is the real part of $\psi$, $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers and $i := \sqrt{-1}$. The function $h := h(x)$ denotes a potential function; we will assume that $h$ is a smooth function. $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$, and $n$ is the unit outward normal vector to $\partial\Omega$.

When $h$ is a constant, system (1.1) and (1.2) reduces to the well-known Navier–Stokes. Papers [1,2] showed the following regularity criteria:

$$\int_0^T \frac{\|u(t)\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}}{\log(e + \|u(t)\|_{L^p_{\mathrm{w}}})} \mathrm{d}t < \infty \ \text{ with } \ 3 < p \le \infty, \tag{1.8}$$

or

$$u \in L^2(0, T; \mathrm{BMO}), \tag{1.9}$$

or

$$\int_0^T \frac{\|\nabla u(t)\|_{L^q_{\mathrm{w}}}^{\frac{2q}{2q-3}}}{\log(e + \|\nabla u(t)\|_{L^q_{\mathrm{w}}})} \mathrm{d}t < \infty \ \text{ with } \ \frac{3}{2} < q \le \infty, \tag{1.10}$$

or

$$\nabla u \in L^1(0, T; \mathrm{BMO}). \tag{1.11}$$

Here $L^q_{\mathrm{w}}$ is the usual weak $L^q$ space (see Definition 1.1 for details), and BMO is the space of bounded mean oscillation whose norm is defined by

$$\|f\|_{\mathrm{BMO}} := \|f\|_{L^2} + [f]_{\mathrm{BMO}},$$

with

$$[f]_{\text{BMO}} := \sup_{\substack{x\in\Omega \\ r\in(0,d)}} \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}|\mathrm{d}y,$$
$$f_{\Omega_r(x)} := \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} f(y)\mathrm{d}y,$$

$\Omega_r(x) := B_r(x)\cap\Omega$, $B_r(x)$ is the ball with center $x$ and radius $r$, and $d$ is the diameter of $\Omega$. $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

On the other hand, when $u = 0$, system (1.3), (1.4) and (1.5) reduces to the time-dependent Ginzburg–Landau, which has received many studies [3–12]. Paper [4] showed the existence of global weak solutions. Paper [10,12] proved the uniqueness of weak solutions.

The aim of this paper is to prove some regularity criteria of the problem in a bounded domain. We will prove

**Theorem 1.1** *Let* $u_0 \in H_0^1 \cap H^2$, $\psi_0, A_0 \in H^1$ *with* $|\psi_0| \le 1$, $\operatorname{div} u_0 = \operatorname{div} A_0 = 0$ *in* $\Omega$. *Let* $(u, \pi, \psi, A, \phi)$ *be a local strong solution to the problem* (1.1)–(1.7). *If* (1.8) *or* (1.9) *holds true with* $0 < T < \infty$, *then the solution* $(u, \pi, \psi, A, \phi)$ *can be extended beyond* $T > 0$.

**Theorem 1.2** *Let* $u_0 \in H_0^1 \cap H^3$, $\psi_0, A_0 \in H^1$ *with* $|\psi_0| \le 1$, $\operatorname{div} u_0 = \operatorname{div} A_0 = 0$ *in* $\Omega$. *Let* $(u, \pi, \psi, A, \phi)$ *be a local strong solution to the problem* (1.1)–(1.7). *If* (1.10) *or* (1.11) *holds true with* $0 < T < \infty$, *then the solution* $(u, \pi, \psi, A, \phi)$ *can be extended beyond* $T > 0$.

***Remark 1.1*** We can prove similar results under the Lorentz gauge.

**Definition 1.1** Let $f \in L^{p,q}$ be such that

$$\left(\frac{p}{q}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty,$$

where $f^*(t)$ is the nonincreasing function equimeasurable with $|f|$ on $(0, \infty)$. We say that $f$ belongs to the Lorentz space $L^{p,\infty} \equiv L^p_{\mathrm{w}}$ if

$$\operatorname{mes}\{x \in \Omega : |f(x)| > \alpha\} \le A\alpha^{-p} \ \text{ for all } \ \alpha > 0.$$

In the following proofs, we will use the following Gagliardo–Nirenberg inequality [13]:

$$\|u\|_{L^{\frac{2r}{r-2},2}} \le C\|u\|_{L^2}^{1-\frac{3}{r}} \|u\|_{H^1}^{\frac{3}{r}} \ \text{ with } \ 3 < r < \infty, \tag{1.12}$$

and the generalized Hölder inequality [14]:

$$\|uv\|_{L^{p,q}} \leq C\|u\|_{L^{p_1,q_1}}\|v\|_{L^{p_2,q_2}} \tag{1.13}$$

with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$.

In the following proofs, we will also use the following three lemmas:

**Lemma 1.1** *We have*

$$\|f\|_{L^\infty(\Omega)} \leq C(1+\|f\|_{\mathrm{BMO}(\Omega)}\log^{\frac{1}{2}}(e+\|f\|_{W^{1,m}(\Omega)})) \tag{1.14}$$

*for all* $f \in W_0^{1,m}(\Omega)$ *with* $3 < m < \infty$.

***Proof*** When $\Omega := \mathbb{R}^3$, (1.14) is proved by Ogawa [15]. For a bounded domain $\Omega$ in $\mathbb{R}^3$, we define

$$\tilde{f} := \begin{cases} f & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \Omega^c := \mathbb{R}^3 \setminus \Omega. \end{cases}$$

Then we have [16, p.71]:

$$\|\tilde{f}\|_{W^{1,m}(\mathbb{R}^3)} = \|f\|_{W^{1,m}(\Omega)},$$

and it is obvious that

$$\|\tilde{f}\|_{L^\infty(\mathbb{R}^3)} = \|f\|_{L^\infty(\Omega)}, \|\tilde{f}\|_{\mathrm{BMO}(\mathbb{R}^3)} \leq C\|f\|_{\mathrm{BMO}(\Omega)}.$$

Thus (1.14) is proved. □

**Lemma 1.2** *([17]). We have*

$$\|u\|_{L^4(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}\|u\|_{\mathrm{BMO}(\Omega)}. \tag{1.15}$$

**Lemma 1.3** *([18]). There holds the following logarithmic Sobolev inequality:*

$$\|\nabla f\|_{L^\infty} \leq C(1+\|\nabla f\|_{\mathrm{BMO}}\log(e+\|f\|_{W^{s,p}}) \ \ \textit{with} \ \ s > 1+\frac{3}{p} \tag{1.16}$$

*for any* $f \in W^{s,p}(\Omega)$ *and* $\Omega \subset \mathbb{R}^3$.

Applying div to (1.3) and using (1.5), we see that

$$-\Delta\phi = \operatorname{div}\operatorname{Re}\left\{\frac{i}{k}\overline{\psi}\nabla\psi + |\psi|^2 A\right\}. \tag{1.17}$$

## 2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. It is easy to show the local well-posedness of strong solutions; we only need to establish a priori estimates.

First, we set

$$f := |\psi|, \psi := f e^{i\theta}, \quad \text{and} \quad V := -A + \nabla\theta.$$

Then we have

$$\eta \partial_t f + u \cdot \nabla f = \frac{1}{k^2}\Delta f - f(f^2 - 1 + V^2). \tag{2.1}$$

Testing (2.1) by $(f-1)_+$ and using (1.1), we see that

$$\begin{aligned} &\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (f-1)_+^2 \mathrm{d}x + \frac{1}{k^2}\int |\nabla (f-1)_+|^2 \mathrm{d}x \\ &= -\int f(f^2 - 1 + V^2)(f-1)_+ \mathrm{d}x \le 0, \end{aligned}$$

which gives

$$(f-1)_+ = 0,$$

and thus

$$|\psi| \le 1. \tag{2.2}$$

Testing (1.3) by $\overline{\psi}$, taking the real parts and using (1.1), we get

$$\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\psi|^2 \mathrm{d}x + \int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 \mathrm{d}x + \int |\psi|^4 \mathrm{d}x = \int |\psi|^2 \mathrm{d}x,$$

which leads to

$$\int |\psi|^2 \mathrm{d}x + \int_0^T \int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.3}$$

Testing (1.4) by $A$, using (1.5), (2.2) and (2.3), we find that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^2 \mathrm{d}x + \int |\mathrm{rot}\, A|^2 \mathrm{d}x &= -\mathrm{Re}\int \left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi} A \mathrm{d}x \\ &\le \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2} \|\psi\|_{L^\infty} \|A\|_{L^2} \\ &\le \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2} \|A\|_{L^2}, \end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} \le C. \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that

$$\int_0^T \int |\psi A|^2 \mathrm{d}x\mathrm{d}t \le \|\psi\|_{L^\infty(0,T;L^\infty)} \int_0^T \int |A|^2 \mathrm{d}x\mathrm{d}t \le C,$$

whence

$$\|\psi\|_{L^2(0,T;H^1)} \le C. \tag{2.5}$$

Testing (1.4) by $-\Delta A$, using (1.5), (1.6), (2.2) and (2.3), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, A|^2 \mathrm{d}x + \int |\Delta A|^2 \mathrm{d}x \\
&\quad = \int \mathrm{Re}\left[\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right]\Delta A \mathrm{d}x \\
&\quad \le \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2} \|\psi\|_{L^\infty}\|\Delta A\|_{L^2} \\
&\quad \le \frac{1}{2}\|\Delta A\|_{L^2}^2 + C\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} \le C. \tag{2.6}$$

Here we used the well-known facts

$$\|A\|_{H^1} \le C(\|A\|_{L^2} + \|\mathrm{rot}\, A\|_{L^2}), \tag{2.7}$$

and

$$\|A\|_{H^2} \le C(\|A\|_{L^2} + \|\Delta A\|_{L^2}) \tag{2.8}$$

due to $\mathrm{div}\, A = 0$ in $\Omega$ and $A \cdot n = 0$, $\mathrm{rot}\, A \times n = 0$ on $\partial\Omega$.

$$\|\nabla\phi\|_{L^2(0,T;L^2)} \le C\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2(0,T;L^2)} \|\psi\|_{L^\infty(0,T;L^\infty)} \le C. \tag{2.9}$$

Testing (1.2) by $u$ and using (1.1) and (2.2), we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x + \int |\nabla u|^2\mathrm{d}x = \int |\psi|^2\nabla h\cdot u\mathrm{d}x \le \|\psi\|_{L^\infty}^2\|\nabla h\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2},$$

which yields

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \le C. \tag{2.10}$$

(I) Let (1.8) hold true.

Testing (1.2) by $-\Delta u + \nabla\pi$, using (1.1), (2.2), (1.12) and (1.13), we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla u|^2\mathrm{d}x + \int |\nabla\pi - \Delta u|^2\mathrm{d}x\\
&= -\int u\cdot\nabla u\cdot(\nabla\pi - \Delta u)\mathrm{d}x + \int |\psi|^2\nabla h(\nabla\pi - \Delta u)\mathrm{d}x\\
&\le \|u\|_{L^p_{\mathrm{w}}}\|\nabla u\|_{L^{\frac{2p}{p-2},2}}\|\nabla\pi - \Delta u\|_{L^2} + \|\psi\|_{L^\infty}^2\|\nabla h\|_{L^2}\|\nabla\pi - \Delta u\|_{L^2}\\
&\le \|u\|_{L^p_w}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|\nabla\pi - \Delta u\|_{L^2}^{1+\frac{3}{p}} + C\|\nabla\pi - \Delta u\|_{L^2}\\
&\le \frac{1}{2}\|\nabla\pi - \Delta u\|_{L^2}^2 + C\|u\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}\|\nabla u\|_{L^2}^2 + C\\
&\le \frac{1}{2}\|\nabla\pi - \Delta u\|_{L^2}^2 + \frac{C\|u\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}}{\log(e + \|u\|_{L^p_{\mathrm{w}}})}\|\nabla u\|_{L^2}^2\log(e + \|u\|_{L^p_{\mathrm{w}}}) + C\\
&\le \frac{1}{2}\|\nabla\pi - \Delta u\|_{L^2}^2 + \frac{C\|u\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-2}}}{\log(e + \|u\|_{L^p_{\mathrm{w}}})}\|\nabla u\|_{L^2}^2\log(e + y) + C,
\end{aligned} \tag{2.11}$$

which gives

$$\int |\nabla u|^2\mathrm{d}x + \int_0^t\int |\Delta u|^2\mathrm{d}x\mathrm{d}s \le C(e + y)^{C_0\epsilon} \tag{2.12}$$

with

$$y(t) := \sup_{[t_0,t]}\|u(\cdot,s)\|_{W^{1,m}} \text{ and } 3 < m \le 6,$$

for any $0 < t_0 \le t \le T$, where $C_0$ is an absolute constant, provided that

$$\int_{t_0}^T \frac{\|u(t)\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}}{\log(e + \|u\|_{L^p_{\mathrm{w}}})}\mathrm{d}t \le \epsilon << 1. \tag{2.13}$$

Here we have used the well-known $H^2$-estimate of Stokes system:

$$\|u\|_{H^2} \le C\|\nabla \pi - \Delta u\|_{L^2}. \tag{2.14}$$

Integrating (2.11) over $(t_0, t)$ and using (2.12) and (2.13), we obtain

$$\int_{t_0}^{t} \int |\partial_t u|^2 \mathrm{d}x\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{2.15}$$

Equation (1.3) can be rewritten as

$$\begin{aligned} &\eta\partial_t\psi + u\cdot\nabla\psi + i\eta k\phi\psi - \frac{1}{k^2}\Delta\psi + \frac{2i}{k}A\cdot\nabla\psi + |A|^2\psi + |\psi|^2\psi - \psi \\ &\quad = 0. \end{aligned} \tag{2.16}$$

Testing (2.16) by $-\Delta\overline{\psi}$ and taking the real parts, using (2.2), (2.6), (2.11), (1.14) and (1.13), we have

$$\begin{aligned} &\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\mathrm{d}x + \frac{1}{k^2}\int |\Delta\psi|^2\mathrm{d}x \\ &\quad = \mathrm{Re}\int u\cdot\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x \\ &\qquad + \mathrm{Re}\int i\eta k\phi\psi\Delta\overline{\psi}\mathrm{d}x - \mathrm{Re}\int \frac{2i}{k}A\cdot\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x \\ &\qquad + \mathrm{Re}\int |A|^2\psi\Delta\overline{\psi}\mathrm{d}x + \mathrm{Re}\int |\psi|^2\psi\Delta\overline{\psi}\mathrm{d}x + \int |\nabla\psi|^2\mathrm{d}x \\ &\quad \le \|u\|_{L^p_{\mathrm{w}}}\|\nabla\psi\|_{L^{\frac{2p}{p-2},2}}\|\Delta\psi\|_{L^2} + C\|\phi\|_{L^2}\|\Delta\psi\|_{L^2} \\ &\qquad + C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2} \\ &\qquad + C\|A\|_{L^4}^2\|\Delta\psi\|_{L^2} + C\|\psi\|_{L^2}\|\Delta\psi\|_{L^2} + \|\nabla\psi\|_{L^2}^2 \\ &\quad \le C\|u\|_{L^p_{\mathrm{w}}}\|\nabla\psi\|_{L^2}^{1-\frac{3}{p}}\|\Delta\psi\|_{L^2}^{1+\frac{3}{p}} + C\|\phi\|_{L^2}\|\Delta\psi\|_{L^2} \\ &\qquad + C\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2} + C\|\Delta\psi\|_{L^2} \\ &\quad \le \frac{1}{32k^2}\|\Delta\psi\|_{L^2}^2 + C\|u\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}\|\nabla\psi\|_{L^2}^2 + C\|\phi\|_{L^2}^2 + C \\ &\quad \le \frac{1}{32k^2}\|\Delta\psi\|_{L^2}^2 + C\frac{\|u\|_{L^p_{\mathrm{w}}}^{\frac{2p}{p-3}}}{\log(e+\|u\|_{L^p_{\mathrm{w}}})}\log(e+y)\|\nabla\psi\|_{L^2}^2 \\ &\qquad + C\|\phi\|_{L^2}^2 + C, \end{aligned} \tag{2.17}$$

which implies

$$\int |\nabla\psi|^2\mathrm{d}x + \int_{t_0}^{t}\int |\Delta\psi|^2\mathrm{d}x\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{2.18}$$

Here we have used the Gagliardo–Nirenberg inequalities

$$\|\nabla\psi\|_{L^4}^2 \le C\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}, \tag{2.19}$$
$$\|\nabla\psi\|_{L^2}^2 \le C\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}, \tag{2.20}$$

and the fact

$$\|\psi\|_{H^2} \le C\|\Delta\psi\|_{L^2}. \tag{2.21}$$

Similarly, testing (2.16) by $\partial_t\overline{\psi}$ and taking the real parts, we obtain

$$\int_{t_0}^{t}\int |\partial_t\psi|^2 \mathrm{d}x\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{2.22}$$

Taking $\partial_t$ to (1.2), testing by $\partial_t u$, using (1.1), (2.2), (2.12), (2.15) and (2.22), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t u|^2\mathrm{d}x + \int |\nabla\partial_t u|^2\mathrm{d}x \\
&= -\int \partial_t u\cdot\nabla u\cdot\partial_t u\mathrm{d}x + \int \partial_t|\psi|^2\nabla h\partial_t u\mathrm{d}x \\
&\le \|\nabla u\|_{L^2}\|\partial_t u\|_{L^4}^2 + C\|\psi\|_{L^\infty}\|\partial_t\psi\|_{L^2}\|\nabla h\|_{L^\infty}\|\partial_t u\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t u\|_{L^2}^{\frac{3}{2}} + C\|\partial_t\psi\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2 \\
&\le \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\partial_t u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2
\end{aligned} \tag{2.23}$$

which leads to

$$\int |\partial_t u|^2\mathrm{d}x + \int_{t_0}^{t}\int |\nabla\partial_t u|^2\mathrm{d}x\mathrm{d}s \le C(e+y)^{C_0\epsilon}. \tag{2.24}$$

On the other hand, Eq. (1.2) can be rewritten as

$$-\Delta u + \nabla\pi = f := |\psi|^2\nabla h - \partial_t u - u\cdot\nabla u. \tag{2.25}$$

Then we have

$$\begin{aligned}
\|u\|_{H^2} &\le C\|f\|_{L^2} = C\||\psi|^2\nabla h - \partial_t u - u\cdot\nabla u\|_{L^2} \\
&\le C + C\|\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} \\
&\le C + C\|\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}\cdot\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

whence

$$\begin{aligned}\|u\|_{H^2} &\leq C + C\|\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}^3\\ &\leq C(e+y)^{C_0\epsilon},\end{aligned}\tag{2.26}$$

which implies

$$\|u\|_{L^\infty(0,T;H^2)} \leq C.\tag{2.27}$$

(2.18) and (2.27) give

$$\|\psi\|_{L^\infty(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^2)} \leq C.\tag{2.28}$$

(1.17), (2.6) and (2.28) lead to

$$\|\phi\|_{L^\infty(0,T;H^1)} + \|\phi\|_{L^2(0,T;H^2)} \leq C.\tag{2.29}$$

(II) Let (1.9) hold true.

We still have (2.9) (with $p = \infty$) and using Lemma 1.1, we get (2.12). Provided that

$$\int_{t_0}^{T} \|u(t)\|_{\mathrm{BMO}}^2 \mathrm{d}t \leq \epsilon << 1.$$

Similar to (2.17), we have

$$\begin{aligned}&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\mathrm{d}x + \frac{1}{k^2}\int |\Delta\psi|^2\mathrm{d}x\\ &\quad= \mathrm{Re}\int u\cdot\nabla\psi\cdot\Delta\psi\mathrm{d}x + \text{the same other terms}.\end{aligned}\tag{2.30}$$

Now we bound the first term of RHS of (2.30) as follows:

$$\begin{aligned}&\mathrm{Re}\int u\cdot\nabla\psi\cdot\Delta\psi\mathrm{d}x \leq \|u\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2}\\ &\leq \|u\|_{L^2}^{\frac{1}{2}}\|u\|_{\mathrm{BMO}}^{\frac{1}{2}}\cdot\|\psi\|_{L^\infty}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}}\cdot\|\Delta\psi\|_{L^2}\\ &\leq C\|u\|_{\mathrm{BMO}}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{3}{2}} \leq \frac{1}{32k^2}\|\Delta\psi\|_{L^2}^2 + C\|u\|_{\mathrm{BMO}}^2.\end{aligned}\tag{2.31}$$

And thus we have

$$\int |\nabla\psi|^2\mathrm{d}x + \int_0^t\int(|\Delta\psi|^2 + |\partial_t\psi|^2)\mathrm{d}x\mathrm{d}s \leq C.\tag{2.32}$$

Then we still have (2.27) and (2.29).

This completes the proof. □

## 3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We only need to establish some a priori estimates.

We still have (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) and (2.10).

(I). Let (1.10) hold true.

We still have (2.17). We bound the first term of the RHS of (2.17) as follows:

$$\begin{aligned}\mathrm{Re}\int u\cdot\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x &= \sum_{i,j}\mathrm{Re}\int u_i\partial_i\psi\partial_j^2\overline{\psi}\mathrm{d}x = -\sum_{i,j}\mathrm{Re}\int \partial_j u_i\partial_i\psi\partial_j\overline{\psi}\mathrm{d}x\\ &\le C\|\nabla u\|_{L^p_{\mathrm{w}}}\|\nabla\psi\|^2_{L^{\frac{2q}{q-1},2}} \le C\|\nabla u\|_{L^q_{\mathrm{w}}}\|\nabla\psi\|_{L^2}^{2-\frac{3}{q}}\|\Delta\psi\|_{L^2}^{\frac{3}{q}}\\ &\le \frac{1}{32k^2}\|\Delta\psi\|^2_{L^2}+C\|\nabla u\|_{L^q_{\mathrm{w}}}^{\frac{2q}{2q-3}}\|\nabla\psi\|^2_{L^2}\le\frac{1}{32k^2}\|\Delta\psi\|^2_{L^2}\\ &\quad +C\frac{\|\nabla u\|_{L^q_{\mathrm{w}}}^{\frac{2q}{2q-3}}}{\log(e+\|\nabla u\|_{L^q_{\mathrm{w}}})}\log(e+z)\|\nabla\psi\|^2_{L^2}\end{aligned}\tag{3.1}$$

with

$$z(t):=\sup_{[t_0,t]}\|u(\cdot,s)\|_{H^3}.$$

The other terms can be bounded as before.

Then we have

$$\int|\nabla\psi|^2\mathrm{d}x+\int_{t_0}^t\int(|\Delta\psi|^2+|\partial_t\psi|^2)\mathrm{d}x\mathrm{d}s\le C(e+z)^{C_0\epsilon}\tag{3.2}$$

provided that

$$\int_{t_0}^T\frac{\|\nabla u(t)\|_{L^q_{\mathrm{w}}}^{\frac{2q}{2q-3}}}{\log(e+\|\nabla u\|_{L^q_{\mathrm{w}}})}\mathrm{d}t\le\epsilon<<1.\tag{3.3}$$

Similarly to (2.23), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\partial_t u|^2\mathrm{d}x+\int|\nabla\partial_t u|^2\mathrm{d}x\\ &=-\int\partial_t u\cdot\nabla u\cdot\partial_t u\mathrm{d}x+\int\partial_t|\psi|^2\cdot\nabla h\cdot\partial_t u\mathrm{d}x\\ &\le C\|\nabla u\|_{L^q_{\mathrm{w}}}\|\partial_t u\|^2_{L^{\frac{2q}{q-1},2}}+C\|\partial_t\psi\|_{L^2}\|\partial_t u\|_{L^2}\\ &\le C\|\nabla u\|_{L^q_{\mathrm{w}}}\|\partial_t u\|_{L^2}^{2-\frac{3}{q}}\|\nabla\partial_t u\|_{L^2}^{\frac{3}{q}}+C\|\partial_t\psi\|_{L^2}\|\partial_t u\|_{L^2}\end{aligned}$$

$$\leq \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^q_{\rm w}}^{\frac{2q}{2q-3}}\|\partial_t u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}\|\partial_t u\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C\frac{\|\nabla u\|_{L^q_{\rm w}}^{\frac{2q}{2q-3}}}{\log(e+\|\nabla u\|_{L^q_{\rm w}})}\log(e+z)\|\partial_t u\|_{L^2}^2$$
$$+C\|\partial_t\psi\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2,$$

which gives

$$\int|\partial_t u|^2\mathrm{d}x + \int_0^t\int|\nabla\partial_t u|^2\mathrm{d}x\mathrm{d}s \leq C(e+z)^{C_0\epsilon}. \tag{3.4}$$

Testing (1.2) by $\partial_t u$, using (1.1), (2.10) and (3.4), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x + \int|\partial_t u|^2\mathrm{d}x = -\int u\cdot\nabla u\cdot\partial_t u\mathrm{d}x + \int|\psi|^2\nabla h\cdot\partial_t u\mathrm{d}x$$
$$\leq \|u\|_{L^6}\|\nabla u\|_{L^2}\|\partial_t u\|_{L^3} + C\|\partial_t u\|_{L^2}$$
$$\leq C\|\nabla u\|_{L^2}^4 + C\|\nabla\partial_t u\|_{L^2}^2 + C,$$

which gives

$$\int|\nabla u|^2\mathrm{d}x \leq C(e+z)^{C_0\epsilon}. \tag{3.5}$$

Applying $\partial_t$ to (1.2), testing by $-\Delta\partial_t u + \nabla\partial_t\pi$, using (1.1), (2.2), (3.2), (3.4) and (3.5), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\partial_t u|^2\mathrm{d}x + \int|\nabla\partial_t\pi - \Delta\partial_t u|^2\mathrm{d}x$$
$$= -\int(\partial_t u\cdot\nabla u + u\cdot\nabla\partial_t u)(\nabla\partial_t\pi - \Delta\partial_t u)\mathrm{d}x$$
$$+\int\partial_t|\psi|^2\cdot\nabla h(\nabla\partial_t\pi - \Delta\partial_t u)\mathrm{d}x$$
$$\leq (\|\partial_t u\|_{L^\infty}\|\nabla u\|_{L^2} + \|u\|_{L^6}\|\nabla\partial_t u\|_{L^3})\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}$$
$$+C\|\partial_t\psi\|_{L^2}\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}$$
$$\leq C\|\nabla\partial_t u\|_{L^2}^{\frac{1}{2}}\|\Delta\partial_t u\|_{L^2}^{\frac{1}{2}}\cdot\|\nabla u\|_{L^2}\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}$$
$$+C\|\partial_t\psi\|_{L^2}\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla\partial_t u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}^2,$$

which implies

$$\int|\nabla\partial_t u|^2\mathrm{d}x + \int_{t_0}^t\int|\nabla\partial_t\pi - \Delta\partial_t u|^2\mathrm{d}x\mathrm{d}s \leq C(e+z)^{C_0\epsilon}. \tag{3.6}$$

Here we have used the fact

$$\|\partial_t u\|_{H^2} \le C\|\nabla\partial_t\pi - \Delta\partial_t u\|_{L^2}. \tag{3.7}$$

From (2.25), (2.2), (3.2), (3.6) and (3.5), we have

$$\begin{aligned}
\|u\|_{H^3} &\le C(\|\nabla f\|_{L^2} + \|u\|_{L^2})\\
&\le C\|\nabla(|\psi|^2\nabla h)\|_{L^2} + C\|\nabla\partial_t u\|_{L^2} + C\|\nabla(u\cdot\nabla u)\|_{L^2} + C\\
&\le C\|\nabla\psi\|_{L^2} + C\|\nabla\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla^2 u\|_{L^3} + C\|\nabla u\|_{L^4}^2 + C\\
&\le C\|\nabla\psi\|_{L^2} + C\|\nabla\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac14}\|u\|_{H^3}^{\frac34}\\
&\quad + C\|\nabla u\|_{L^2}^{\frac54}\|u\|_{H^3}^{\frac34} + C,
\end{aligned}$$

and therefore

$$\begin{aligned}
\|u\|_{H^3} &\le C\|\nabla\psi\|_{L^2} + C\|\nabla\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}^5 + C\\
&\le C(e+z)^{C_0\epsilon},
\end{aligned}$$

which implies

$$\|u\|_{L^\infty(0,T;H^3)} \le C. \tag{3.8}$$

We still have (2.29).

(II) Let (1.11) hold true.

Similarly to (3.1) for $q=\infty$ and using Lemma 1.2, we still have (3.2), provided that

$$\int_{t_0}^{t} \|\nabla u\|_{\mathrm{BMO}} \mathrm{d}t \le \epsilon << 1. \tag{3.9}$$

We still have (3.4) (for $q=\infty$ and Lemma 1.2) and (3.5).

We still have (3.6), (3.8), (2.29) and (2.32).

This completes the proof. □

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## 4 Appendix

In this appendix, we will prove the local well-posedness of strong solutions to the problem (1.1)–(1.7).

First, we give the definition of strong solutions to the problem.

**Definition 4.1** (*strong solutions*). $(u, \pi, \psi, A, \phi)$ is called a strong solution to the problem (1.1)–(1.7) in $\Omega \times (0, T)$ under the Coulomb gauge if

$$\begin{aligned}&(u, \psi, A)(\cdot, 0) = (u_0, \psi_0, A_0)(\cdot) \in H_0^1 \cap H^2 \times H^1 \times H^1 \ \text{ with}\\ &|\psi_0| \le 1, \operatorname{div} u_0 = \operatorname{div} A_0 = 0 \ \text{ in } \ \Omega,\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&u \in L^\infty(0, T; H_0^1 \cap H^2) \cap L^2(0, T; H^3),\\ &\pi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),\\ &\psi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), |\psi| \le 1 \ \text{ in } \ \Omega \times (0, T),\\ &A \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),\\ &\phi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),\\ &\partial_t u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1),\\ &\partial_t \psi, \partial_t A \in L^2(0, T; L^2),\end{aligned} \tag{4.2}$$

and the equations

$$(1.1) - (1.5) \ \textit{holds almost everywhere on} \ \Omega \times (0, T). \tag{4.3}$$

In this appendix, we will prove

**Theorem 4.1** *Let* (4.1) *hold true. Then the problem* (1.1)–(1.7) *has a unique strong solution* $(u, \pi, \psi, A, \phi)$ *satisfying (4.2) for some* $0 < T \le \infty$.

The proof of the uniqueness part is standard with regularity (4.2), and thus we omit the details here. We will use the Galerkin method to show the existence part; the key step of the Galerkin method is to show the a priori estimates. Thus we only need to show the a priori estimates.

***Proof of Theorem 4.1*** We still have (2.2), (2.3), (2.4), (2.5), (2.6), (2.9) and (2.10). Similar to (2.11), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla u|^2 \mathrm{d}x + \int |\nabla \pi - \Delta u|^2 \mathrm{d}x\\ &= -\int u \cdot \nabla u \cdot (\nabla \pi - \Delta u)\mathrm{d}x + \int |\psi|^2 \nabla h \cdot (\nabla \pi - \Delta u)\mathrm{d}x\\ &\le \|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla \pi - \Delta u\|_{L^2} + \|\psi\|_{L^\infty}^2 \|\nabla h\|_{L^2}\|\nabla \pi - \Delta u\|_{L^2}\\ &\le C\|\nabla u\|_{L^2} \cdot \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \pi - \Delta u\|_{L^2}^{\frac{3}{2}} + C\|\nabla \pi - \Delta u\|_{L^2}\\ &\le \frac{1}{2}\|\nabla \pi - \Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C,\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} \le C \tag{4.4}$$

for some $0 < T \le \infty$.

Similar to (2.17), we get

$$\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\mathrm{d}x + \frac{1}{k^2}\int |\Delta\psi|^2\mathrm{d}x \\
&= \mathrm{Re}\int u\cdot\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x + \mathrm{Re}\int i\eta k\phi\psi\Delta\overline{\psi}\mathrm{d}x - \mathrm{Re}\int \frac{2i}{k}A\cdot\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x \\
&\quad + \mathrm{Re}\int |A|^2\psi\Delta\overline{\psi}\mathrm{d}x + \mathrm{Re}\int |\psi|^2\psi\Delta\overline{\psi}\mathrm{d}x + \int |\nabla\psi|^2\mathrm{d}x \\
&\le C\|u\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2} + C\|\phi\|_{L^2}\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2} \\
&\quad + C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2} + C\|A\|_{L^4}^2\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2} \\
&\quad + C\|\psi\|_{L^\infty}^2\|\psi\|_{L^2}\|\Delta\psi\|_{L^2} + \|\nabla\psi\|_{L^2}^2 \\
&\le C\|\psi\|_{L^\infty}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{3}{2}} + C\|\phi\|_{L^2}\|\Delta\psi\|_{L^2} + C\|\Delta\psi\|_{L^2} + \|\nabla\psi\|_{L^2}^2 \\
&\le \frac{1}{2k^2}\|\Delta\psi\|_{L^2}^2 + C + C\|\phi\|_{L^2}^2 + C\|\nabla\psi\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|\psi\|_{L^\infty(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^2)} \le C \tag{4.5}$$

for some $0 < T \le \infty$.

Similar to (2.22), we have

$$\|\partial_t\psi\|_{L^2(0,T;L^2)} \le C \tag{4.6}$$

for some $0 < T \le \infty$.

Similar to (2.23), we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t u|^2\mathrm{d}x + \int |\nabla\partial_t u|^2\mathrm{d}x \\
&= -\int \partial_t u\cdot\nabla u\cdot\partial_t u\mathrm{d}x + \int \partial_t|\psi|^2\nabla h\cdot\partial_t u\mathrm{d}x \\
&\le \|\nabla u\|_{L^2}\|\partial_t u\|_{L^4}^2 + C\|\psi\|_{L^\infty}\|\partial_t\psi\|_{L^2}\|\nabla h\|_{L^\infty}\|\partial_t u\|_{L^2} \\
&\le C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t u\|_{L^2}^{\frac{3}{2}} + C\|\partial_t\psi\|_{L^2}\|\partial_t u\|_{L^2} \\
&\le \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|\partial_t u\|_{L^\infty(0,T;L^2)} + \|\partial_t u\|_{L^2(0,T;H^1)} \le C \tag{4.7}$$

for some $0 < T \le \infty$.

We still have (2.27), (2.28) and (2.29).

This completes the proof. □

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