# Moving Frames and Differential Invariants on Fully Affine Planar Curves

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## Abstract

In this paper, by the affine analogue of the fundamental theorem for Euclidean planar curves, we classify the affine curves with constant affine curvatures. Note that we use the fully affine group and not the equi-affine subgroup consisting of area-preserving affine transformations. (Caution: much of the literature omits the "equi-" in their treatment.) According to the equivariant method of moving frames, explicit formulas for the generating affine differential invariants and invariant differential operators are constructed. At the same time, by using the fact that the affine transformation group $GA(2, \mathbb{R})$ can factor as a product of two subgroup $B \cdot SE(2, \mathbb{R})$ and the moving frame of the subgroup $SE(2, \mathbb{R})$, we build the moving frame of $GA(2, \mathbb{R})$ and obtain the relations among invariants of group $GA(2, \mathbb{R})$ and its subgroup $SE(2, \mathbb{R})$. Applying the affine curvature to recognize affine equivalent objects is considered in the last part of this paper.

**Keywords** Arc length parameter · Affine curvature · Maurer–Cartan invariant · Moving frame

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## 1 Introduction

In Euclidean differential geometry, the Frenet–Serret frame is a moving frame defined on a curve which can be constructed purely from the velocity and acceleration of the curve. The torsion and curvature are obtained by differentiating the frame, which describe completely how the frame evolves in time along the curve [4]. Special affine curvature, also known as the equi-affine curvature, is a particular type of curvature that is defined on a plane curve and remains unchanged under a special affine transformation (an affine transformation that preserves area) [16]. Hu and Liu studied centro-affine invariant arc length and centro-affine curvature functions of a curve in affine space directly by the parameter transformations and the centro-affine transformations [7,13]. Yang and Yu defined the notion of affine curvatures on a discrete planar curve and studied the applications in self-affine fractals [24]. By using the equivariant method of moving frames, Olver gave the complete systems of functionally independent differential invariants for curves under centro-affine transformations and equi-affine transformations [21,22]. Applications of full affine invariance to image processing can be found, for instance, in [25]. In this paper, we introduce and study the fully affine invariant arc length and fully affine curvature functions on a planar curve.

The arrangement of this paper is as follows: In Sect. 2, we introduce moving frames and the definitions. The equivariant method of moving frames is applied to generate affine differential invariants and invariant differential operators. In Sect. 3, we use the traditional method to obtain the fully affine invariants and then study the arc length and affine curvature. A classification for the planar curves with the constant affine curvatures is also given. Section 4 expresses the differential invariants for the affine transformation group in terms of the differential invariants of its subgroup. Finally, in Sect. 5, we investigate some applications of the affine curvature invariants.

## 2 Preliminaries

### 2.1 The Equivariant Method of Moving Frames

Let us first recall the basic equivariant moving frame method introduced by Fels and Olver [5,6]. For more details, see [5,6,14,15,19]. Assume $G$ is an $r$-dimensional Lie group acting smoothly on an $m$-dimensional manifold $M$ with a left action, such that

$$G \times M \to M, \quad h \cdot (g \cdot z) = (hg) \cdot z.$$

A *right (resp. left) group-based moving frame* is a smooth, $G$-equivariant map $\rho : M \to G$, with respect to the action on $M$ and the inverse right (resp. left) action of $G$ on itself, specifically,

$$\rho : M \to G, \quad \rho(g \cdot z) = \rho(z)g^{-1} \quad (\text{resp. } \rho(g \cdot z) = g\rho(z)).$$

In many cases, right moving frames can be easier to compute.

Given a Lie group $G$ acting on a manifold $M$, a moving frame exists in a neighborhood of a point $z \in M$ if and only if $G$ acts freely and regularly near $z$; that is, the existence of a moving frame on the open subset $U \subset M$ is guaranteed if:

1. the intersection of the orbits with $U$ has the same dimension as that of the group $G$ and further foliate $U$ (the regularity of the foliation is necessary);
2. there exists a submanifold $\mathcal{K} \subset U$ that intersects the orbits of $U$ transversally, and the intersection $\mathcal{O} \cap \mathcal{K}$ contains a single point, and this submanifold $\mathcal{K}$ is known as the *cross section* and has dimension equal to $\dim(M) - \dim(G)$;
3. given a point $z \in \mathcal{O}(z) \cap U$, where $\mathcal{O}(z)$ is the orbit through $z$, the group element $h = \rho(z) \in G$ which takes $z$ to a point $h \cdot z \in \mathcal{O}(z) \cap \mathcal{K}$ on the cross section is unique.

Given local coordinates $Z = (z_1, \ldots, z_m)$ on $M$, let $\omega(g, z) = g \cdot z$ be the formulae for the transformed coordinates under the group transformation. The right moving frame $g = \rho(z)$ associated with the coordinate cross section $K = \{z_1 = c_1, \ldots, z_r = c_r\}$ is found by solving the *normalization equations*

$$\omega_1(g, z) = c_1, \ldots, \omega_r(g, z) = c_r, \tag{2.1}$$

for the group parameters $(g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. The remaining coordinates are the non-constant invariants.

**Theorem 2.1** *If $g = \rho(z)$ is the moving frame solution to Eq. (2.1), then*

$$I_1(z) = \omega(\rho(z) \cdot z), \ldots, I_{z-r}(z) = \omega_m(\rho(z) \cdot z), \tag{2.2}$$

*form a complete system of functionally independent invariants.*

**Definition 2.2** The invariantization of a scalar function $F : M \to \mathbb{R}$ with respect to a right moving frame is the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

However, most interesting group actions (Euclidean, affine, projective, etc.) are not free, and freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $\dim(M) = m < r = \dim(G)$. There are two basic methods for converting a non-free (but effective) action into a free action. The first is to look at the product action of $G$ on a sufficiently large Cartesian product $M^{\times(n+1)}$, leading to *joint invariants* [18], which are especially interesting in classical algebra and numerical analysis [20,23]. The second is to prolong the group action to a jet space $\mathbf{J}^n$ of suitably high order, which is the natural setting for the traditional moving frame theory, and leads to the classical *differential invariants* for the group [22]. Combining the two methods of prolongation and product will lead to *joint differential invariants* [18].

**Definition 2.3** Given a smooth manifold $M$ of dimension $m$ and an integer $p < m$, the $k$th-order jet bundle $J^k = J^k(M, p)$ is a fiber bundle over $M$, such that the fiber of a point $z \in M$ consists of the set of equivalence classes of $p$-dimensional submanifolds of $M$ under the equivalence relation of $k$th-order contact at the point $z$.

Assume the manifold $M$ is given by the local coordinates $(x^1, \cdots, x^p, u^1, \cdots, u^q)$ in some neighborhood where the submanifold $S$ can be represented as a graph $u = u(x)$. The *fundamental differential invariants* are obtained by invariantization of the individual jet coordinate functions,

$$H^i = \iota(x^i), \quad I_J^\alpha = \iota(u_J^\alpha), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad \#J \geq 0. \tag{2.3}$$

The specification of independent and dependent variables on $M$ further splits the differential one forms on the infinite-order jet bundle $J^\infty$ into *horizontal one forms*, spanned by $dx^1, \cdots, dx^p$ and *contact one form*, spanned by the *basic contact one forms*

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha dx^i, \quad \alpha = 1, \ldots, q, \quad 0 \leq \#J. \tag{2.4}$$

A connected transformation group $G$ can be prescribed by its infinitesimal generators. A basis for the infinitesimal generators of the effectively acting $r$-dimensional transformation group is

$$\mathbf{v}_\sigma = \sum_{i=1}^p \xi_\sigma^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi_\sigma^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \quad \sigma = 1, \ldots, r, \tag{2.5}$$

which can be identified with a basis of its Lie algebra $\mathfrak{g}$. The corresponding prolonged infinitesimal generator

$$\mathrm{pr}\, \mathbf{v}_\sigma = \sum_{i=1}^p \xi_\sigma^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{k=\#J\geq 0} \varphi_{J,\sigma}^\alpha(x, u^{(k)})\frac{\partial}{\partial u^\alpha}, \quad \sigma = 1, \ldots, r,$$

$$\tag{2.6}$$

generates the prolongation of the associated one-parameter subgroup acting on jet bundles and

$$\varphi_{J,\sigma}^\alpha = D_J \left( \varphi_\sigma^\alpha - \sum_{i=1}^p \xi_\sigma^i u_i^\alpha \right) + \sum_{i=1}^p \xi_\sigma^i u_{J,i}^\alpha.$$

In particular, specializing the recurrence relations to the cross section coordinates produces the following formulas that enable one to easily determine the formulas for the invariantized Maurer–Cartan forms by merely solving a system of $r = \dim G$ linear equations.

**Theorem 2.4** [22] *Let $I_1 = \iota(z_1), \ldots, I_r = \iota(z_r)$ be the phantom invariants related to the cross section coordinates $z_1, \ldots, z_r$. Then*

$$0 = dI_\varsigma = d\iota(z_\varsigma) = \iota(dz_\varsigma) + \sum_{k=1}^r \nu^k \iota[\mathrm{pr}\, \mathbf{v}_k(z_\varsigma)], \quad \varsigma = 1, \ldots, r. \tag{2.7}$$

When we only consider curves, $p = 1$, the invariant Maurer–Cartan forms can be rewritten in terms of the invariant horizontal and contact forms

$$v^k = R^k \varpi + \sum_{\alpha, J} S_\alpha^{k, J} \vartheta_J^\alpha, \tag{2.8}$$

where $\varphi = \iota(dx)$ and $\vartheta_J^\alpha = \iota(\theta_J^\alpha)$. The coefficient $R^k$ are known as the Maurer–Cartan invariants and $S_\alpha^{K, J}$ are also differential invariants. The Maurer–Cartan invariants can be written as $\mathcal{R} = \{R^\sigma, \sigma = 1, \ldots, r\}$.

**Theorem 2.5** [22] *If the moving frame has order n, then the set of fundamental differential invariants*

$$\mathcal{I}^{(n+1)} = \{H_i, I_J^\alpha \mid i = 1, \ldots p, \ \alpha = 1, \ldots, q, \ \#J \le n + 1\}$$

*of order $\le n + 1$ forms a generating set.*

**Theorem 2.6** [22] *The differential invariants $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating set.*

## 2.2 Recurrence and the Algebra of Affine Differential Invariants

According to the Erlangen program, a Klein geometry is the theory of geometric invariants of a transformation group. A geometric invariant is an invariant of a transformation group which is not changed in any differentiable coordinate transformation. For example, Euclidean geometry studies the theory of geometric invariants under rigid motion. Affine geometry is the theory of geometric invariants under affine transformation group. The nice thing about Klein geometry for curves is that we have a very precise notion of group arc length and group curvature. According to [3], several group arc lengths and curvatures for planar curves are specified in Table 1. In Table 1,

$$P = 3u_{xx}u_{xxxx} - 5u_{xxx}^2, \quad Q = 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3.$$

Let us now implement the moving frame calculation for the affine group acting on plane curves based on Sect. 2.1. We write the standard action of the affine group on the plane in a slightly more convenient form:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{where} \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1, \quad \lambda \ne 0. \tag{2.9}$$

**Table 1** The arc length and curvature for several Klein geometries

| Group name | Arc length | Curvature |
|---|---|---|
| $E(2)$ | $\sqrt{1 + u_x^2}\,dx$ | $(1 + u_x^2)^{-3/2}u_{xx}$ |
| $Sim(2)$ | $(1 + u_x^2)^{-1}u_{xx}dx$ | $\left[(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2\right]u_{xx}^{-2}$ |
| $SA(2)$ | $u_{xx}^{1/3}dx$ | $u_{xx}^{-8/3}P$ |
| $A(2)$ | $u_{xx}^{-1}\sqrt{P}dx$ | $P^{-3/2}Q$ |

By a direct computation, the prolonged affine transformations up to order 6 are given by

$$y = \lambda\alpha x + \lambda\beta u + a,$$
$$v = \lambda\gamma x + \lambda\delta u + b,$$
$$v_y = \frac{\gamma + \delta u_x}{\alpha + \beta u_x},$$
$$v_{yy} = \frac{u_{xx}}{\lambda(\alpha + \beta u_x)^3},$$
$$v_{yyy} = \frac{(\alpha + \beta u_x)u_{xxx} - 3\beta u_{xx}^2}{\lambda^2(\alpha + \beta u_x)^5},$$
$$v_{yyyy} = \frac{(\alpha + \beta u_x)^2 u_{xxxx} - 10\beta u_{xx}u_{xxx}(\alpha + \beta u_x) + 15\beta^2 u_{xx}^3}{\lambda^3(\alpha + \beta u_x)^7},$$
$$v_{yyyyy} = \frac{A}{\lambda^4(\alpha + \beta u_x)^9},$$
$$v_{yyyyyy} = \frac{B}{\lambda^5(\alpha + \beta u_x)^{11}},$$

where

$$A = (\alpha + \beta u_x)^3 u_{xxxxx} - 15\beta u_{xx}u_{xxxx}(\alpha + \beta u_x)^2 - 10\beta u_{xxx}^2(\alpha + \beta u_x)^2$$
$$+ 105\beta^2 u_{xx}^2 u_{xxx}(\alpha + \beta u_x) - 105\beta^3 u_{xx}^4,$$
$$B = (\alpha + \beta u_x)^4 u_{xxxxxx} - 21\beta u_{xx}u_{xxxxx}(\alpha + \beta u_x)^3 - 35\beta u_{xxx}u_{xxxx}(\alpha + \beta u_x)^3$$
$$+ 210\beta^2 u_{xx}^2 u_{xxxx}(\alpha + \beta u_x)^2$$
$$+ 280\beta^2 u_{xx}u_{xxx}^2(\alpha + \beta u_x)^2 - 1260\beta^3 u_{xx}^3 u_{xxx}(\alpha + \beta u_x) + 945\beta^4 u_{xx}^5.$$

To construct a moving frame, we use the cross section normalizations

$$y = 0, \quad v = 0, \quad v_y = 0, \quad v_{yy} = \frac{9}{2}, \quad v_{yyy} = 0, \quad v_{yyyy} = \frac{3}{2}. \tag{2.10}$$

Generally, the choice of cross section normalizations should be $v_{yy} = 1$ and $v_{yyyy} = 1$. In order to produce the fully affine curvature in agreement with the following section, here we use $v_{yy} = \frac{9}{2}$ and $v_{yyyy} = \frac{3}{2}$.

Solving for the group parameters (omitting the translational components $a, b$ since they play no role) yields

$$\delta^4 = \frac{81(3u_{xx}u_{xxxx} - 5u_{xxx}^2)}{4u_{xx}^4}, \quad \lambda = \frac{2}{9}\delta^3 u_{xx}, \quad \gamma = -\delta u_x, \quad \beta = \frac{u_{xxx}}{3\delta u_{xx}^2},$$
$$\alpha = \frac{1}{\delta} - \frac{u_{xxx}u_x}{3\delta u_{xx}^2}, \tag{2.11}$$

which prescribes the right-equivariant moving frame. Then, the affine arc length element is given as

$$ds = \frac{\sqrt{3u_{xx}u_{xxxx} - 5u_{xxx}^2}}{u_{xx}}dx. \tag{2.12}$$

By a direct calculation,

$$v_{yyyyy} = \frac{9}{2(3u_{xx}u_{xxxx} - 5u_{xxx}^2)^{\frac{3}{2}}}\left(u_{xxxxx}u_{xx}^2 - 5u_{xx}u_{xxx}u_{xxxx} + \frac{40}{9}u_{xxx}^3\right) = \kappa,$$

$$v_{yyyyyy} = \frac{9\left(u_{xxxxxx}u_{xx}^3 - 7u_{xx}^2u_{xxx}u_{xxxxx} + \frac{35}{3}u_{xx}u_{xxx}^2u_{xxxx} - \frac{35}{9}u_{xxx}^4\right)}{2(3u_{xx}u_{xxxx} - 5u_{xxx}^2)^2}.$$

So

$$\iota(x) = H = 0, \quad \iota(u) = I_0 = 0, \quad \iota(u_x) = I_1 = 0, \quad \iota(u_{xx}) = I_2 = \frac{9}{2},$$

$$\iota(u_{xxx}) = I_3 = 0, \quad \iota(u_{xxxx}) = I_4 = \frac{3}{2}, \quad \iota(u_{xxxxx}) = \kappa = I_5. \tag{2.13}$$

As for the recurrence formulas, the prolonged infinitesimal generators of the planar affine group action on curve jets are

$$\begin{aligned}
&\text{pr } \mathbf{v}_1 = \partial_x, \\
&\text{pr } \mathbf{v}_2 = \partial_u, \\
&\text{pr } \mathbf{v}_3 = x\partial_x - u_x\partial_{u_x} - 2u_{xx}\partial_{u_{xx}} - 3u_{xxx}\partial_{u_{xxx}} - 4u_{xxxx}\partial_{u_{xxxx}} - \cdots, \\
&\text{pr } \mathbf{v}_4 = u\partial_x - u_x^2\partial_{u_x} - 3u_xu_{xx}\partial_{u_{xx}} - (3u_{xx}^2 + 4u_xu_{xxx})\partial_{u_{xxx}} \\
&\qquad\qquad - (10u_{xx}u_{xxx} + 5u_xu_{xxxx})\partial_{u_{xxxx}} \\
&\qquad\qquad - (10u_{xxx}^2 + 15u_{xx}u_{xxxx} + 6u_xu_{xxxxx})\partial_{u_{xxxxx}} - \cdots, \\
&\text{pr } \mathbf{v}_5 = x\partial_u + \partial_{u_x}, \\
&\text{pr } \mathbf{v}_6 = u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}} + u_{xxx}\partial_{u_{xxx}} + \cdots.
\end{aligned}$$

Then, using the phantom recurrence formulas (2.7),

$$0 = d\iota(x) = \iota(dx) + \sum_{i=1}^{6}v^i\iota(\text{pr }\mathbf{v}_i(x)) = \varpi + v^1,$$

$$0 = d\iota(u) = \iota(du) + \sum_{i=1}^{6}v^i\iota(\text{pr }\mathbf{v}_i(u)) = \iota(u_xdx + \theta) + v^2 = \vartheta + v^2,$$

$$0 = d\iota(u_x) = \iota(du_x) + \sum_{i=1}^{6}v^i\iota(\text{pr }\mathbf{v}_i(u_x)) = \iota(u_{xx}dx + \theta_x) + v^5 = \frac{9}{2}\varpi + \vartheta_1 + v^5,$$

$$0 = d\iota(u_{xx}) = \iota(du_{xx}) + \sum_{i=1}^{6}v^i\iota(\text{pr }\mathbf{v}_i(u_{xx})) = \iota(u_{xxx}dx + \theta_{xx}) - 9v^3 + \frac{9}{2}v^6$$

$$= \vartheta_2 - 9v^3 + \frac{9}{2}v^6,$$

$$0 = \mathrm{d}\iota(u_{xxx}) = \iota(\mathrm{d}u_{xxx}) + \sum_{i=1}^{6} v^i \iota(\mathrm{pr}\ \mathbf{v}_i(u_{xxx})) = \iota(u_{xxxx}\mathrm{d}x + \theta_{xxx}) - \frac{243}{4}v^4$$

$$= \frac{3}{2}\varpi + \vartheta_3 - \frac{243}{4}v^4,$$

$$0 = \mathrm{d}\iota(u_{xxxx}) = \iota(\mathrm{d}u_{xxxx}) + \sum_{i=1}^{6} v^i \iota(\mathrm{pr}\ \mathbf{v}_i(u_{xxxx})) = \iota(u_{xxxxx}\mathrm{d}x + \theta_{xxxx})$$

$$- 6v^3 + \frac{3}{2}v^6 = I_5\varpi + \vartheta_4 - 6v^3 + \frac{3}{2}v^6. \tag{2.14}$$

These can be solved for the Maurer–Cartan forms

$$v^1 = -\varpi, \qquad v^2 = -\vartheta, \qquad v^3 = \frac{I_5}{3}\varpi - \frac{\vartheta_2}{9} + \frac{\vartheta_4}{3},$$

$$v^4 = \frac{2}{81}\varpi + \frac{4}{243}\vartheta_3, \qquad v^5 = -\frac{9}{2}\varpi - \vartheta_1, \qquad v^6 = \frac{2}{3}I_5\varpi - \frac{4}{9}\vartheta_2 + \frac{2}{3}\vartheta_4. \tag{2.15}$$

From these expressions, we can obtain the Maurer–Cartan invariants

$$R^1 = -1, \quad R^2 = 0, \quad R^3 = \frac{I_5}{3}, \quad R^4 = \frac{2}{81}, \quad R^5 = -\frac{9}{2}, \quad R^6 = \frac{2I_5}{3}. \tag{2.16}$$

The nonzero coefficients related to the vertical forms are

$$S_0^2 = -1, \quad S_2^3 = -\frac{1}{9}, \quad S_4^3 = \frac{1}{3}, \quad S_3^4 = \frac{4}{243}, \quad S_1^5 = -1, \quad S_2^6 = -\frac{4}{9}, \quad S_4^6 = \frac{2}{3}. \tag{2.17}$$

Then we find the horizontal derivatives

$$\mathrm{d}_{\mathcal{H}}I_5 = \left(I_6 - I_5^2 - \frac{5}{2}\right)\varpi, \qquad \mathrm{d}_{\mathcal{H}}I_6 = \left(I_7 - \frac{4}{3}I_5 I_6 - \frac{7}{3}I_5\right)\varpi,$$

which yield

$$I_6 = \kappa_s + \kappa^2 + \frac{5}{2}, \qquad I_7 = \kappa_{ss} + \frac{10}{3}\kappa\kappa_s + \frac{4}{3}\kappa^3 + \frac{17}{3}\kappa.$$

## 3 Planar Curves with Constant Affine Curvatures

In this section, we obtain the affine arc length and affine curvature for plane curves in affine differential geometry. Note that we use the full affine group and not the equi-affine subgroup consisting of area-preserving affine transformations. (Caution: much of the literature omits the "equi-" in their treatment.) The affine plane $\mathbb{R}^2$ is referred

to in terms of a coordinate system $\{x, y\}$. The allowable coordinate changes are given by affine maps of the form

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{where} \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0. \tag{3.1}$$

Consider a smooth curve, parameterized by

$$\mathbf{x}(t) = (x(t) \ \ y(t))^T, \tag{3.2}$$

where $x(t), y(t)$ are smooth functions of $t$ defined over a certain interval $I$. It is convenient to denote by $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ the tangent vector. Now we seek a new parameter $\sigma$ for the curve, whereby $\mathbf{x}' = \mathrm{d}\mathbf{x}/\mathrm{d}\sigma$ to distinguish $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$.

We assume that the given curve $\mathbf{x}(t)$ satisfies the condition

$$[\dot{\mathbf{x}} \ \ \ddot{\mathbf{x}}] = \det (\dot{\mathbf{x}} \ \ \ddot{\mathbf{x}}) \neq 0, \tag{3.3}$$

for any $t$, where, from here on, we use the notation $[\ldots] = \det(\ldots)$. Accordingly, it follows that

$$\dot{\mathbf{x}} = \mathbf{x}' \frac{\mathrm{d}\sigma}{\mathrm{d}t},$$

$$\ddot{\mathbf{x}} = \mathbf{x}'' \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \mathbf{x}' \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2},$$

$$\dddot{\mathbf{x}} = \mathbf{x}''' \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^3 + 3\mathbf{x}'' \frac{\mathrm{d}\sigma}{\mathrm{d}t} \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \mathbf{x}' \frac{\mathrm{d}^3\sigma}{\mathrm{d}t^3},$$

$$\ddddot{\mathbf{x}} = \mathbf{x}'''' \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^4 + 6\mathbf{x}''' \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + 4\mathbf{x}'' \frac{\mathrm{d}\sigma}{\mathrm{d}t} \frac{\mathrm{d}^3\sigma}{\mathrm{d}t^3} + 3\mathbf{x}'' \left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)^2 + \mathbf{x}' \frac{\mathrm{d}^4\sigma}{\mathrm{d}t^4}. \tag{3.4}$$

By a direct computation, we find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \frac{3 [\mathbf{x}' \ \mathbf{x}''] [\mathbf{x}' \ \mathbf{x}''''] - 5 [\mathbf{x}' \ \mathbf{x}''']^2 + 12 [\mathbf{x}' \ \mathbf{x}''] [\mathbf{x}'' \ \mathbf{x}''']}{[\mathbf{x}' \ \mathbf{x}'']^2}$$

$$= \frac{3 [\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] [\dot{\mathbf{x}} \ \ddddot{\mathbf{x}}] - 5 [\dot{\mathbf{x}} \ \dddot{\mathbf{x}}]^2 + 12 [\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] [\ddot{\mathbf{x}} \ \dddot{\mathbf{x}}]}{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]^2}. \tag{3.5}$$

**Definition 3.1** The curve $\mathbf{x}(t)$ is called degenerate at the point $t$ if

$$3 [\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] [\dot{\mathbf{x}} \ \ddddot{\mathbf{x}}] - 5 [\dot{\mathbf{x}} \ \dddot{\mathbf{x}}]^2 + 12 [\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] [\ddot{\mathbf{x}} \ \dddot{\mathbf{x}}] = 0. \tag{3.6}$$

We note that, according to the identity (3.5), the degeneracy condition (3.6) holds for all parameterizations of the curve. Let us first classify the degenerate affine planar curves.

**Theorem 3.2** *The curve* $\mathbf{x}(t)$ *is degenerate for all* $t \in I$ *if and only if it is locally affine equivalent to a fixed point, a straight line segment or an arc of a parabola* $y = x^2$.

**Proof** If $[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] = 0$, by taking the derivative both sides of it, we have $[\dot{\mathbf{x}} \ \dddot{\mathbf{x}}] = 0$. In this case, according to Eq. (3.8) it is easy to see that the curve is degenerate. On the other hand, $[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] = 0$ implies it is a point or straight line.

If $[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] \neq 0$, and $\mathbf{x}$ is degenerate, we choose $\sigma$ to be the equi-affine arc length parameter [16]. By the theory of equi-affine planar curves [16], it follows that

$$[\mathbf{x}' \ \mathbf{x}''] = 1, \qquad [\mathbf{x}'' \ \mathbf{x}'''] = \kappa, \qquad \mathbf{x}''' = -\kappa \mathbf{x}', \tag{3.7}$$

where $\kappa$ denotes the equi-affine curvature. Substituting (3.7) into the degeneracy condition (3.6) (with derivatives with respect to $\sigma$ replacing those with respect to $t$) produces $\kappa = 0$, which implies the curve is equi-affine equivalent to an arc of a parabola $y = x^2$, [16], which is degenerate according to (3.6). This completes the proof of this theorem. $\qquad \square$

From here on, we will assume that the curve $\mathbf{x}(t)$ is non-degenerate, meaning that

$$3[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}][\dot{\mathbf{x}} \ \dddot{\mathbf{x}}] - 5[\dot{\mathbf{x}} \ \dddot{\mathbf{x}}]^2 + 12[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}][\ddot{\mathbf{x}} \ \dddot{\mathbf{x}}] \neq 0, \tag{3.8}$$

for all $t \in I$. We will not discuss isolated degenerate points on curves here. According to the deduced conclusion by the method of moving frames in previous section, by Eq. (2.12) we obtain the affine arc length parameter $\sigma$ as the following formula

$$\frac{d\sigma}{dt} = \sqrt{\frac{\left| 3[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}][\dot{\mathbf{x}} \ \dddot{\mathbf{x}}] - 5[\dot{\mathbf{x}} \ \dddot{\mathbf{x}}]^2 + 12[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}][\ddot{\mathbf{x}} \ \dddot{\mathbf{x}}] \right|}{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]^2}}, \tag{3.9}$$

which implies that

$$\frac{\left| 3[\mathbf{x}' \ \mathbf{x}''][\mathbf{x}' \ \mathbf{x}''''] - 5[\mathbf{x}' \ \mathbf{x}''']^2 + 12[\mathbf{x}' \ \mathbf{x}''][\mathbf{x}'' \ \mathbf{x}'''] \right|}{[\mathbf{x}' \ \mathbf{x}'']^2} = 1. \tag{3.10}$$

**Definition 3.3** This parameter $\sigma$ in Eq. (3.9) is called the *affine arc length parameter* of the non-degenerate curve parameterized by $\mathbf{x}(t)$.

Differentiating with respect to the affine arc length parameter yields

$$\mathbf{x}''' = -\kappa_1 \mathbf{x}' - \kappa_2 \mathbf{x}'', \qquad \mathbf{x}'''' = (-\kappa_1' + \kappa_1 \kappa_2)\mathbf{x}' + (-\kappa_2' - \kappa_1 + \kappa_2^2)\mathbf{x}'', \tag{3.11}$$

where

$$\kappa_1 = \frac{[\mathbf{x}'' \ \mathbf{x}''']}{[\mathbf{x}' \ \mathbf{x}'']}, \qquad \kappa_2 = -\frac{[\mathbf{x}' \ \mathbf{x}''']}{[\mathbf{x}' \ \mathbf{x}'']}. \tag{3.12}$$

From Eq. (3.10), it is easy to obtain the syzygy

$$\kappa_1 = \frac{2\kappa_2^2 + 3\kappa_2' + \epsilon}{9}, \tag{3.13}$$

where

$$\begin{aligned}
\epsilon &= \mathrm{sign}(3\,[\mathbf{x}' \ \mathbf{x}'']\,[\mathbf{x}' \ \mathbf{x}'''] - 5\,[\mathbf{x}' \ \mathbf{x}''']^2 + 12\,[\mathbf{x}' \ \mathbf{x}'']\,[\mathbf{x}'' \ \mathbf{x}''']) \\
&= \mathrm{sign}(3\,[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]\,[\dot{\mathbf{x}} \ \dddot{\mathbf{x}}] - 5\,[\dot{\mathbf{x}} \ \dddot{\mathbf{x}}]^2 + 12\,[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]\,[\ddot{\mathbf{x}} \ \dddot{\mathbf{x}}]),
\end{aligned} \tag{3.14}$$

and $\mathrm{sign}(\cdot)$ is the signum function.

On the other hand, according to Eqs. (3.4) and (3.12), the expression of $\kappa_2$ in terms of a general parameter $t$ is given by

$$\kappa_2 = -\frac{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]}{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]}\frac{dt}{d\sigma} - 3\frac{d\sigma}{dt}\frac{d^2 t}{d\sigma^2} = -\frac{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]}{[\dot{\mathbf{x}} \ \ddot{\mathbf{x}}]}\frac{dt}{d\sigma} - 3\frac{d}{dt}\left(\frac{dt}{d\sigma}\right), \tag{3.15}$$

where $dt/d\sigma$ can be deduced from Eq. (3.9).

**Remark 3.4** Observe that $\epsilon$ is constant on a non-degenerate curve, because $\epsilon = 0$ implies the curve has a point where it is degenerate. Furthermore, $d\sigma/dt$ remains positive in Eq. (3.9), while $\kappa_2$ can change sign under the orientation-reversing change of parameter $\tilde{t} = -t$. In fact, one can verify that $\epsilon$, $|\kappa_2|$, $d\kappa_2/d\alpha$ are all invariant under affine transformations and reparameterizations from Eqs. (3.5), (3.9) and (3.15).

In addition, if the curve is described as a graph, $\mathbf{x} = (t, u(t))^T$, then

$$\frac{d\sigma}{dt} = \frac{\sqrt{\epsilon(3u_{tt}u_{tttt} - 5u_{ttt}^2)}}{\epsilon_1 u_{tt}}, \qquad \frac{d^2 t}{d\sigma^2} = \frac{13u_{tt}^2 u_{ttt}u_{tttt} - 10u_{ttt}^3 u_{tt} - 3u_{tt}^3 u_{ttttt}}{2\epsilon(3u_{tt}u_{tttt} - 5u_{ttt}^2)^2}. \tag{3.16}$$

Given

$$\epsilon = \mathrm{sign}(3u_{tt}u_{tttt} - 5u_{ttt}^2), \qquad \epsilon_1 = \mathrm{sign}(u_{tt}),$$

we have

$$\kappa_2 = \frac{9\epsilon\epsilon_1\left(u_{tt}^2 u_{ttttt} - 5u_{tt}u_{ttt}u_{tttt} + \frac{40}{9}u_{ttt}^3\right)}{2(\epsilon(3u_{tt}u_{tttt} - 5u_{ttt}^2))^{\frac{3}{2}}}. \tag{3.17}$$

Furthermore,

$$\frac{d\kappa_2}{d\sigma} = \frac{d\kappa_2}{dt}\frac{dt}{d\sigma} = \frac{9u_{tt}A}{2(\epsilon(3u_{tt}u_{tttt} - 5u_{ttt}^2))^3}, \tag{3.18}$$

where

$$A = 3u_{tt}^3 u_{tttt} u_{tttttt} - 5u_{tt}^2 u_{ttt}^2 u_{tttttt} + 24u_{tt}^2 u_{ttt} u_{tttt} u_{ttttt}$$
$$- 5u_{tt} u_{ttt}^3 u_{ttttt} - \frac{9}{2} u_{tt}^3 u_{ttttt}^2 - 15u_{tt}^2 u_{ttt}^3 u_{tttt} - \frac{5}{2} u_{tt} u_{ttt}^2 u_{tttt}^2 + 5u_{ttt}^4 u_{tttt}.$$

Next, we are able to define the fully affine curvature.

**Definition 3.5** The *affine curvature* of a curve on the plane is defined by $\kappa = |\kappa_2|$, where $\kappa_2$ is given by Eq. (3.12) with the affine arc length parameter $\sigma$ or Eq. (3.15) with a general parameter $t$.

The following states the fundamental theorem for affine plane curves.

**Theorem 3.6** *Given a smooth positive function $\kappa = \kappa(\sigma)$ on an interval $I$ and a constant $\epsilon \in \{1, -1\}$, there exists a planar curve with $\sigma$ as its affine arc length parameter and $\kappa$ as its affine curvature. Such a curve is unique up to an affine transformation.*

**Proof** Firstly, for the given positive function $\kappa$ and the fixed value $\epsilon$, it is feasible that

$$\kappa_2 = \kappa \quad \text{or} \quad \kappa_2 = -\kappa, \quad \text{and} \quad \kappa_1 = \frac{2\kappa_2^2 + 3\kappa_2' + \epsilon}{9}.$$

If $\kappa_2 = \kappa$, Eqs. (3.11) and (3.13) can be written as

$$\frac{d^3 \mathbf{x}}{d\sigma^3} + \kappa \frac{d^2 \mathbf{x}}{d\sigma^2} + \frac{2\kappa^2 + 3\kappa_\sigma + \epsilon}{9} \frac{d\mathbf{x}}{d\sigma} = 0. \tag{3.19}$$

The change of parameter $\sigma_1 = -\sigma$ converts Eq. (3.19) into

$$\frac{d^3 \mathbf{x}}{d\sigma_1^3} - \kappa \frac{d^2 \mathbf{x}}{d\sigma_1^2} + \frac{2\kappa^2 - 3\kappa_{\sigma_1} + \epsilon}{9} \frac{d\mathbf{x}}{d\sigma_1} = 0. \tag{3.20}$$

On the other hand, if $\kappa_2 = -\kappa$, Eq. (3.11) can be changed to

$$\frac{d^3 \mathbf{x}}{d\sigma^3} - \kappa \frac{d^2 \mathbf{x}}{d\sigma^2} + \frac{2\kappa^2 - 3\kappa_\sigma + \epsilon}{9} \frac{d\mathbf{x}}{d\sigma} = 0. \tag{3.21}$$

It is easy to see the solution sets for (3.20) and (3.21) are same.

The first equation in (3.11) can be viewed as a linear second-order ordinary differential equation for the components of $\mathbf{x}'(t)$, and hence, we can write the general solution as a linear combination

$$\mathbf{x}' = \mathbf{C}_1 \phi_1(t) + \mathbf{C}_2 \phi_2(t), \tag{3.22}$$

of two linearly independent solutions $\phi_1(t)$, $\phi_2(t)$, where $\mathbf{C}_1$, $\mathbf{C}_2$ are arbitrary constant vectors. Integration yields

$$\mathbf{x} = \mathbf{C}_1 \int \phi_1(t)dt + \mathbf{C}_2 \int \phi_2(t)dt + \mathbf{C}_3, \tag{3.23}$$

where $\mathbf{C}_3$ is another arbitrary constant vector. Then $\mathbf{x}$ is affine equivalent to

$$\left( \int \phi_1(t) dt, \int \phi_2(t) dt \right)^T. \tag{3.24}$$

Now, let us prove that the curve is unique up to an affine transformation of the plane.

Assume $(\mathbf{x}_1)$ and $(\mathbf{x}_2)$ are two curves, and they have the same value $\epsilon$ and affine curvatures $\kappa(\sigma)$ at the corresponding points. By a suitable affine transformation, we can allow $\mathbf{x}_1(\sigma_0) = \mathbf{x}_2(\sigma_0)$, $\mathbf{x}_1'(\sigma_0) = \mathbf{x}_2'(\sigma_0)$ and $\mathbf{x}_1''(\sigma_0) = \mathbf{x}_2''(\sigma_0)$ at the point which affine arc length parameter $\sigma = \sigma_0$.

$\{\mathbf{x}_1', \mathbf{x}_1''\}$ and $\{\mathbf{x}_2', \mathbf{x}_2''\}$ are the basic vectors of curves $(\mathbf{x}_1)$ and $(\mathbf{x}_2)$, respectively. They are the solutions of the differential equation system (3.19) with the same initial conditions. According to the existence theorem of solutions for differential equation system, the solutions $\mathbf{x}_1(\sigma)$ and $\mathbf{x}_2(\sigma)$ are totally same, that is, $\mathbf{x}_1 = \mathbf{x}_2$. $\qquad\square$

Let us consider the special case where $\kappa$ is a constant function, which implies that $\kappa_2$ is constant, and hence, $\kappa_1 = \dfrac{2\kappa_2^2 + \epsilon}{9}$ is also constant. We conclude that

**Theorem 3.7** *If a non-degenerate affine curve has constant curvature $\kappa$, it is locally affine equivalent to an arc of one of the following curves:*

1. $y = x^\alpha$ *where $x > 0$ and $\alpha \notin \{0, 1, \frac{1}{2}, 2\}$; in this case $\kappa = |2\alpha^2 + \alpha - 1||\alpha - 2|^{-\frac{1}{2}}|2\alpha - 1|^{-\frac{3}{2}}$ and $\epsilon = -\text{sign}(2\alpha^2 - 5\alpha + 2)$.*
2. $y = x \log x$; *in this case $\kappa = 2$ and $\epsilon = 1$.*
3. *a logarithmic spiral with polar coordinates $\rho = \exp\left(\dfrac{\alpha}{\beta}\theta\right)$, where $\beta = \dfrac{\sqrt{1 - \alpha^2}}{3}$, where $0 \leq \alpha < 1$, while $\rho$ is the radial coordinate and $\theta$ the polar angle; in this case $\kappa = 2\alpha$ and $\epsilon = 1$.*

From Eqs. (3.11) and (3.13), one concludes the following:

**Corollary 3.8** *A non-degenerate affine curve has affine curvature $\kappa = 0$ if and only if it is locally affine equivalent to an arc of either the unit circle $x^2 + y^2 = 1$, where $\epsilon = 1$, or the rectangular hyperbola $y = x^{-1}$, where $\epsilon = -1$.*

The following theorem, due originally to Cartan, characterizes curves whose curvature is constant, which in turn is a special case of a more general result characterizing submanifolds with all constant differential invariants under transformation groups [17].

**Theorem 3.9** [17] *Let $G$ be an ordinary transformation group acting on $X \subset \mathbb{R}^2$. Let $\mathcal{C} \subset X$ be an analytic curve with $G$-invariant signature set $\mathcal{S}$. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ *has constant $G$-invariant curvature $\kappa$.*
(ii) $\mathcal{S}$ *degenerates to a point.*

(iii) $\mathcal{C}$ *is the orbit of a one-parameter subgroup of G.*

(iv) $\mathcal{C}$ *admits a one-parameter symmetry group.*

Thus, the curves with constant affine curvatures in Theorem 3.7 are the orbits of the following one-parameter subgroups. (It is worth noting that these curves correspond to the list of adjoint-inequivalent one-parameter subgroups of the affine group.)

**Theorem 3.10** 1. *The curve* $y = x^\alpha, x > 0, \alpha \notin \{0, 1, \frac{1}{2}, 2\}$ *is the orbit of the one-parameter subgroup* $\left\{ \exp(Xt) | X = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \right\}$,

2. *The curve* $y = x \ln x$ *is the orbit of the one-parameter subgroup* $\left\{ \exp(Xt) | X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$,

3. *The logarithmic spiral with polar coordinates* $\rho = \exp\left(\frac{\alpha}{\beta}\theta\right)$, *where* $\beta = \frac{\sqrt{1 - \alpha^2}}{3}$ *with* $0 \leq \alpha < 1$ *is the orbit of the one-parameter subgroup* $\left\{ \exp(Xt) | X = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \right\}$.

***Remark 3.11*** By Eqs. (3.7) and (3.9), the fully affine arc length element $d\sigma$ can be expressed in terms of the equi-affine differential invariants as $\frac{d\sigma}{dv} = 3\sqrt{|\mu|}$, where $v$ is the equi-affine arc length element and $\mu$ is the equi-affine curvature. Then from Eqs. (3.7) and (3.15), the fully affine curvature can be written in terms of the equi-affine curvature and its derivatives as $\left(\frac{1}{\sqrt{|\mu|}}\right)_v$. Furthermore, the affine arc length element $d\sigma$ can be expressed in terms of the Euclidean arc length element $s$ and differential invariants as $\frac{d\sigma}{ds} = \sqrt{\frac{|3\phi\phi_{ss} - 5\phi_s^2 + 9\phi^4|}{\phi^2}}$, where $s$ is the Euclidean arc length element and $\phi$ is the Euclidean curvature. The fully affine curvature can be written in terms of the Euclidean curvature and its derivatives as: $\left| \frac{\phi_s}{\sqrt{|3\phi\phi_{ss} - 5\phi_s^2 + 9\phi^4|}} + 3\left(\sqrt{\frac{\phi^2}{|3\phi\phi_{ss} - 5\phi_s^2 + 9\phi^4|}}\right)_s \right|$. In fact, according to [11], we have $\frac{dv}{ds} = \phi^{\frac{1}{3}}$ and $\mu = \frac{3\phi\phi_{ss} - 5\phi_s^2 + 9\phi^4}{9\phi^{\frac{8}{3}}}$.

## 4 Moving Frame Construction for the Affine Group that Factors as a Product

In fact, it will be simpler to construct the moving frames and then obtain differential invariants if the acting group has fewer parameters. Thus, it is desirable to use the results obtained for a subgroup $A \subset G$ to construct a moving frame and differential invariants for the entire group $G$. The inductive algorithms [10,11] presented by Kogan allow us, in the case when the group $G$ factors as a product, to extend a moving frame for a subgroup to the entire group. One also obtains at the same time the relations among the invariants of group $G$ and its subgroup $A$. It is worth remarking that, in order to obtain such relations, the algorithm does not require the explicit formulae for

the invariants of either $G$ and $A$, but only the corresponding moving frames which lead to these invariants.

In this section, we use the algorithm [10,11] to make an induction from the Euclidean action on plane curves to the affine action. Firstly, let us review the action of the special Euclidean group $SE(2) = SO(2) \ltimes \mathbb{R}^2$ on plane curve $u = u(x)$ [10,11]. Its first prolongation, given by

$$
\begin{aligned}
y &= x \cos \alpha - u \sin \alpha + a, \\
v &= x \sin \alpha + u \cos \alpha + b, \\
v_y &= \frac{\sin \alpha + u_x \cos \alpha}{\cos \alpha - u_x \sin \alpha},
\end{aligned}
\tag{4.1}
$$

defines a free action on $J^1(\mathbb{R}^2, 1)$. We could get a moving frame on $J^1(\mathbb{R}^2, 1)$ by choosing a cross section $\{x = 0, u = 0, u_x = 0\}$. Then by setting $\{y = 0, v = 0, v_y = 0\}$, it is simple to find an equivariant map $J^1(\mathbb{R}^2, 1) \to SE(2)$

$$
\alpha = -\arctan u_x, \quad a = -\frac{u_x u + x}{\sqrt{1 + u_x^2}}, \quad b = \frac{u_x x - u}{\sqrt{1 + u_x^2}}.
\tag{4.2}
$$

Therefore, the corresponding element of the special Euclidean group can be written in a matrix form

$$
\rho = \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.
\tag{4.3}
$$

Then the following differential invariants can be obtained directly

$$
\begin{aligned}
I_2^e &= v_{yy} = \frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}}, \\
I_3^e &= v_{yyy} = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3}, \\
I_4^e &= v_{yyyy} = \frac{(1 + u_x^2)^2 u_{xxxx} - 10u_x u_{xx}u_{xxx}(1 + u_x^2) + 15u_x^2 u_{xx}^3}{(1 + u_x^2)^{\frac{9}{2}}}.
\end{aligned}
\tag{4.4}
$$

Now we denote by $I_2^e = \kappa$ the Euclidean curvature. Immediately, from the recurrence formula (2.7), we have

$$
\begin{aligned}
I_3^e &= \kappa_s, \\
I_4^e &= \kappa_{ss} + 3\kappa^3, \\
I_5^e &= \kappa_{sss} + 19\kappa^2\kappa_s,
\end{aligned}
\tag{4.5}
$$

where $ds = \sqrt{1 + u_x^2}dx$ is infinitesimal Euclidean arc length. The contact invariant differential form equals to $\omega = \sqrt{1 + u_x^2}dx = ds$. The dual total vector field $\mathcal{D} =$

$\frac{1}{\sqrt{1+u_x^2}} D_x = \frac{d}{ds}$ provides an invariant differential operator, and any other invariant can be expressed as a function of $I_2^e$ and its derivatives with respect to $\mathcal{D}$.

Now it is time to build a moving frame for the affine group, by using the moving frame obtained above for the special Euclidean group $SE(2, \mathbb{R})$ acting on curves in $\mathbb{R}^2$. Obviously, the affine group transformation $GA(2, \mathbb{R})$ on the plane is the semi-direct product of the general linear group $GL(2, \mathbb{R})$ and translations in $\mathbb{R}^2$. Thus, we prolong it to the first jet bundle and notice that the isotropy group $B$ of the point $z_0^{(1)} = \{x = 0, u = 0, u_x = 0\}$ is given by all linear transformations of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \tag{4.6}$$

where $\alpha\delta \neq 0$. Thus, $GA(2, \mathbb{R}) = B \cdot SE(2, \mathbb{R})$ and $B \cap SE(2, R)$ is finite. Exactly $B \cap SE(2, R) = \{I, -I\}$. Then let us prolong the action of $B$ up to fifth order:

$$y = \alpha x + \beta u,$$
$$v = \delta u,$$
$$v_y = \frac{\delta u_x}{\alpha + \beta u_x},$$
$$v_{yy} = \frac{\alpha\delta u_{xx}}{(\alpha + \beta u_x)^3},$$
$$v_{yyy} = \frac{\alpha\delta(\alpha + \beta u_x)u_{xxx} - 3\alpha\beta\delta u_{xx}^2}{(\alpha + \beta u_x)^5},$$
$$v_{yyyy} = \frac{\alpha\delta(\alpha + \beta u_x)^2 u_{xxxx} - 10\alpha\beta\delta u_{xx}u_{xxx}(\alpha + \beta u_x) + 15\alpha\beta^2\delta u_{xx}^3}{(\alpha + \beta u_x)^7},$$
$$v_{yyyyy} = \frac{A}{(\alpha + \beta u_x)^9},$$

where

$$A = \alpha\delta(\alpha + \beta u_x)^3 u_{xxxxx} - 15\alpha\beta\delta u_{xx}u_{xxxx}(\alpha + \beta u_x)^2 - 10\alpha\beta\delta u_{xxx}^2(\alpha + \beta u_x)^2$$
$$+ 105\alpha\beta^2\delta u_{xx}^2 u_{xxx}(\alpha + \beta u_x) - 105\alpha\beta^3\delta u_{xx}^4.$$

If we restrict these transformations to the Euclidean cross section

$$\mathcal{K}_E^5 = \{z^{(5)} \mid \pi_1^5(z^{(5)}) = z_0^{(1)}\} = \{z^{(5)} \mid x = 0, u = 0, u_x = 0\},$$

then it follows that

$$v_{yy} = \delta\frac{u_{xx}}{\alpha^2},$$
$$v_{yyy} = \delta\frac{\alpha u_{xxx} - 3\beta u_{xx}^2}{\alpha^4},$$
$$v_{yyyy} = \delta\frac{\alpha^2 u_{xxxx} - 10\alpha\beta u_{xx}u_{xxx} + 15\beta^2 u_{xx}^3}{\alpha^6},$$

$$v_{yyyyy} = \delta \frac{\alpha^3 u_{xxxxx} - 15\alpha^2 \beta u_{xx} u_{xxxx} - 10\alpha^2 \beta u_{xxx}^2 + 105\alpha\beta^2 u_{xx}^2 u_{xxx} - 105\beta^3 u_{xx}^4}{\alpha^8}.$$

In fact, the above action is free on the open subset $\{z^{(5)} \in \mathcal{K}_E^5 \mid u_{xx} \neq 0\}$. The cross section

$$\mathcal{K}^5 = \left\{ z^{(5)} \in \mathcal{K}_E^5 \mid u_{xx} = \frac{9}{2}, \ u_{xxx} = 0, \ u_{xxxx} = \frac{3}{2} \right\}$$

to the orbit of $B$ on $\mathcal{K}_E^5$ can generate a moving frame $\rho_B : \mathcal{K}_E^5 \rightarrow B$:

$$\alpha^2 = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{u_{xx}^2}, \quad \beta = \frac{\alpha u_{xxx}}{3u_{xx}^2}, \quad \delta = \frac{9\alpha^2}{2u_{xx}}.$$

Thus, we obtain the corresponding fifth-order invariant for the action of $B$ on $\mathcal{K}_E^5$

$$I_5^b = \frac{9}{2} \frac{u_{xx}^2 u_{xxxxx} - 5u_{xx}u_{xxx}u_{xxxx} + \frac{40}{9}u_{xxx}^3}{(3u_{xx}u_{xxxx} - 5u_{xxx}^2)^{\frac{3}{2}}}, \tag{4.7}$$

which is the *affine curvature* in Eqs. (3.17) and (2.13).

It is clear that $\mathcal{K}^5$ is a cross section to the orbits of the entire group $GA(2, \mathbb{R})$ on the open subset of $J^5$ where $u_{xx} \neq 0$. Then the fifth-order affine invariant can be obtained by invariantization of $I_5^b$ with respect to Euclidean action. So we get the lowest order affine invariant $\mu$ according to Euclidean invariants

$$\mu = I_5^a = \frac{9\left((I_2^e)^2 I_5^e - 5I_2^e I_3^e I_4^e + \frac{40}{9}(I_3^e)^3\right)}{2(3I_2^e I_4^e - 5(I_3^e)^2)^{\frac{3}{2}}}. \tag{4.8}$$

This equation can be rewritten by using the Euclidean curvature $I_2^e = \kappa$ and its derivatives in Eq. (4.5). The moving frame for the affine group corresponding to the cross section $\mathcal{K}^5$ is the product of two matrices

$$\begin{pmatrix} \frac{\sqrt{3u_{xx}u_{xxxx}-5u_{xxx}^2}}{u_{xx}} & \frac{u_{xxx}\sqrt{3u_{xx}u_{xxxx}-5u_{xxx}^2}}{3u_{xx}^3} & 0 \\ 0 & \frac{9(3u_{xx}u_{xxxx}-5u_{xxx}^2)}{2u_{xx}^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

We can obtain the affine contact invariant horizontal form $d\theta$ in terms of the Euclidean arc length $ds$ (see [10,11] for the detailed definitions of the notations):

$$d\theta = \sigma_B^* \sigma_E^* \pi_H \omega_E^* d_H \omega_E^*(x),$$

where the Euclidean invariantization of $d_H \omega_B^*(x) = (\alpha + \beta u_x)dx$ equals to $\alpha ds$ and hence

$$d\theta = \sigma_B^*(\alpha ds) = \frac{\sqrt{3I_2^e I_4^e - 5(I_3^e)^2}}{I_2^e}ds. \tag{4.10}$$

The form $d\theta$ is the affine arc length and can be written in the standard coordinates as

$$d\theta = \frac{\sqrt{3u_{xx}u_{xxxx} - 5u_{xxx}^2}}{u_{xx}}dx,$$

which is coincident to Eq. (2.12).

## 5 Applications of Affine Curvature

In this section, we will employ the affine curvature defined in the previous sections to determine whether two smooth (or discrete) planar curves are affine equivalent. Furthermore, once we can confirm that the two smooth (or discrete) curves are affine equivalent, the least squares method will be utilized for identifying the most suitable affine transformation between them. After applying the affine transformation to transform one of the two curves, the accuracy can be calculated and evaluated. In Sect. 5.1, we introduce a numerical algorithm to assess the degree of equivalence between two planar curves. In Sect. 5.2, the affine signature curves will be applied to judge whether there exist an affine transformation between two smooth planar curves. In Sect. 5.3, the discrete case is considered.

### 5.1 Measuring Numerically How Well Two Curves Match

As we know, the inner product between two (sufficiently nice) functions $f(x)$ and $g(x)$ over the interval $[a, b]$,

$$\langle f(x), g(x) \rangle := \int_c^b f(x)g(x)dx,$$

often is considered as a standard way to compare the degree of equivalence between these two functions, and from which we get

$$||f(x) - g(x)|| = \sqrt{\int_a^b (f(x) - g(x))^2 dx},$$

where $||f(x) - g(x)||$ can be defined as the distance between the functions $f(x)$ and $g(x)$, and denoted by $\text{dist}(f(x), g(x))$. Thus, if $\text{dist}(f(x), g(x)) = 0$, immediately we can conclude that

$$f(x) \equiv g(x), \quad \forall x \in [a, b].$$

Now we can apply this concept to compare two discrete curves. Let us say, the curve $A$ consists of $N$ points and all the points are

$$(x_{a_1}, y_{a_1})^{\text{Trans}}, \ (x_{a_2}, y_{a_2})^{\text{Trans}}, \ \ldots, \ (x_{a_N}, y_{a_N})^{\text{Trans}}.$$

Similarly, the curve $B$ consists of $M$ points

$$(x_{b_1}, y_{b_1})^{\text{Trans}}, \ (x_{b_2}, y_{b_2})^{\text{Trans}}, \ \ldots, \ (x_{b_M}, y_{b_M})^{\text{Trans}}.$$

Locally, we can always assume that $x_{a_1} < x_{a_2} < \cdots < x_{a_N}$ and $x_{b_1} < x_{b_2} < \cdots < x_{b_M}$. For the sake of convenience, we take the following notations to indicate the maps of Curve $A$ and Curve $B$, that is,

$$y_{a_i} = f(x_{a_i}), \ \ y_{b_j} = g(x_{b_j}), \ \ i = 1, 2, \ldots, N, \ \ j = 1, 2, \ldots, M.$$

Two new sets $\{x'_{a_i}\}$ and $\{x'_{b_i}\}$ of $x$-coordinates are created by choosing $x'_{a_i} \in \{x_{a_i}\}$ and $x'_{b_i} \in \{x_{b_i}\}$ such that $x'_{a_1} < x'_{b_1} < x'_{a_2} < x'_{b_2} < \cdots$ or $x'_{b_1} < x'_{a_1} < x'_{b_2} < x'_{a_2} < \cdots$. Then, we need to generate a common set of points for both curves with $x$-coordinates from the following set.

$$\{x_1, x_2, \ldots, x_K\} = \{x'_{a_1}, x'_{a_2}, \ldots, x'_{a_{N'}}\} \cup \{x'_{b_1}, x'_{b_2}, \ldots, x'_{b_{M'}}\},$$

where $x_1 < x_2 < \cdots < x_K$ and $\max\{M', N'\} \leq K \leq M' + N'$. The next step is to define the maps for the set $\{x_1, x_2, \cdots, x_K\}$, and here we take

$$f(x_i) = \begin{cases} \frac{(x_i - x'_{a_{l-1}}) f(x'_{a_l}) + (x'_{a_l} - x_i) f(x'_{a_{l-1}})}{x'_{a_l} - x'_{a_{l-1}}}, & x'_{a_{l-1}} \leq x_i \leq x'_{a_l}, \ \ 1 < l \leq N', \\ g(x_i) & x_i < x'_{a_1}, \\ g(x_i) & x_i > x'_{a_{N'}}, \end{cases} \tag{5.1}$$

and

$$g(x_i) = \begin{cases} \frac{(x_i - x'_{b_{l-1}}) g(x'_{b_l}) + (x'_{b_l} - x_i) g(x'_{b_{l-1}})}{x'_{b_l} - x'_{b_{l-1}}}, & x'_{b_{l-1}} \leq x_i \leq x'_{b_l}, \ \ 1 < l \leq M', \\ f(x_i) & x_i < x'_{b_1}, \\ f(x_i) & x_i > x'_{b_{M'}}. \end{cases} \tag{5.2}$$

The missing $y$-coordinates (if any) for each curve are obtained via interpolating neighboring points. Notice that in the second and third entries, $f$ and $g$ are reversed. For the new function $f(x)$, if $x_i < x'_{a_1}$ or $x_i > x'_{a_N}$, then $x_i$ is the first one or last one. So $x_i \in \{x'_{b_i}\}$, and we take its value from $g(x)$. The same operation is performed for the new function $g(x)$. Thus, we can calculate the difference between Curve $A$ and Curve $B$ by

$$\text{dist}_1 = \max\{|f(x_i) - g(x_i)|, i = 1, 2, \ldots, K\} \text{ or } \text{dist}_2 = \frac{1}{K}\sqrt{\sum_{i=1}^{K}|f(x_i) - g(x_i)|^2}.$$

(5.3)

The value of difference rate can be given by

$$\text{error}_1 = \frac{\text{dist}_1}{\frac{1}{N}\sum_{i=1}^{N}|y_{a_i}|} \text{ or } \text{error}_2 = \frac{\text{dist}_2}{\frac{1}{N}\sqrt{\sum_{i=1}^{N}y_{a_i}^2}}.$$

(5.4)

**Remark 5.1** The possible signature metrics for comparison of digital signatures are Hausdorff, Monge–Kantorovich transport, electrostatic repulsion, latent semantic analysis, histograms, geodesic distance, diffusion metric, Gromov–Hausdorff, etc [1,2]. In [8], Hoff and Olver used signature similarity coefficient to measure the closeness of the two signatures. In this paper, in order to simple show the practicability of the signature curve in matching curves up to an affine transformation, we directly use the $L^2$ inner product and norm to compare signatures. Practical issues require the appropriate metric to compare signatures, which is an important issue we are studying.

### 5.2 Detecting the Affine Equivalence Between Two Smooth Planar Curves

On the basis of the previous sections, it is clear that, by a certain generating set of invariants and a collection of invariant differential operators, all of the essential differential invariants can be produced, while the differential invariants often are used to find the global equivalence of two curves and symmetries of the curve. Certainly, the equivalence and symmetry problem can be solved using the signature curves [9,18,22]. As introduced in Sect. 2.1, $S$ is the submanifold of manifold $M$. Let $n$ be the order of stabilization of the group action and $\{I_1, I_2, \ldots, I_{N_k}\}$ be a complete list of differential invariants for any $k \geq n$. The restriction of the invariants to the $k$-jet of $S$, $j^k S$, is denoted by $\tilde{I}_\alpha = I_\alpha|_{J^k S}$, and we define $\phi^k : S \to \{\tilde{I}_1, \ldots, \tilde{I}_{N_k}\}$. Accordingly, the $k^{\text{th}}$-order *signature curve* $\mathcal{C}^k$ of the submanifold $S$ is the immersed manifold $Im(\phi^k) \subset \mathbb{R}^{N_k}$.

In fact, if two curves $C_1$ and $C_2$ are affine equivalent, they have same $\kappa = |\kappa_2|$, $\epsilon$, $(\kappa_2)_\sigma$ appearing in Eqs. (3.14), (3.15) and (3.18). Hence, the curves $(\kappa, \epsilon)$ and $(\kappa, (\kappa_2)_\sigma)$ generated by curves $C_1$ and $C_2$ are same. On the contrary, $\epsilon$, $\kappa$ decide a unique curve up to an affine transformation from Theorem 3.6. If we have exact functions for $(\kappa, \epsilon)$ and $(\kappa, (\kappa_2)_\sigma)$, the curve can be identified uniquely up to an affine transformation. Thus, the curves $(\kappa, \epsilon)$ and $(\kappa, (\kappa_2)_\sigma)$ are considered as the *affine signature curves*.

In the following, through an example, we elaborate the process to assess the affine equivalence of two curves and identify the affine transformation between them.

**Example 5.2** Firstly, we list three parametric curves expressed by the following equations
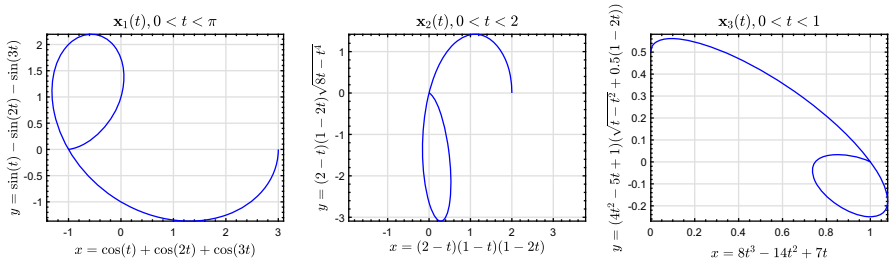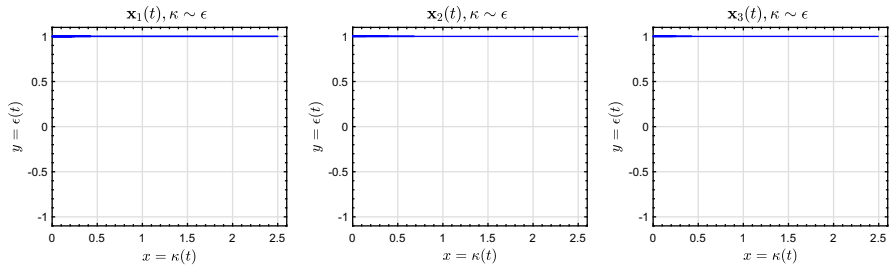
**Fig. 1** The images of the curves



**Fig. 2** The relations between $\kappa$ and $\epsilon$

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t + \cos 2t + \cos 3t \\ \sin t - \sin 2t - \sin 3t \end{pmatrix}, \qquad 0 < t < \pi, \tag{5.5}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} (2-t)(1-t)(1-2t) \\ (2-t)(1-2t)\sqrt{8t-t^4} \end{pmatrix}, \qquad 0 < t < 2, \tag{5.6}$$

and

$$\mathbf{x}_3(t) = \begin{pmatrix} 8t^3 - 14t^2 + 7t \\ (4t^2 - 5t + 1)(\sqrt{t-t^2} + 0.5(1-2t)) \end{pmatrix}, \qquad 0 < t < 1. \tag{5.7}$$

Our goal is to make sure whether there are some affine transformations between them. Generally, we should first draw graphs of these curves which can clearly show topological and geometrical properties. Sometimes, we can know two curves are not affine equivalent directly from their graphs according to the properties preserved by the affine transformation.

Definitely, their graphs are shown in Fig. 1. However, in this example, directly observing the graphs in Fig. 1, it is difficult to determine affine equivalence among them.

Thus, we need to further investigate their signature curves which are shown in Figs. 2 and 3. In Fig. 2, these three graphs are almost same. But in Fig. 3, we do not hesitate to tell that the curve $\mathbf{x}_2(t)$ can not be affine equivalent to either of the two curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$, since the middle graphs of Fig. 3 look obviously different from the other two. Of course, it seems that the left and right of Fig. 3 are the same.
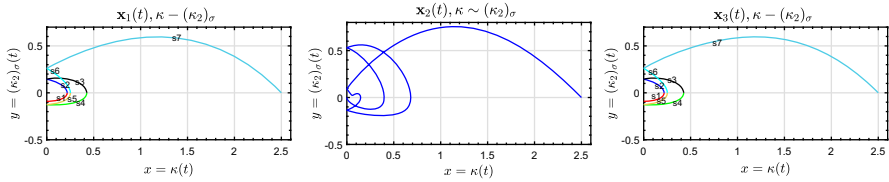
**Fig. 3** The relations between $(\kappa_2)_\sigma$ and $\kappa$

**Table 2** The ranges of parameter $t$ for seven parts

|  | $\mathbf{x}_1(t)$ | $\mathbf{x}_3(t)$ |
|---|---|---|
| s1 | (0,  0.4180312) | (0,  0.04305502) |
| s2 | (0.0.4180312,  0.8050989) | (0.04305502,  0.15348005) |
| s3 | (0.8050989,  1.3450748) | (0.15348005,  0.38809517) |
| s4 | (1.3450748,  1.8195723) | (0.38809517,  0.62310890) |
| s5 | (1.8195723,  2.1421153) | (0.62310890,  0.77037101) |
| s6 | (2.1421153,  2.4224184) | (0.77037101,  0.87617499) |
| s7 | (2.4224184,  $\pi$) | (0.87617499,  1) |

With the help of Figs. 2 and 3, the possibility of the curve $\mathbf{x}_2(t)$ is excluded, and we could move ahead to estimate if there exist an affine transformation between $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$.

By a direct computation, we obtain both maximum values of $\kappa$ for the curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$ are the same value 2.5. At the same time, according to Eqs. (3.8) and (3.14), $\epsilon$ is constant for a non-degenerate affine curve. Thus, it is obvious that the left and right of Fig. 2 are totally the same.

In order to verify the left and right of Fig. 3 are the same graph, we need to use the method stated in Sect. 5.1. Exactly, if they are same, we can conclude that the two curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$ are affine equivalent, and we will further obtain the affine transformation between them from some corresponding points.

Let us first calculate the difference and the value of difference rate in Eqs. (5.3) and (5.4) between the left and the right pictures in Fig. 3. According to the signature curves in Fig. 3, we should divide them into seven parts on the basis of the monotonicity marked by different colors in Fig. 3. By careful comparison and analysis, it is easy to obtain the ranges of parameter $t$ for these seven parts which are listed in Table 2.

Then we choose 50001 points at each part of s1, s2,... s7 in Fig. 3. Using the method stated in Sect. 5.1, we obtain the difference and the value of difference rate for these seven parts which are shown in Table 3.

From the data listed in Table 3, it is apparent that all the values are very small and almost zero. If we increase the number of points, all the values in Table 3 will further decrease. Inevitably, there are some errors in numerical calculation and the values for difference rate are considered to be within acceptable limits, so we can conclude that the curves $\{\mathbf{x}_1(t), 0 < t < \pi\}$ and $\{\mathbf{x}_3(t), 0 < t < 1\}$ are affine equivalent.

Now let us search for the affine transformation between the curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$. According to the method and Eqs. (5.1)–(5.2) in Sect. 5.1, we have obtained three series

**Table 3** The distances and errors for seven parts

|  | dist$_1$ | dist$_2$ | error$_1$ | error$_2$ |
|---|---|---|---|---|
| s1 | $8.3377 \times 10^{-07}$ | $1.4162 \times 10^{-11}$ | $1.6427 \times 10^{-05}$ | $6.8278 \times 10^{-08}$ |
| s2 | $8.9892 \times 10^{-07}$ | $1.2962 \times 10^{-11}$ | $1.0874 \times 10^{-05}$ | $4.2581 \times 10^{-08}$ |
| s3 | $5.7532 \times 10^{-07}$ | $1.0202 \times 10^{-11}$ | $5.2672 \times 10^{-06}$ | $2.6494 \times 10^{-08}$ |
| s4 | $7.5898 \times 10^{-08}$ | $2.9708 \times 10^{-12}$ | $8.4158 \times 10^{-07}$ | $9.5412 \times 10^{-09}$ |
| s5 | $3.6450 \times 10^{-07}$ | $6.9472 \times 10^{-12}$ | $4.5309 \times 10^{-06}$ | $2.4653 \times 10^{-08}$ |
| s6 | $3.6608 \times 10^{-07}$ | $6.5532 \times 10^{-12}$ | $3.0857 \times 10^{-06}$ | $1.4534 \times 10^{-08}$ |
| s7 | $7.5166 \times 10^{-08}$ | $2.5005 \times 10^{-12}$ | $1.6544 \times 10^{-07}$ | $1.4606 \times 10^{-09}$ |

$$\{\kappa_1, \kappa_2, \ldots, \kappa_K\},$$
$$\{f(\kappa_1), f(\kappa_2), \ldots, f(\kappa_K)\},$$

and

$$\{g(\kappa_1), g(\kappa_2), \ldots, g(\kappa_K)\}.$$

At the same time, it is easy to collect the values $(t_1)_i \in (0, \pi)$ and $(t_2)_i \in (0, 1)$ which correspond to the curvature $\kappa_i, i = 1, 2, \ldots, K$ for curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$, respectively.

In order to compute it more accurately, we fix a small number $e$ and choose these corresponding points $(t_1)_i$ and $(t_2)_i$ which satisfying the condition $|f(\kappa_i) - g(\kappa_i)| < e$. In the above case, if we choose $e = 10^{-8}$, we still gain 634793 points which satisfy the condition.

Since there exists an affine transformation between $\{\mathbf{x}_1(t), 0 < t < \pi\}$ and $\{\mathbf{x}_3(t), 0 < t < 1\}$ from above analysis, we assume

$$\mathbf{x}_1(t_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}_3(t_2) + \begin{pmatrix} e \\ f \end{pmatrix}, \quad 0 < t_1 < \pi, \ 0 < t_2 < 1, \tag{5.8}$$

which implies

$$\begin{pmatrix} (\mathbf{x}_3(t_2))_x & (\mathbf{x}_3(t_2))_y & 0 & 0 & 1 & 0 \\ 0 & 0 & (\mathbf{x}_3(t_2))_x & (\mathbf{x}_3(t_2))_y & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} (\mathbf{x}_1(t_1))_x \\ (\mathbf{x}_1(t_1))_y \end{pmatrix}, \tag{5.9}$$

where $(\mathbf{x})_x$, $(\mathbf{x})_y$ denote the $x$-coordinate and $y$-coordinate of the vector $\mathbf{x}$, respectively. Then using the least square method at the corresponding points $(t_1)_i$ and $(t_2)_i$ which satisfying that $|f(\kappa_i) - g(\kappa_i)| < e$, we can get the most suitable solution for the parameters $a, b, c, d, e, f$.
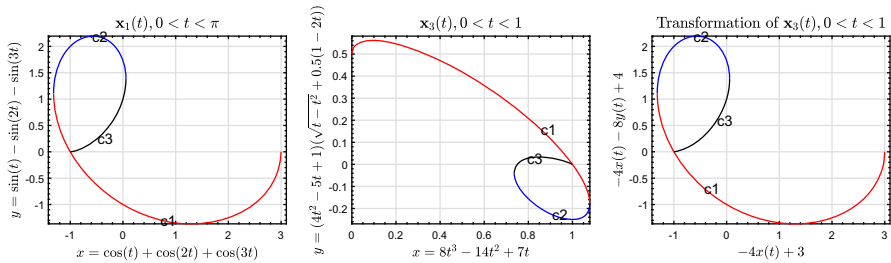
**Fig. 4** Comparison between $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$ after the affine transformation

**Table 4** The ranges of $t$ for c1, c2 and c3 in Fig. 4

|              | c1                   | c2                          | c3                       |
| ------------ | -------------------- | --------------------------- | ------------------------ |
| $\mathbf{x}_1(t)$ | (0,  1.2929432)      | (1.2929432,  2.2238639)     | (2.2238639,  $\pi$)      |
| $\mathbf{x}_3(t)$ | (0,  0.36285406)     | (0.36285406,  0.80381261)   | (0.80381261,  1)         |

**Table 5** The distances and errors between the left and right (Fig. 4)

|     | $\mathrm{dist}_1$             | $\mathrm{dist}_2$             | $\mathrm{error}_1$            | $\mathrm{error}_2$            |
| --- | ----------------------------- | ----------------------------- | ----------------------------- | ----------------------------- |
| c1  | $8.6938 \times 10^{-07}$      | $3.4209 \times 10^{-13}$      | $1.0149 \times 10^{-06}$      | $7.2097 \times 10^{-10}$      |
| c2  | $4.9645 \times 10^{-07}$      | $5.1244 \times 10^{-13}$      | $2.6618 \times 10^{-07}$      | $3.4371 \times 10^{-10}$      |
| c3  | $2.2899 \times 10^{-04}$      | $9.5435 \times 10^{-10}$      | $4.2645 \times 10^{-04}$      | $1.3826 \times 10^{-06}$      |

In the above case, there are more than 508210 points which satisfy the condition. Then using them and the least square method, we can find that the solution of the parameters $a, b, c, d, e, f$ is

$$a = -4.0000000, \ b = 0.0000000, \ c = -4.0000000, \ d = -8.0000000,$$
$$e = 3.0000000, \ f = 4.0000000. \tag{5.10}$$

Now by using these parameters, we transform the middle graph $\mathbf{x}_3(t), 0 < t < 1$ to the right graph of Fig. 4 and then compare it with the original graph $\mathbf{x}_1(t), 0 < t < \pi$ as shown in the left of Fig. 4.

Exactly, to make sure the left graph and the right graph of Fig. 4 are the same, we still need to use the method stated in Sect. 5.1. According to the monotonicity as shown in Fig. 4, it is better to divide it into three parts. By a careful calculation, we can obtain the ranges of $t$ for Parts c1, c2 and c3 in Fig. 4 and the results are shown in Table 4.

In each unit-length interval of $x$-axis, we choose 600001 points as an example. By using the method stated in Sect. 5.1, we obtain the distances and errors in Eqs. (5.3) and (5.4) between the left graph and the right graph of Fig. 4 which are shown in Table 5.

**Remark 5.3** It seems that the results of Part c3 in Table 5 are slightly bigger than these of Part c1 and Part c2, and that is because some sections in Part c3 are abrupt with respect to $x$-axis, which is unfavorable for interpolation. In fact, in this part, the relation of $x$ values and $y$ values is the one-to-one correspondence for $\mathbf{x}_1(t)$ and the transformation of $\mathbf{x}_3(t)$. Then if we exchange $x$ and $y$ axis, the desired effects of Part c3 are discernible, which are

$$\text{dist}_1 = 2.7672 \times 10^{-09}, \ \text{dist}_2 = 6.9579 \times 10^{-15}, \ \text{error}_1 = 7.7556 \times 10^{-09},$$
$$\text{error}_2 = 1.4592 \times 10^{-11}.$$

To sum up, the above result again shows that there exists an affine transformation between $\mathbf{x}_1(t), \ 0 < t < \pi$ and $\mathbf{x}_3(t), \ 0 < t < 1$.

### 5.3 Applying the Affine Curvatures to the Discrete Point Sequence

In this part, we first prove that the fitting cubic curve of a discrete point set is affine invariant under an affine transformation, and then identify the discrete points with the affine curvatures along the fitting cubic curves.

Given $n$ points in the plane, represented as $(x_1, y_1)^{\text{Trans}}, (x_2, y_2)^{\text{Trans}}, \cdots, (x_n, y_n)^{\text{Trans}}$, which generate a matrix

$$M_1 = \begin{pmatrix} x_1^3 & x_1^2 y_1 & x_1 y_1^2 & y_1^3 & x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^3 & x_2^2 y_2 & x_2 y_2^2 & y_2^3 & x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^3 & x_n^2 y_n & x_n y_n^2 & y_n^3 & x_n^2 & x_n y_n & y_n^2 & x_n & y_n & 1 \end{pmatrix}. \tag{5.11}$$

Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{10}$ are the ten column vectors of the matrix $M_1$, that is,

$$M_1 = (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4 \ \mathbf{v}_5 \ \mathbf{v}_6 \ \mathbf{v}_7 \ \mathbf{v}_8 \ \mathbf{v}_9 \ \mathbf{v}_{10}). \tag{5.12}$$

Thus, another matrix $M_2$ can be constructed by using these column vectors, and

$$M_2 = (\mathbf{v}_5 \ \mathbf{v}_6 \ \mathbf{v}_7 \ \mathbf{v}_8 \ \mathbf{v}_9 \ \mathbf{v}_{10}). \tag{5.13}$$

In particular, we can prove that

**Theorem 5.4** *If the ranks of the matrices $M_1$ and $M_2$ satisfy that*

$$\text{rank}(M_1) = 9, \ \text{rank}(M_2) = 6, \tag{5.14}$$

*then there exists a unique cubic curve*

$$A_1 x^3 + A_2 x^2 y + A_3 x y^2 + A_4 y^3 + A_5 x^2 + A_6 x y + A_7 y^2 + A_8 x + A_9 y + A_{10} = 0,$$
$$A_1^2 + A_2^2 + A_3^2 + A_4^2 \neq 0 \tag{5.15}$$

*passing through these n points $(x_1, y_1)^{\text{Trans}}, (x_2, y_2)^{\text{Trans}}, \cdots, (x_n, y_n)^{\text{Trans}}$.*

**Proof** Since $\text{rank}(M_2) = 6$, the solution vector $(A_5, A_6, \cdots, A_{10})$ of the equations

$$A_5 x_i^2 + A_6 x_i y_i + A_7 y_i^2 + A_8 x_i + A_9 y_i + A_{10} = 0, \quad i = 1, 2, \ldots, n$$

is $A_5 = A_6 = \cdots = A_{10} = 0$.

On the other hand, $\text{rank}(M_1) = 9$ implies that the unit vector $(A_1, A_2, \cdots, A_{10})$ satisfying the equations

$$A_1 x_i^3 + A_2 x_i^2 y_i + A_3 x_i y_i^2 + A_4 y_i^3 + A_5 x_i^2 + A_6 x_i y_i + A_7 y_i^2$$
$$+ A_8 x_i + A_9 y_i + A_{10} = 0, \quad i = 1, 2, \cdots, n$$

is unique up to a sign. Therefore, it is obvious that the conclusion of this theorem holds. □

Let us consider an affine transformation

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 \\ y &= \lambda_4 x' + \lambda_5 y' + \lambda_6 \end{aligned}, \qquad \text{where } \lambda_1 \lambda_5 - \lambda_2 \lambda_4 \neq 0, \qquad (5.16)$$

on a plane. Clearly, by a computation, the relation between $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{10})$ and $(\mathbf{v}'_1, \mathbf{v}'_2, \cdots, \mathbf{v}'_{10})$ follows

$$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}) = (\mathbf{v}'_1, \mathbf{v}'_2, \cdots, \mathbf{v}'_{10}) \begin{pmatrix} N & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ N_2 & N_1 & \mathbf{0} & \mathbf{0} \\ N_3 & N_4 & N_5 & \mathbf{0} \\ N_6 & N_7 & N_8 & 1 \end{pmatrix}, \qquad (5.17)$$

where

$$N = \begin{pmatrix} \lambda_1^3 & \lambda_1^2 \lambda_4 & \lambda_4^2 \lambda_1 & \lambda_4^3 \\ 3\lambda_1^2 \lambda_2 & \lambda_1^2 \lambda_5 + 2\lambda_1 \lambda_2 \lambda_4 & \lambda_4^2 \lambda_2 + 2\lambda_1 \lambda_4 \lambda_5 & 3\lambda_4^2 \lambda_5 \\ 3\lambda_1 \lambda_2^2 & \lambda_2^2 \lambda_4 + 2\lambda_1 \lambda_2 \lambda_5 & \lambda_1 \lambda_5^2 + 2\lambda_2 \lambda_4 \lambda_5 & 3\lambda_4 \lambda_5^2 \\ \lambda_2^3 & \lambda_2^2 \lambda_5 & \lambda_5^2 \lambda_2 & \lambda_5^3 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} \lambda_1^2 & \lambda_1 \lambda_4 & \lambda_4^2 \\ 2\lambda_1 \lambda_2 & \lambda_1 \lambda_5 + \lambda_2 \lambda_4 & 2\lambda_4 \lambda_5 \\ \lambda_2^2 & \lambda_2 \lambda_5 & \lambda_5^2 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 3\lambda_1^2 \lambda_3 & \lambda_1^2 \lambda_6 + 2\lambda_1 \lambda_3 \lambda_4 & \lambda_3 \lambda_4^2 + 2\lambda_1 \lambda_4 \lambda_6 & 3\lambda_4^2 \lambda_6 \\ 6\lambda_1 \lambda_2 \lambda_3 & 2(\lambda_1 \lambda_2 \lambda_6 + \lambda_1 \lambda_3 \lambda_5 + \lambda_2 \lambda_3 \lambda_4) & 2(\lambda_3 \lambda_4 \lambda_5 + \lambda_2 \lambda_4 \lambda_6 + \lambda_1 \lambda_5 \lambda_6) & 6\lambda_4 \lambda_5 \lambda_6 \\ 3\lambda_2^2 \lambda_3 & \lambda_2^2 \lambda_6 + 2\lambda_2 \lambda_3 \lambda_5 & \lambda_3 \lambda_5^2 + 2\lambda_2 \lambda_5 \lambda_6 & 3\lambda_5^2 \lambda_6 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 3\lambda_1 \lambda_3^2 & \lambda_3^2 \lambda_4 + 2\lambda_1 \lambda_3 \lambda_6 & \lambda_1 \lambda_6^2 + 2\lambda_3 \lambda_4 \lambda_6 & 3\lambda_4 \lambda_6^2 \\ 3\lambda_2 \lambda_3^2 & \lambda_3^2 \lambda_5 + 2\lambda_2 \lambda_3 \lambda_6 & \lambda_2 \lambda_6^2 + 2\lambda_3 \lambda_5 \lambda_6 & 3\lambda_5 \lambda_6^2 \end{pmatrix},$$

$$N_4 = \begin{pmatrix} 2\lambda_1 \lambda_3 & \lambda_1 \lambda_6 + \lambda_3 \lambda_4 & 2\lambda_4 \lambda_6 \\ 2\lambda_2 \lambda_3 & \lambda_2 \lambda_6 + \lambda_3 \lambda_5 & 2\lambda_5 \lambda_6 \end{pmatrix},$$

$$N_5 = \begin{pmatrix} \lambda_1 & \lambda_4 \\ \lambda_2 & \lambda_5 \end{pmatrix}, \quad N_6 = \begin{pmatrix} \lambda_3^3 & \lambda_3^2 \lambda_6 & \lambda_3 \lambda_6^2 & \lambda_6^3 \end{pmatrix}, \quad N_7 = \begin{pmatrix} \lambda_3^2 & \lambda_3 \lambda_6 & \lambda_6^2 \end{pmatrix}, \quad N_8 = \begin{pmatrix} \lambda_3 & \lambda_6 \end{pmatrix}.$$

Hence, it yields that

**Theorem 5.5** *A cubic curve*

$$A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x^2 + A_6xy + A_7y^2 + A_8x + A_9y + A_{10} = 0,$$
$$A_1^2 + A_2^2 + A_3^2 + A_4^2 \neq 0$$

*is still a cubic curve under an affine transformation.*

Let us also verify the following theorem.

**Theorem 5.6** *Assume two point sets $P = \{P_1, P_2, \ldots, P_n\}$ and $Q = \{Q_1, Q_2, \ldots, Q_n\}$ are affine equivalent, where $n \geq 9$. If the $n$ points $P_1, P_2, \cdots, P_n$ satisfy the conditions of Theorem 5.4, which determine uniquely a cubic curve $C_p$, then the $n$ points $Q_1, Q_2, \cdots, Q_n$ also determine uniquely a cubic curve $C_q$, and the cubic curves $C_p$ and $C_q$ are affine equivalent with the same correspondence as that between $P$ and $Q$.*

**Proof** Under the affine transformation (5.16), we have the relation in Eq. (5.17). By a direct calculation, it appears that

$$\det(N) = (\lambda_1\lambda_5 - \lambda_2\lambda_4)^6 \neq 0, \quad \det(N_1) = (\lambda_1\lambda_5 - \lambda_2\lambda_4)^3 \neq 0,$$
$$\det(N_5) = \lambda_1\lambda_5 - \lambda_2\lambda_4 \neq 0.$$

Then, again from Eq. (5.17), we can conclude that

$$\text{rank} \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{10}\} = \text{rank} \{\mathbf{v}_1', \mathbf{v}_2', \cdots, \mathbf{v}_{10}'\}$$

and

$$\text{rank} \{\mathbf{v}_5, \mathbf{v}_6, \cdots, \mathbf{v}_{10}\} = \text{rank} \{\mathbf{v}_5', \mathbf{v}_6', \cdots, \mathbf{v}_{10}'\}.$$

Finally, combining of Theorem 5.4 and Theorem 5.5 yields the conclusion of this theorem. □

**Remark 5.7** In fact, by using the above method, we can consider further the similar conclusions for quartic curve, quintic curve, and so on. On the other hand, if $\text{rank}(M_2) = 6$ and $6 \leq \text{rank}(M_1) < 9$, there are more than one cubic curves passing through these points, and after an affine transformation, we need to verify them respectively.

Next let us consider how to describe the affine curvature of the discrete curve. Firstly, we fit piecewise the discrete points by using the cubic curves. In fact, it is convenient to find the derivatives of the implicit functions. Then according to Eqs. (3.16)-(3.18), we can obtain the affine invariants $\epsilon$, $\kappa$, $\frac{d\kappa_2}{d\sigma}$ of these points along the fitting cubic curve, and these points can be identified by the affine invariants $\epsilon$, $\kappa$, $\frac{d\kappa_2}{d\sigma}$. By this way, it is more convenient to find the correspondence relation between two affine equivalent point sets. In the following example, we show the relations of the affine curvatures among a smooth curve, the fitting cubic curves for the discrete points on this smooth curve and the fitting cubic curves for the smooth curve after an affine transformation.
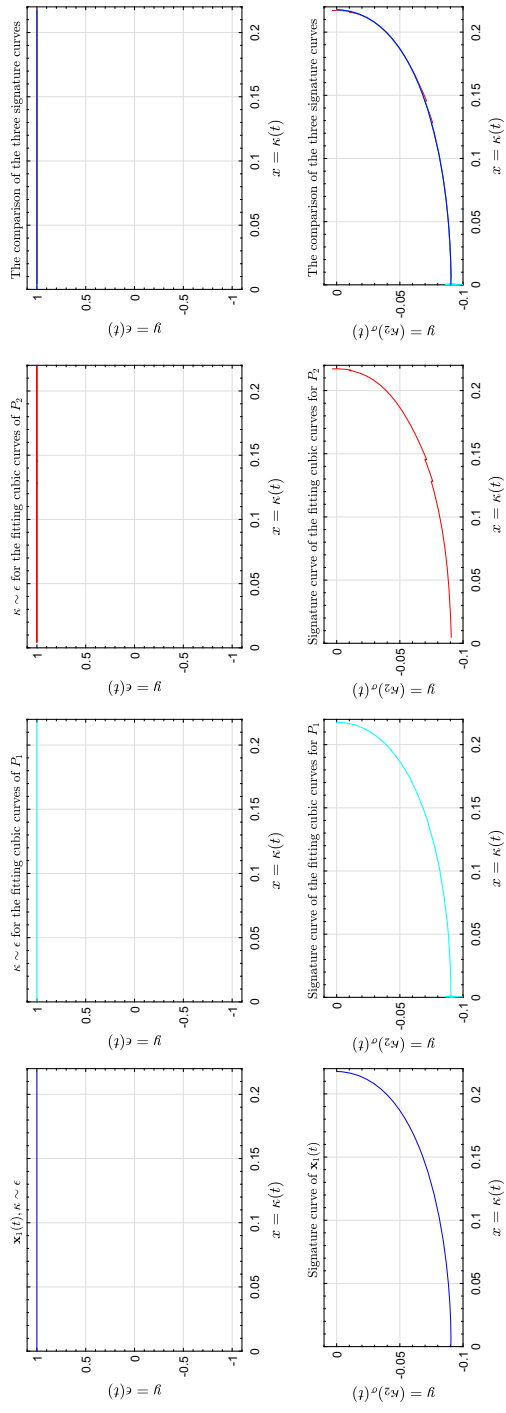
**Fig. 5** The comparison of signature curves between the smooth curve and the fitting cubic curves

**Example 5.8** From Sect. 5.2, we know the curves $\{\mathbf{x}_1(t), 0 < t < 0.4180312\}$ and $\{\mathbf{x}_3(t), 0 < t < 0.04305502\}$ in Example 5.2 are affine equivalent. It is convenient to obtain some discrete points on these two curves. We choose 6000 equidistant points for parameter $t$ on the intervals $(0, 0.4180312)$ and $(0, 0.04305502)$ and generate two discrete point sets $P_1$ and $P_2$ on the curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$, respectively.

Next, we fit piecewise the discrete points in the sets $P_1$ and $P_2$ to the cubic curves by using Theorem 5.4 and calculate their smooth affine curvatures at every point along the fitting cubic curves. Their signature curves are shown in Fig. 5.

In every row of Fig. 5, the left ones are the signature curves of the smooth curve $\mathbf{x}_1(t)$, and the middle twos are the signature curves of the fitting cubic curves to the discrete point sequence $P_1$ and $P_2$. For the sake of direct comparison and contrast, we put the left three results together into one graph as shown in the right ones of Fig. 5. From the visual point of view in Fig. 5, it is no doubt that the fitting affine curvatures are very close to the origin affine curvatures on the curves $\mathbf{x}_1(t)$ and $\mathbf{x}_3(t)$ in Example 5.2, and the fitting effect is satisfactory. Of course, in order to obtain a more accurate result, we need to apply the similar numerical method as that used in Sect. 5.2 and find the affine transformation through those corresponding points whose affine curvatures are the nearest and by using the least square method mentioned in Sect. 5.2. Finally, we can compare the difference between one point set $P_1$ and the image of another point set $P_2$ after the affine transformation obtained.

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