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Two-Distance Vertex-Distinguishing Index of Sparse Subcubic Graphs

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Abstract
The 2-distance vertex-distinguishing index $\chi'_{\mathrm{d2}}(G)$ of a graph G is the minimum number of colors required for a proper edge coloring of G such that any pair of vertices at distance two have distinct sets of colors. It was conjectured that every subcubic graph G has $\chi'_{\mathrm{d2}}(G) \leq 5$. In this paper, we confirm this conjecture for subcubic graphs with maximum average degree less than $\frac{8}{3}$.

Keywords Subcubic graph · Maximum average degree · Edge coloring · 2-Distance vertex-distinguishing index · AVD edge coloring

Mathematics Subject Classification 05C15

1 Introduction

All graphs considered in this paper are finite and simple. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, maximum degree $\Delta(G)$, and minimum degree $\delta(G)$. Let $N_G(v)$ denote the set of neighbors of a vertex v in G, and let $d_G(v) = |N_G(v)|$ denote the degree of v in G. A vertex of degree k (at most k, at least k, resp.) is called a k-*vertex* (k^--*vertex*, k^+-*vertex*, resp.). The *distance*, denoted by $d(u, v)$ between two vertices u and v is the length of a shortest path connecting them. If no confusion arises, we abbreviate $\Delta(G)$ to Δ.

A *proper edge k-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index*,

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denoted $\chi'(G)$, of a graph G is the smallest integer k such that G has a proper edge k-coloring. For a vertex $v \in V(G)$, let $C_\phi(v)$ denote the set of colors assigned to the edges incident to v, that is,

$$C_\phi(v) = \{\phi(uv) | uv \in E(G)\}.$$

The coloring ϕ is called *2-distance vertex-distinguishing* (or a 2DVDE-coloring, in short) if $C_\phi(u) \neq C_\phi(v)$ for any pair of vertices u and v with $d(u, v) = 2$. Let $\chi'_{\mathrm{d2}}(G)$ denote the *2-distance vertex-distinguishing index* of G, which is the smallest integer k such that G has a 2DVDE-coloring using k colors.

The 2-distance vertex-distinguishing edge coloring of graphs can be thought of as a special case of the r-strong edge coloring of graphs, see [1]. Let $r \geq 1$ be an integer. The *r-strong chromatic index* $\chi'_{\mathrm{s}}(G, r)$ of a graph G is the minimum number of colors required for a proper edge coloring of G such that any two vertices u and v with $d(u, v) \leq r$ have $C_\phi(x) \neq C_\phi(y)$. In particular, when $r = 1$, we have $\chi'_s(G, 1) = \chi'_a(G)$, which is called the *neighbor-distinguishing index* of G. Zhang, Liu, and Wang [12] first investigated this parameter and proposed the following conjecture:

Conjecture 1 *If G is a graph different from a 5-cycle, then $\chi'_{\mathrm{a}}(G) \leq \Delta + 2$.*

Balister et al. [2] confirmed Conjecture 1 for bipartite graphs and subcubic graphs. Using a probabilistic analysis, Hatami [3] showed that every graph G with $\Delta > 10^{20}$ has $\chi'_{\mathrm{a}}(G) \leq \Delta + 300$. Akbari et al. [1] proved that every graph G satisfies $\chi'_{\mathrm{a}}(G) \leq 3\Delta$. Zhang et al. [11] proved that every graph G has $\chi'_{\mathrm{a}}(G) \leq 2.5(\Delta + 2)$. Wang et al. [9] improved these upper bounds to $\chi'_{\mathrm{a}}(G) \leq 2.5\Delta$ if $\Delta \geq 7$, and to $\chi'_{\mathrm{a}}(G) \leq 2\Delta$ if $\Delta \leq 6$. The currently best known upper bound that $\chi'_{\mathrm{a}}(G) \leq 2\Delta + 2$ for any graph G was obtained by Vučković [6].

It follows from the definition that $\chi'_{\mathrm{d2}}(G) \geq \chi'(G) \geq \Delta$, and moreover $\chi'_{\mathrm{d2}}(G) \geq \Delta + 1$ if G contains two vertices of maximum degree at distance 2. The 2-distance vertex-distinguishing index for special graphs such as cycles, paths, trees, complete graphs, complete bipartite graphs, and unicycle graphs has been determined in [8]. Using an algorithmic analysis, Wang et al. [7] proved that every outerplanar graph G satisfies $\chi'_{\mathrm{d2}}(G) \leq \Delta + 8$. Additionally, it was shown in [4] that if G is a bipartite outerplanar graph, then $\chi'_{\mathrm{d2}}(G) \leq \Delta + 2$.

A *cubic graph* is a 3-regular graph, and a *subcubic graph* is a graph of maximum degree at most 3. The *maximum average degree* of a graph G is defined as

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} \mid H \subseteq G\right\}.$$

Very recently, Victor et al. [5] showed that every subcubic graph G satisfies $\chi'_{\mathrm{d2}}(G) \leq 6$, and raised the following conjecture:

Conjecture 2 *For a subcubic graph G, $\chi'_{\mathrm{d2}}(G) \leq 5$.*

Note that if Conjecture 2 were true, then the upper bound 5 is tight. In this paper, we confirm partially this conjecture by showing the following result:

Theorem 1 *If H is a subcubic graph with* $\mathrm{mad}(H) < \frac{8}{3}$, *then* $\chi'_{\mathrm{d2}}(H) \leq 5$.

To prove Theorem 1, we need to apply repeatedly the following easy fact (see [10]):

Lemma 2 *Let G be a graph.*

(1) *If v is a leaf of G, then* $\mathrm{mad}(G - v) \leq \mathrm{mad}(G)$.
(2) *If e is an edge of G, then* $\mathrm{mad}(G - e) \leq \mathrm{mad}(G)$.

Let G be a subcubic graph and v be a 3-vertex of G. For $0 \leq i \leq 3$, v is called a 3_i*-vertex* if v is adjacent to exactly i 2-vertices. For a subgraph H of G and a 2DVDE-coloring ϕ of H, we say, in short, that ϕ is a *legal coloring* of H. Two vertices $u, v \in V(G)$ with $d(u, v) = 2$ are called *conflict* with respect to the coloring ϕ if $C_\phi(u) = C_\phi(v)$.

2 Proof of Theorem 1

The proof is by contradiction. Let H be a minimum counterexample that minimizes $|E(H)| + |V(H)|$. Then, $\Delta(H) \leq 3$, $\mathrm{mad}(H) < \frac{8}{3}$, and $\chi'_{\mathrm{d2}}(H) > 5$. It is easy to note that H is connected, for otherwise by the minimality of H, we can 5-2DVDE-color independently each connected component of H using the same set of colors and consider the resulting coloring as a 5-2DVDE-coloring of H. Let H' denote the graph obtained by deleting all 1-vertices of H. Then, H' is clearly connected, $\Delta(H') \leq 3$, and $\mathrm{mad}(H') \leq \mathrm{mad}(H) < \frac{8}{3}$ by Lemma 2. Moreover, by the minimality of H, any of its subgraph obtained by edge deletion can be legally colored with at most five colors. We first list some structural properties of H'. In the subsequent proofs, we routinely construct 5-2DVDE-colorings of H without verifying in detail that H is legally-5-colored since this can be supplied in a straightforward manner. In the following, we always let $C = \{1, 2, \ldots, 5\}$ denote a set of five colors. Given a 5-2DVDE-coloring ϕ of a subgraph G of H using the color set C, for a vertex $v \in V(G)$, we denote simply $C_\phi(v)$ by $C(v)$.

Claim 1 $\delta(H') \geq 2$.

Proof Suppose to the contrary that $\delta(H') \leq 1$. If $\delta(H') = 0$, then H' is isomorphic to K_1 and so H is isomorphic to the star $K_{1,n-1}$ with $|V(H)| = n$. Obviously, we can color the edges of $K_{1,n-1}$ with distinct colors, so $\chi'_{\mathrm{d2}}(H) = \Delta(H) \leq 3$, which contradicts the hypothesis on H. Assume that $\delta(H') = 1$, and let u be a 1-vertex of H' adjacent to a vertex v. Then, $d_H(u) \in \{2, 3\}$, let u_1 be another neighbor of u different from v in H, and let $G = H - uu_1$. By the minimally of H, G has a 5-2DVDE-coloring ϕ using the color set C. Observe that $|C(u) \cup C(v)| \leq 4$ since $d_H(v) \leq 3$ and v is adjacent to u. Therefore, to extend ϕ to H, it suffices to color uu_1 with a color in $C - C(u) - C(v)$. This contradicts the choice of H. □

Claim 2 H' contains no two adjacent 2-vertices.

Proof Suppose to the contrary that H' contains two adjacent 2-vertices u and v. Let $N_{H'}(u) = \{v, u_1\}$ and $N_{H'}(v) = \{u, v_1\}$. Then, $d_H(u), d_H(v) \in \{2, 3\}$. We discuss the following two cases by symmetry.

Case 1. $d_H(u) = d_H(v) = 2$.

Consider the graph $G = H - uv$. By the minimally of H, G has a 5-2DVDE-coloring ϕ using the color set C. We assume that uv cannot be colored with any color in C. Therefore, at least one of u_1 and v_1 is a 3-vertex; otherwise, we color uv with a color in $C - C(u_1) - C(v_1)$. Without loss of generality, assume that $d_G(u_1) = 3$, $N_G(u_1) = \{u, u_2, u_3\}$, and $C(u_1) = \{1, 2, 3\}$ such that $\phi(uu_1) = 1$ and $\phi(u_1u_2) = 2$. We discuss two possibilities:

- Let $d_G(v_1) = 2$, say $N_G(v_1) = \{v, v_2\}$. If $d_G(v_2) = 2$, then the proof is reduced to the previous case by replacing uv with vv_1. Otherwise, $d_G(v_2) = 3$, we may color uv with a color in $\{4, 5\} - \{\phi(vv_1)\}$. This contradicts the assumption that uv cannot be colored.
- Let $d_G(v_1) = 3$, say $N_H(v_1) = \{v, v_2, v_3\}$. First assume that at least two of u_2, u_3, v_2, v_3 are 3-vertices. By symmetry, we have the following two possibilities. If $d_G(u_2) = d_G(u_3) = 3$, then we color uv with a color in $\{2, 3, 4, 5\} - C(v_1)$. If $d_G(u_2) = d_G(v_2) = 3$, then we color uv with a color in $\{2, 4, 5\} - \{\phi(vv_1), \phi(v_1v_3)\}$. Next assume that at most one of u_2, u_3, v_2, v_3 is of degree 3, say, $d_G(u_2) = d_G(u_3) = d_G(v_2) = 2$ and $2 \leq d_G(v_3) \leq 3$. It is easy to see that $\{4, 5\} \subset C(v_1)$ because uv can not be legally colored.

First suppose that $d_G(v_3) = 2$. If $\phi(vv_1) = 2$, then $\phi(v_1v_2) = 4$ and $\phi(v_1v_3) = 5$. It follows that $C(u_2) = \{1, 2\}$, $C(u_3) = \{1, 3\}$, $C(v_2) = \{2, 4\}$, and $C(v_3) = \{2, 5\}$. It suffices to recolor vv_1 with 1 and color uv with 4. If $\phi(vv_1) \in \{1, 3\}$, we have a similar discussion. So assume that $\phi(vv_1) \in \{4, 5\}$, say $\phi(vv_1) = 4$. Then, $C(u_2) = \{1, 2\}$ and $C(u_3) = \{1, 3\}$. Without loss of generality, assume that $\phi(v_1v_3) = 5$ and so $\phi(v_1v_2) \in \{1, 2, 3\}$. Let z be the other neighbor of v_3 different from v_1. Then, we must have $\phi(zv_3) = 4$. Now we recolor uu_1 with 4 and color uv with a color in $\{1, 2, 3\}$ such that v does not conflict with v_2.

Next suppose that $d_G(v_3) = 3$. A similar and easier proof can be established.

Case 2. $d_H(u) = 3$ and $d_H(v) \in \{2, 3\}$.

Let $N_H(u) = \{v, u_1, x\}$ with $d_H(x) = 1$. If $d_H(v) = 3$, then we furthermore assume that $N_H(v) = \{u, v_1, y\}$ with $d_H(y) = 1$. Consider the graph $G = H - ux$. By the minimality of H, G has a 5-2DVDE-coloring ϕ using the color set C. Assume that ux cannot be colored with any color in C. We have to consider two cases as follows.

Assume that u_1 is a 2-vertex of G. Then, $N_G(u_1) = \{u, u_2\}$. If u_2 is a 2-vertex, then we can color ux with a color in $C - C(u_1) - \{\phi(uv), \phi(vv_1)\}$, which is a contradiction. Otherwise, u_2 is a 3-vertex. If $\phi(uv) \neq \phi(u_1u_2)$, then we color ux with a color in $C - C(u_1) - \{\phi(uv), \phi(vv_1)\}$. If $\phi(uv) = \phi(u_1u_2)$, then $C - C(u_1) - \{\phi(uv), \phi(vv_1)\}$ contains at least two colors, so that we can choose one of them to color ux.

Assume that $d_G(u_1) = 3$ and $N_G(u_1) = \{u, u_2, u_3\}$. If $d_G(v_1) = 2$, then we color ux with a color in $C - C(u_1) - \{\phi(uv)\}$. Thus, assume that $d_G(v_1) = 3$. Without loss of generality, we may assume that $\phi(uv) = 1$, $\phi(uu_1) = 2$, $C(u_2) = \{1, 2, 4\}$, $C(u_3) = \{1, 2, 3\}$, and $C(v_1) = \{1, 2, 5\}$. There are two possibilities to be handled.

- Let $d_H(v) = 2$. If $C(u_1) = \{2, 3, 4\}$, it suffices to recolor uu_1 with 5 and color ux with 4. So assume that $C(u_1) = \{1, 2, 4\}$, and hence, it suffices to recolor uv with 4 and color ux with 5.

- Let $d_H(v) = 3$. Then, $N_G(v) = \{u, y, v_1\}$. Let $N_G(v_1) = \{v, v_2, v_3\}$. If $C(u_1) = \{2, 3, 4\}$, then we recolor uu_1 with 5 and color ux with 4. So assume that $C(u_1) = \{1, 2, 4\}$ by symmetry. Note that $\phi(vv_1) \in \{2, 5\}$. If $\phi(vv_1) = 2$, then it follows immediately that $\phi(vy) \in \{3, 5\}$, we switch the colors of vy and vu and color ux with 4. Now suppose that $\phi(vv_1) = 5$, and furthermore, let $\phi(v_1v_2) = 2$. Then, $\phi(vy) \in \{2, 3, 4\}$. If $\phi(vy) = 2$, then we recolor vu with 3 or 4 such that v does not conflict with v_2, and color ux with 5. If $\phi(vy) \in \{3, 4\}$, then after switching the colors of vy and vu, we color ux with 5. □

The proof of Claims 3–5 below will be given in the subsequent sections.

Claim 3 H' contains no 3_3-vertex.

Claim 4 H' contains no 2-vertex adjacent to two 3_2-vertices.

Claim 5 H' contains no 3_2-vertex.

We define an initial weight function $w(v) = d_{H'}(v)$ for every vertex $v \in V(H')$. Then, we redistribute weights according to the following rule:
(R) Every 3_1-vertex sends the weight of $\frac{1}{3}$ to the uniquely adjacent 2-vertex.

The sum of all charges is kept fixed when the discharging is in process. Once the discharging is finished, a new charge function w' is produced. Nevertheless, we can show that $w'(v) \geq \frac{8}{3}$ for all $v \in V(H')$. In fact, let $v \in V(H')$. By Claims 1–5, v is either a 2-vertex or a 3_1-vertex or a 3_0-vertex. If v is a 3_0-vertex, then $w'(v) = 3$. If v is a 3_1-vertex, then $w'(v) = 3 - \frac{1}{3} = \frac{8}{3}$. If v is a 2-vertex, then $w'(v) = 2 + 2.\frac{1}{3} = \frac{8}{3}$. This leads to the following obvious contradiction:

$$\frac{8}{3} = \frac{\frac{8}{3}|V(H')|}{|V(H')|} \leq \frac{\sum_{v \in V(H')} w'(v)}{|V(H')|} = \frac{\sum_{v \in V(H')} w(v)}{|V(H')|} = \frac{2|E(H')|}{|V(H')|} \leq \mathrm{mad}(H') < \frac{8}{3}.$$

This completes the proof of Theorem 1. □

3 Proof of Claim 3

Assume to the contrary that H' contains a 3-vertex x adjacent to three 2-vertices u, v, w (see Fig. 1). Let $N_{H'}(u) = \{x, u_1\}$, $N_{H'}(v) = \{x, v_1\}$, and $N_{H'}(w) = \{x, w_1\}$. By Claims 1 and 2, $d_{H'}(u_1) = d_{H'}(v_1) = d_{H'}(w_1) = 3$. Note that $d_H(u), d_H(v), d_H(w) \in \{2, 3\}$. Setting $N_{H'}(u_1) = \{u, u_2, u_3\}$, we discuss two cases below.

Case 1. $d_H(u) = d_H(v) = d_H(w) = 2$.

Let $G = H - ux$, which admits a 5-2DVDE-coloring ϕ with $\phi(xv) = 1$ and $\phi(xw) = 2$. Assume that xu cannot be colored with any color in C. Let us deal with the following cases, depending on the color of uu_1.

(1) $\phi(uu_1) \in \{1, 2\}$, say $\phi(uu_1) = 2$ by symmetry.

(1.1) Suppose that at least one of u_2 and u_3 is a 3-vertex in G, say $d_G(u_3) = 3$. By symmetry, the proof splits into two cases.

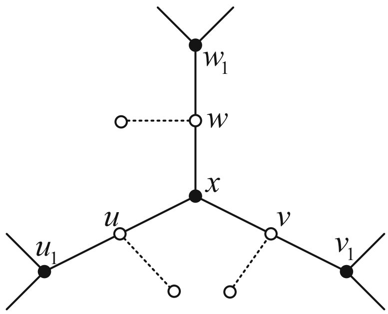


Fig. 1 The configurations in the proof of Claim 3

(1.1.1) Let $\phi(ww_1) = 1$. Without loss of generality, assume that $C(v_1) = \{1, 2, 4\}$ and $C(w_1) = \{1, 2, 5\}$. It follows that $C(u_1) = \{1, 2, 3\}$, or $C(u_2) = \{2, 3\}$. Recolor xw with 4 and color xu with 5. If $\phi(vv_1) \neq 4$, we are done. Otherwise, we recolor xv with 3.

(1.1.2) Let $\phi(ww_1) \in \{3, 4, 5\}$, and assume $\phi(ww_1) = 5$ by symmetry. Similarly, we can assume that $C(v_1) = \{1, 2, 4\}$; and $C(u_1) = \{1, 2, 3\}$ or $C(u_2) = \{2, 3\}$. Recolor xv with 3 and color ux with 1 or 4 such that x does not conflict with u_1.

(1.2) Suppose that $d_G(u_2) = d_G(u_3) = 2$. There are two subcases below by symmetry.

(1.2.1) $\{C(u_2), C(u_3)\} = \{\{2, 3\}, \{2, 4\}\}$.

Assume that $C(v_1) = \{1, 2, 5\}$. If $C(w_1) \neq \{1, 2, 3\}$, then we first recolor uu_1 with 5 and color xu with 3. Otherwise, $C(w_1) = \{1, 2, 3\}$, recolor uu_1 with 5 and color xu with 3.

Assume $C(w_1) = \{1, 2, 5\}$, then a similar strategy as in the previous case is applied.

Assume now that $C(v_1) \neq \{1, 2, 5\}$ and $C(w_1) \neq \{1, 2, 5\}$. If $\phi(ww_1) \neq 5$, then we color xu with 5. Otherwise, assume that $\phi(ww_1) = 5$. Recolor $uu_1 = 5$, color ux with 3 or 4 such that x does not conflict with v_1.

(1.2.2) At most one of $C(u_2)$ and $C(u_3)$ is $\{2, i\}$ for some $i \in \{3, 4, 5\}$, say $C(u_2) = \{2, 3\}$ by symmetry.

Assume that $\phi(ww_1) \in \{4, 5\}$, say $\phi(ww_1) = 4$. Then, it is immediate to derive that $C(v_1) = \{1, 2, 5\}$. We first recolor xv with 3 and color xu with 5. If $C(u_1) \neq \{2, 3, 5\}$, we are done. Otherwise, we recolor ux with 1.

Assume that $\phi(ww_1) \notin \{4, 5\}$. Furthermore, suppose that $C(v_1) = \{1, 2, 5\}$ and $C(w_1) = \{1, 2, 4\}$. This implies that $\phi(ww_1) = 1$. Recolor xv with 3 and color xu with 4 or 5 such that x does not conflict with u_1.

(2) $\phi(uu_1) \notin \{1, 2\}$, say $\phi(uu_1) = 3$ by symmetry.

We have to handle three possibilities by symmetry.

(2.1) $d_G(u_2) = d_G(u_3) = 3$. Assume that $C(v_1) = \{1, 2, 4\}$ and $C(w_1) = \{1, 2, 5\}$. Recolor xw with 4 and color xu with 5. If v does not conflict with w, then we are done. Otherwise, we know that $\phi(ww_1) = 1$ and $\phi(vv_1) = 4$. In this case, we keep $\phi(xw) = 2$, and then we recolor xv with 5 and xu with 4.

(2.2) $d_G(u_2) = 2$ and $d_G(u_3) = 3$.

If $C(u_2) \notin \{\{3, 4\}, \{3, 5\}\}$, then the proof can be analogously given as in Case (2.1). Otherwise, without loss of generality, assume that $C(u_2) = \{3, 4\}$, and further

$C(v_1) = \{1, 2, 5\}$. If $\phi(ww_1) \neq 4$, then we recolor xv with 4 and ux with 1 or 5 such that x does not conflict with w_1. If $\phi(ww_1) = 4$, then we recolor xv with 3 and color ux with 1.

(2.3) $d_G(u_2) = d_G(u_3) = 2$.

If $3 \notin C(u_2) \cup C(u_3)$, then the proof is similar to that of Case (2.1).

Assume that $3 \in C(u_2)$ and $3 \notin C(u_3)$ (if $3 \in C(u_3)$ and $3 \notin C(u_2)$, we have a similar proof). If $\phi(u_1u_2) \in \{1, 2\}$, say $\phi(u_1u_2) = 1$, then we assume that $C(v_1) = \{1, 2, 5\}$ and $C(w_1) = \{1, 2, 4\}$. Recolor xw with 3 and color xu with 4 or 5, say 4, such that x does not conflict with u_1. If $\phi(ww_1) \neq 4$, we are done. Otherwise, we recolor xv with 3 and xw with 5. If $\phi(u_1u_2) \in \{4, 5\}$, say $\phi(u_1u_2) = 4$, then at least one of v_1 and w_1 has color set {1,2,5}, say v_1. Recolor xv with 4 and ux with 1 or 5 such that x does not conflict with w_1. If $\phi(ww_1) \neq 4$ or $\phi(vv_1) \neq 2$, we are done. Otherwise, $\phi(ww_1) = 4$ and $\phi(vv_1) = 2$, we recolor xv with 3, and color ux with 1.

Assume that $3 \in C(u_2) \cap C(u_3)$. If $C(u_1) = \{1, 2, 3\}$, then we may assume that $C(v_1) = \{1, 2, 4\}$ and $C(w_1) = \{1, 2, 5\}$. Recolor xw with 4 and color xu with 5. If v and w are not conflicting, we are done. Otherwise, $\phi(vv_1) = 4$ and $\phi(ww_1) = 1$, it suffices to recolor xv with 3. If $C(u_1) \neq \{1, 2, 3\}$, say $1 \notin C(u_1)$, we recolor uu_1 with 1 and return to a case similar to (1.2.2).

Case 2. At least one of u, v, w is a 3-vertex in H, say $d_H(u) = 3$.

Let $N_H(u) = \{x, u_1, u_4\}$. Let $G = H - uu_4$, which admits a 5-2DVDE-coloring ϕ such that $\phi(xu) = 1$ and $\phi(uu_1) = 2$. In view of the number of 2-vertices in the set $\{v, w, u_2, u_3\}$ in G, we need to consider four cases by symmetry.

(1) $d_G(u_2) = d_G(w) = 2$. We color uu_4 with a color in $\{3, 4, 5\}$ such that u does not conflict with u_3 and v.

(2) $d_G(u_2) = d_G(u_3) = 2$. We color uu_4 with a color in $\{3, 4, 5\}$ such that u does not conflict with w and v.

(3) $d_G(u_2) = 2$ and $d_G(u_3) = d_G(v) = d_G(w) = 3$. Let $N_G(v) = \{x, v_1, v_2\}$ with $d_G(v_2) = 1$ and $N_G(w) = \{x, w_1, w_2\}$ with $d_G(w_2) = 1$. By Claim 2, $d_G(v_1) = d_G(w_1) = 3$. Hence we assume that $C(u_3) = \{1, 2, 3\}$, $C(v) = \{1, 2, 5\}$, and $C(w) = \{1, 2, 4\}$. If $C(x) = \{1, 4, 5\}$, we recolor ux with 3 and color uu_4 with 5. Otherwise, assume that $C(x) = \{1, 2, 4\}$ by symmetry. Then, $\phi(vv_1) \in \{1, 5\}$, if $\phi(vv_1) = 1$, exchange the color of vv_2 and vx, then recolor ux with 3 and color uu_4 with 4. Now if $\phi(vv_1) = 5$, observe that $\phi(ww_1) \in \{1, 2\}$. So if $\phi(ww_1) = 1$, we recolor ux with 3 and color uu_4 with 5. Otherwise, $\phi(ww_1) = 2$, exchange the color of vv_2 and vx, then recolor ux with 3 and color uu_4 with 5.

(4) $d_G(v) = d_G(w) = d_G(u_2) = d_G(u_3) = 3$. Let us consider two possibilities below.

(4.1) $C(v) \notin \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$. Assume by symmetry that $C(w) = \{1, 2, 3\}$, $C(u_2) = \{1, 2, 4\}$, and $C(u_3) = \{1, 2, 5\}$. If $C(u_1) = \{2, 4, 5\}$, then we recolor uu_1 with 3 and color uu_4 with 4 or 5 such that u does not conflict with v. So assume that $C(u_1) = \{1, 2, 4\}$ by symmetry. This implies that $\phi(u_1u_2) = 4$ and $\phi(u_1u_3) = 1$. Noting that $\phi(xw) \in \{2, 3\}$, we have to handle two situations as follows.

• Let $\phi(xw) = 2$. Then, $\phi(vx) \in \{3, 4, 5\}$.

First suppose that $\phi(vx) = 3$. If we can recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and uu_4 with 4, we are done. Otherwise, we assume that

$C(v_1) = \{2, 3, 4\}$ and $C(w_1) = \{2, 3, 5\}$, then we recolor ww_2 with 2, wx with 1, ux with 5, and color uu_4 with 4.

Next suppose that $\phi(vx) \in \{4, 5\}$, say $\phi(vx) = 4$ by symmetry. If we can recolor ux with 3 and uu_4 with 5, or recolor ux with 5 and uu_4 with 3, we are done. Otherwise, assume that $C(v_1) = \{2, 4, 5\}$ and $C(w_1) = \{2, 3, 4\}$, then we recolor ww_2 with 2, wx with 1, ux with 5, and color uu_4 with 3.

• Let $\phi(xw) = 3$. Then, $\phi(vx) \in \{2, 4, 5\}$. If $\phi(vx) = 2$, then we use the same strategy as in the previous case to color uu_4. So assume that $\phi(vx) = 4$, say. If $C(v_1) \neq \{3, 4, 5\}$, then we recolor ux with 5 and color uu_4 with 3. If $C(v_1) = \{3, 4, 5\}$, then we exchange the color of ww_2 and xw. Then, we color ux with 5, and color uu_4 with 4.

(4.2) $C(v) \in \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(v) = \{1, 2, 3\}$. We need to discuss two subcases.

(4.2.1) $C(w) \notin \{\{1, 2, 4\}, \{1, 2, 5\}\}$. Then, we may assume that $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 5\}$. If $C(u_1) = \{2, 4, 5\}$, then we recolor uu_1 with 3 and color uu_4 with 4 or 5 such that u does not conflict with w. So assume that $C(u_1) = \{1, 2, 4\}$ by symmetry. Since $\phi(xv) \in \{2, 3\}$, we have two possibilities.

• Let $\phi(vx) = 2$. Then, $\phi(wx) \in \{3, 4, 5\}$. First assume that $\phi(wx) = 3$. If we can legally recolor ux with 4 and uu_4 with 5, or recolor ux with 5 and uu_4 with 4, we are done. Otherwise, it is easy to see that $C(v_1) = \{2, 3, 4\}$ and $C(w_1) = \{2, 3, 5\}$ (up to symmetry). It suffices to recolor vv_2 with 2, vx with 1, ux with 5, and color uu_4 with 4. Next assume that $\phi(wx) \in \{4, 5\}$, say $\phi(wx) = 4$. If we can legally recolor ux with 3 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 3, we are done. Otherwise, we derive that $C(v_1) = \{2, 3, 4\}$ and $(w_1) = \{2, 4, 5\}$ (up to symmetry). It suffices to recolor vv_2 with 2, vx with 1, ux with 5, and color uu_4 with 3.

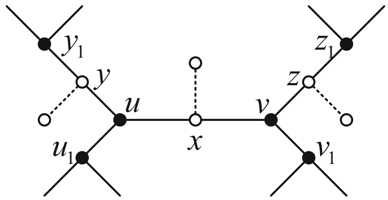
• Let $\phi(vx) = 3$. Then, $\phi(wx) \in \{2, 4, 5\}$. First assume that $\phi(wx) = 2$. If we can legally recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with w, or recolor ux with 4 and color uu_4 with 3 or 5 such that u does not conflict with w, we are done. Otherwise, it follows that $C(v_1) = \{2, 3, 4\}$ and $C(w_1) = \{2, 3, 5\}$, say. It suffices to recolor vx with 5, ux with 4 and uu_4 with 3 or 5 such that u does not conflict with w. Next, assume that $\phi(wx) \in \{4, 5\}$, say $\phi(wx) = 4$. If we can legally recolor ux with 5 and color uu_4 with 3, we are done. Otherwise, we derive that $C(w_1) = \{3, 4, 5\}$. When $\phi(vv_2) = 1$, we recolor vv_2 with 3, vx with 1, ux with 3, and color uu_3 with 5. When $\phi(vv_2) = 2$, we recolor vv_2 with 3, vx with 2, ux with 5, and color uu_4 with 3.

(4.2.2) $C(w) \in \{\{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(w) = \{1, 2, 4\}$. Without loss of generality, we suppose that $C(u_2) = \{1, 2, 5\}$. Since $\phi(vx) \in \{2, 3\}$, we need to discuss two subcases.

• Let $\phi(vx) = 2$. Then, $\phi(wx) = 4$. If we can legally recolor ux with 3 and color uu_4 with 4 or 5 such that u does not conflict with u_3, or recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3, we are done. Otherwise, we may assume that $C(v_1) = \{2, 3, 4\}$ and $C(w_1) = \{2, 4, 5\}$, then we recolor ww_2 with 4, xw with 1, ux with 4, and color uu_4 with 3 or 5 such that u does not conflict with u_3.

• Let $\phi(vx) = 3$. Then, $\phi(wx) \in \{2, 4\}$. If $\phi(wx) = 4$, then we recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3. So assume that $\phi(wx) = 2$. If we can legally recolor ux with 4 and color uu_4 with 3 or 5 such that u

Fig. 2 The configuration in the proof of Claim 4



does not conflict with u_3, or recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3, we are done. Otherwise, $C(v_1) = \{2, 3, 5\}$ and $C(w_1) = \{2, 3, 4\}$, say. Now it suffices to recolor vv_2 with 3, vx with 1, ux with 3, and color uu_4 with 4 or 5 such that u does not conflict with u_3. □

4 Proof of Claim 4

Assume to the contrary that H' contains a 2-vertex x adjacent to two 3_2-vertices u and v (see Fig. 2). Let $N_{H'}(u) = \{x, y, u_1\}$ with $d_{H'}(y) = 2$, $N_{H'}(v) = \{x, z, v_1\}$ with $d_{H'}(z) = 2$, $N_{H'}(y) = \{u, y_1\}$, and $N_{H'}(z) = \{v, z_1\}$. By Claims 1 and 2, $d_{H'}(u_1) = d_{H'}(v_1) = d_{H'}(y_1) = d_{H'}(z_1) = 3$. Let $N_{H'}(u_1) = \{u, u_2, u_3\}$, $N_{H'}(v_1) = \{v, v_2, v_3\}$, $N_{H'}(y_1) = \{y, y_2, y_3\}$, and $N_{H'}(z_1) = \{z, z_2, z_3\}$. By Claim 3, at most one of y_2 and y_3 has degree two; and at most one of z_2 and z_3 has degree two. So assume, without loss of generality, that $d_{H'}(y_3) = d_{H'}(z_3) = 3$. We discuss two cases, depending on the degree of x, y, z in H.

Case 1. $d_H(x) = d_H(y) = d_H(z) = 2$.

Consider the graph $G = H - xu$, which has a 5-2DVDE-coloring ϕ using the color set C. We assume that xu cannot be colored with any color in C. Let $\phi(uu_1) = 1$ and $\phi(uy) = 2$. We discuss three possibilities according to the degree of u_2 and u_3 in G.

(1) $d_G(u_2) = d_G(u_3) = 2$. Without loss of generality, we assume that $\phi(vx) = \phi(zz_1) = 3$, $\phi(vz) = 5$, and $C(y_1) = \{1, 2, 4\}$. Then, it suffices to recolor uy with 5 and color ux with 4.

(2) $d_G(u_2) = 3$ and $d_G(u_3) = 2$. If $\{1, 2\} \subset C(v)$, say $C(v) = \{1, 2, 3\}$, then we may assume that $C(u_2) = \{1, 2, 4\}$, and $C_\phi(y) = \{2, 5\}$ with $\phi(xv) = 2$ or $C(y_1) = \{1, 2, 5\}$. If $C(y_1) = \{1, 2, 5\}$, recolor uy with 3 and color ux with 4. Next suppose $C(y) = \{2, 5\}$ and $\phi(xv) = 2$, if $C(y_2) \neq \{3, 5\}$, we proceed as in the previous case. Otherwise $C(y_1) = \{3, 5\}$, then we recolor uy with 4, color ux with 3 or 5, such that x does not conflict with z.

Now suppose that $\{1, 2\} \not\subset C(v)$. We have to consider two subcases as follows.

(2.1) $\phi(vx) \in \{1, 2\}$, say $\phi(vx) = 2$ (if $\phi(vx) = 1$, our discussion is similar). Then, it follows that $1 \notin \{\phi(vz), \phi(vv_1)\}$, and we may assume that $\phi(zz_1) = 2$, $\phi(vz) = 3$, $C(u_2) = \{1, 2, 4\}$, and $C(y) = \{2, 5\}$ or $C(y_1) = \{1, 2, 5\}$. If $C(y_1) = \{1, 2, 5\}$, recolor uy with 3 and color ux with 4. Next suppose $C(y) = \{2, 5\}$, if $C(y_2) \neq \{3, 5\}$, we proceed as in the previous case. Otherwise $C(y_1) = \{3, 5\}$, then recolor uy with 4, color ux with 5.

(2.2) $\phi(vx) \in \{3, 4, 5\}$, say $\phi(vx) = 3$ by symmetry. If $C(z) \notin \{\{3, 4\}, \{3, 5\}\}$, then we may assume that $C(u_2) = \{1, 2, 4\}$, and $C(y_1) = \{1, 2, 5\}$. It suffices to recolor uy with 3, color ux with 4 or 5 such that u does not conflict with v. So assume that $C(z) \in \{\{3, 4\}, \{3, 5\}\}$, say $C(z) = \{3, 4\}$ by symmetry. Then, at least one of u_2 and y_1 has color set $\{1, 2, 5\}$. If $C(y_1) = \{1, 2, 5\}$, we first suppose $C(u_2) \neq \{1, 2, 3\}$, then we recolor uy with 3 and color ux with 2. If $C(u_2) = \{1, 2, 3\}$, then we recolor uy with 4, and color ux with 2.

Otherwise, $C(u_2) = \{1, 2, 5\}$, we have $\phi(vv_1) \in \{1, 2, 5\}$. First assume that $\phi(vv_1) = 1$. If we can recolor vx with 2 and color ux with 3 or 4 such that u does not conflict with y_1, and x does not conflict with y, or recolor vx with 5 and color ux with 3 or 4 such that u does not conflict with y_1, we are done. Otherwise, we may assume that $C(v_2) = \{1, 2, 4\}$ and $C(v_3) = \{1, 4, 5\}$. When $C(z_2) = \{3, 5\}$, we recolor vz with 2 and vx with 5 and color ux with 3 or 4 such that u does not conflict with y_1. When $C(z_2) \neq \{3, 5\}$, we recolor vz with 5 and vx with 2 and color ux with 3 or 4 such that u does not conflict with y_1 and x does not conflict with y.

If $\phi(vv_1) = 2$ or $\phi(vv_1) = 5$, we have a similar argument.

(3) $d_G(u_2) = d_G(u_3) = 3$. We discuss two possibilities according to the color set of v.

(3.1) $\{1, 2\} \subset C(v)$, say $C(v) = \{1, 2, 3\}$. We discuss the following subcases:

• Assume that $C(y) = \{2, 5\}$. Since ux cannot be colored, we assume $C(u_2) = \{1, 2, 4\}$.

If $C(y_2) = \{3, 5\}$ and $C(z) \neq \{2, 3\}$, then we recolor uy with 4 and color ux with 3 or 5 such that u does not conflict with u_3. Now, suppose $C(y_2) = \{3, 5\}$ and $C(z) = \{2, 3\}$; if $C(u_3) \neq \{1, 2, 3\}$ we color ux with 3 and if we can recolor vx with 4 or 5, we are done. If vx cannot be recolor with 4 or 5, then we may assume that $C(v_2) = \{1, 3, 4\}$ and $C(v_3) = \{1, 3, 5\}$; in this case, if $C(z_3) = \{2, 4\}$, recolor vz with 5, vx with 4 and color ux with 3. If $C(z_3) \neq \{2, 4\}$, recolor vz with 4, vx with 5 and color ux with 3. We next suppose $C(u_3) = \{1, 2, 3\}$, then recolor uy with 4 and color ux with 5.

If $C(y_2) \neq \{3, 5\}$, then we recolor uy with 3 and color ux with 4 or 5 such that u does not conflict with u_3.

• Assume that $C(y_1) \in \{\{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(y_1) = \{1, 2, 5\}$, and $C(y) \neq \{2, 5\}$. Then, at least one of u_2 and u_3, say u_2, has color set $\{1, 2, 4\}$. If $C(u_3) = \{1, 4, 5\}$, then we recolor uy with 3 and color ux with 4. If $C(u_3) \neq \{1, 4, 5\}$, then we recolor uy with 4 and color ux with 5.

• Assume now that $C(y_1) \notin \{\{1, 2, 4\}, \{1, 2, 5\}\}$ and $C(y) \neq \{2, 5\}$. Then, it is easy to see that $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 5\}$ by symmetry. If $C(y_1) \neq \{3, 4, 5\}$, say $3 \notin C(y_1)$, then we recolor uy with 3 and ux with 4. If $C(y_1) = \{3, 4, 5\}$, say $\phi(yy_1) = 3$ and $\phi(y_2y_2) = 4$; if $C(y_2) = \{3, 4\}$, then we recolor uy with 5 and color uy with 4; otherwise, we recolor uy with 4 and color uy with 5.

(3.2) $\{1, 2\} \not\subset C(v)$. In view of the color of xv, we consider three subcases.

(3.2.1) $\phi(vx) = 1$. If $1 \notin C(z)$ or $C(z) = \{1, 2\}$, then we may assume that $C(u_2) = \{1, 2, 3\}$, $C(u_3) = \{1, 2, 4\}$, and $C(y_1) = \{1, 2, 5\}$. Recolor uy with 4 and color ux with 3 or 5 such that u does not conflict with v. Otherwise, let $C(z) = \{1, 5\}$. Then, $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are the color sets of at least two of u_2, u_3, y_1. Assume that $C(y_1) \in \{\{1, 2, 3\}, \{1, 2, 4\}\}$, say $C(y_1) = \{1, 2, 3\}$, and moreover, $C(u_2) = \{1, 2, 4\}$. If $C(u_3) = \{1, 3, 4\}$, then we recolor uy with 5 and color ux with 3 or 4 such that u does not conflict with v. If $C(u_3) \neq \{1, 3, 4\}$, then we recolor uy with 4 and color ux

with 3. If $C(y_1) \notin \{\{1, 2, 3\}, \{1, 2, 4\}\}$, then $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 3\}$. If $C(y_1) \neq \{3, 4, 5\}$, say $4 \notin C(y_1)$, then we recolor uy with 4 and color ux with 3. So suppose that $C(y_1) = \{3, 4, 5\}$, say $\phi(yy_1) = 3$ and $\phi(y_1y_2) = 4$. When $C(y_2) = \{3, 4\}$, we recolor uy with 5 and color ux with 3 or 4 such that u does not conflict with v. When $C(y_2) \neq \{3, 4\}$, we recolor uy with 4 and color ux with 3 or 5 such that u does not conflict with v.

(3.2.2) $\phi(vx) = 2$. If $2 \notin C(z)$ or $C(z) = \{1, 2\}$, then we may assume that $C(u_2) = \{1, 2, 3\}$, $C(u_3) = \{1, 2, 4\}$, and $C(y) = \{2, 5\}$ or $C(y_1) = \{1, 2, 5\}$. First if $C(y_1) = \{1, 2, 5\}$, recolor uy with 4 and color ux with 3. Next suppose $C(y) = \{2, 5\}$, then if $C(y_2) \neq \{4, 5\}$, recolor uy with 4 and color ux with 3. Otherwise, $C(y_2) = \{4, 5\}$ and we recolor uy with 3 and color ux with 4.

So assume that $2 \in C(z)$; furthermore, let $C(z) = \{2, 5\}$. Note that $\phi(vv_1) \in \{3, 4\}$ since $\{1, 2\} \not\subset C(v)$. By symmetry, we may assume that $\phi(vv_1) = 3$. We discuss the following subcases:

- Suppose that $C(y) \in \{\{2, 3\}, \{2, 4\}\}$, say $C(y) = \{2, 3\}$ by symmetry. Then, we may assume that $C(u_2) = \{1, 2, 4\}$.

If $C(y_2) = \{3, 5\}$, we first suppose $C(u_3) \neq \{1, 3, 4\}$; then we recolor uy with 4 and color ux with 3. Next assume $C(u_3) = \{1, 3, 4\}$; then, we assign color 5 to ux, and so if we can recolor vx with 1 or 4 we are done. Otherwise, we may assume $C(v_2) = \{1, 3, 5\}$ and $C(v_3) = \{3, 4, 5\}$. In the latter case, when $C(z_2) \neq \{2, 4\}$, recolor vz with 4 and vx with 1. Otherwise, if $C(z_2) = \{2, 4\}$, recolor vz with 1 and vx with 4.

Now if $C(y_2) \neq \{3, 5\}$, suppose $C(y_1) \neq \{1, 3, 5\}$, then we recolor uy with 5 and color ux with 3 or 4 such that u does not conflict with u_3. If $C(y_1) = \{1, 3, 5\}$ and $C(u_3) \neq \{1, 4, 5\}$, then we recolor uy with 5 and color ux 4. Finally, assume that $C(y_1) = \{1, 3, 5\}$ and $C(u_3) = \{1, 4, 5\}$. If we can recolor xv with 1 and color ux with 5, or recolor xv with 4 and color ux with 5, we are done. Otherwise, it follows that $C(v_2) = \{1, 3, 5\}$ and $C(v_3) = \{3, 4, 5\}$ (up to symmetry), and henceforth when $C(z_2) \neq \{2, 4\}$, recolor vz with 4, vx with 1 and color ux with 5. Otherwise, if $C(z_2) = \{2, 4\}$, recolor vz with 1, vx with 4 and color ux with 5.

- Suppose that $C(y) \notin \{\{2, 3\}, \{2, 4\}\}$, and $C(y_1) \in \{\{1, 2, 3\}, \{1, 2, 4\}\}$, say $C(y_1) = \{1, 2, 3\}$ by symmetry. Then, we may assume that $C(u_2) = \{1, 2, 4\}$. If $C(u_3) = \{1, 3, 4\}$, then we recolor uy with 5 and color ux with 3. If $C(u_3) \neq \{1, 3, 4\}$, then we recolor uy with 4 and color ux with 3.

- Suppose that $C(y) \notin \{\{2, 3\}, \{2, 4\}\}$ and $C(y_1) \notin \{\{1, 2, 3\}, \{1, 2, 4\}\}$. Then, $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 3\}$. If $C(y_1) \neq \{3, 4, 5\}$, say $4 \notin C(y_1)$, then we recolor uy with 4 and color ux with 3. Otherwise, $C(y_1) = \{3, 4, 5\}$, say $\phi(yy_1) = 3$ and $\phi(y_1y_2) = 4$. When $C(y_2) = \{3, 4\}$, we recolor uy with 5 and color ux with 3. When $C(y_2) \neq \{3, 4\}$, we recolor uy with 4 and color ux with 3.

(3.2.3) $\phi(vx) \in \{3, 4, 5\}$, say $\phi(vx) = 3$ by symmetry. We first observe that if $3 \notin C(y_1)$, then it suffices to recolor uy with 3 and reduce the proof to Case (3.2.2). So, assume that $3 \in C(y_1)$ and let us discuss the following two cases.

- $3 \notin C(z)$ or $C(z) \in \{\{1, 3\}, \{2, 3\}\}$. Without loss of generality, assume that $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 5\}$. If $C(y_1) \neq \{3, 4, 5\}$, say $4 \notin C(y_1)$, then we recolor uy with 4 and color ux with 5. Otherwise, $C(y_1) = \{3, 4, 5\}$, say

$\phi(yy_1) = 3$ and $\phi(y_1y_2) = 4$. When $C(y_2) = \{3, 4\}$, we recolor uy with 5 and color ux with 4. When $C(y_2) \neq \{3, 4\}$, we recolor uy with 4 and color ux with 5.

• $3 \in C(z)$ and $C(z) \notin \{\{1, 3\}, \{2, 3\}\}$, say $C(z) = \{3, 5\}$. Then, at least one of u_2 and u_3, say u_2, has color set $\{1, 2, 4\}$. Since $\phi(vv_1) \in \{1, 2, 4\}$, we have some subcases below.

Assume that $\phi(vv_1) = 1$ (if $\phi(vv_1) = 2$, we have a similar discussion). If we can recolor vx with 4 and color ux with 3 or 5, we are done. If vx can be recolored with 4, but neither 3 nor 5 can assign to ux, then this implies that $C(u_3) = \{1, 2, 5\}$ and $C(y_1) = \{1, 2, 3\}$, say. It suffices to recolor uy with 5 and color ux with 3. If vx cannot be recolored with 4, then at least one of v_2 and v_3, say v_3, has color set $\{1, 4, 5\}$. If $C(v_2) \neq \{1, 2, 5\}$, then we recolor vx with 2 and then reduce to Case (3.2.2). So assume that $C(v_2) = \{1, 2, 5\}$. If $C(z_2) \neq \{2, 3\}$, then we recolor zv with 2 and reduce to the previous case. If $C(z_2) = \{2, 3\}$, then we recolor zv with 4 and vx with 2 and then reduce to Case (3.2.2).

Assume that $\phi(vv_1) = 4$. If we can recolor vx with 1 or 2, then the proof is reduced to Cases (3.2.1) and (3.2.2). Otherwise, we may assume that $C(v_2) = \{1, 4, 5\}$ and $C(v_3) = \{2, 4, 5\}$. If $C(z_2) = \{2, 3\}$, then we recolor zv with 1 and reduce to the previous case. If $C(z_2) \neq \{2, 3\}$, then we recolor zv with 2 and reduce the previous cases.

Case 2. At least one of x, y, and z is a 3-vertex in H.

All notations in Case 1 are kept in the following discussion. Since $2 \leq d_H(x) \leq 3$, we need to consider two subcases.

(1) Assume that $d_H(x) = 2$. Then, at least one of y and z, say z, is a 3-vertex in H. Let $N_H(z) = \{v, z_1, z_4\}$ with $d_H(z_4) = 1$. Consider the graph $G = H - zz_4$, which has a 5-2DVDE-coloring ϕ using the color set C such that $\phi(zz_1) = 1$ and $\phi(zv) = 2$. Assume that zz_4 cannot be colored with any color in C. If z_2 is a 2-vertex, then zz_4 can be colored with a color in $\{3, 4, 5\} - \{\phi(vv_1), \phi(z_1z_3)\}$ such that z does not conflict with any of v_1 and z_3. So, z_2 and z_3 must be 3-vertices in G, and we may assume that $C(v_1) = \{1, 2, 3\}$, $C(z_2) = \{1, 2, 4\}$, and $C(z_3) = \{1, 2, 5\}$. If $C(z_1) = \{1, 4, 5\}$, then we recolor zz_1 with 3 and color zz_4 with 5. If $C(z_1) \in \{\{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(z_1) = \{1, 2, 4\}$, then $\phi(vv_1) \in \{1, 3\}$, we deal with two possibilities according to the color of vv_1.

• $\phi(vv_1) = 1$. Let $\phi(v_1v_3) = 3$, so $\phi(vx) \in \{3, 4, 5\}$. If $\phi(vx) = 3$, then we can recolor vz with 4 or 5, and then color zz_4 with 3. Otherwise, it is easy to derive that $C(v_3) = \{1, 3, 4\}$ and $C(u) = \{1, 3, 5\}$, say. It suffices to recolor xv with 2, vz with 3, and color zz_4 with 4.

If $\phi(vx) = 4$ or 5, we have a similar proof.

• $\phi(vv_1) = 3$. Then, $\phi(vx) \in \{1, 4, 5\}$. First assume that $\phi(vx) = 1$. If we can recolor vz with 4 or 5, and color zz_4 with 3, we are done. Otherwise, it follows that $C(v_3) = \{1, 3, 4\}$ and $C(u) = \{1, 3, 5\}$, say. Recolor xv with 4, vz with 5, and color zz_4 with 3. Next assume that $\phi(vx) \in \{4, 5\}$, say $\phi(vx) = 4$. If we can recolor vz with 5, then 3 is assigned to zz_4. Otherwise, we have $C(u) = \{3, 4, 5\}$. It suffices to exchange the colors of vx and vz and color zz_4 with 5.

(2) Assume that $d_H(x) = 3$. Let $N_H(x) = \{u, v, x_1\}$ with $d_H(x_1) = 1$. Let $G = H - xx_1$, which has a 5-2DVDE-coloring ϕ with $\phi(xv) = 1$ and $\phi(xu) = 2$. Assume that xx_1 cannot be colored with any color in C. If $d_G(y) = d_G(z) = 2$, then

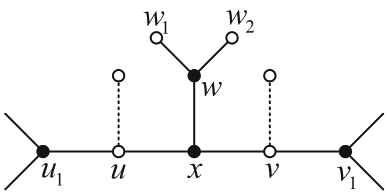


Fig. 3 The configuration in the proof of Claim 5

we color xx_1 with a color in $\{3, 4, 5\}$ such that x does not conflict with u_1 and v_1. So suppose that $d_G(z) = 3$. Without loss of generality, assume that $C(z) = \{1, 2, 3\}$ and $C(v_1) = \{1, 2, 4\}$. Note that either y or u_1 has color set $\{1, 2, 5\}$, say $C(y) = \{1, 2, 5\}$ by symmetry.

If $C(v) = \{1, 3, 4\}$, then we recolor xv with 5 and color xx_1 with 3 or 4 such that x does not conflict with u_1. Otherwise, suppose that $C(v) = \{1, 2, 4\}$ with $\phi(v_1v_2) = 1$ by symmetry. If vx can be recolored with 3, then we color xx_1 with 4 or 5 such that x does not conflict with u_1. Similarly, if vx can be recolored with 5, then we color xx_1 with 3 or 4 such that x does not conflict with u_1. Otherwise, we may assume that $C(z_1) = \{2, 3, 4\}$ and $C(v_3) = \{2, 4, 5\}$. If $C(v_2) = \{1, 3, 4\}$, then we exchange the colors of zz_1 and zz_4, recolor vx with 5, and color xx_1 with 3 or 4 such that x does not conflict with u_1. If $C(v_2) \neq \{1, 3, 4\}$, then we exchange the colors of zz_1 and zz_4, recolor vx with 3, and color xx_1 with 4 or 5 such that x does not conflict with u_1. □

5 Proof of Claim 5

Assume to the contrary that H' contains a 3_2-vertex x adjacent to two 2-vertices u and v (see Fig. 3). Let $N_{H'}(x) = \{u, v, w\}$, $N_{H'}(w) = \{x, w_1, w_2\}$, $N_{H'}(u) = \{x, u_1\}$ and $N_{H'}(v) = \{x, v_1\}$. By Claims 1 and 2, $d_{H'}(u_1) = d_{H'}(v_1) = 3$. Furthermore, let $N_{H'}(u_1) = \{u, u_2, u_3\}$ and $N_{H'}(v_1) = \{v, v_3, v_4\}$. By Claims 3 and 4, $d_{H'}(u_2) = d_{H'}(u_3) = d_{H'}(v_3) = d_{H'}(v_4) = 3$. We deal with three cases depending on the degree of u and v in H.

Case 1. $d_H(u) = d_H(v) = 2$.

Let $G = H - xu$, which admits a 5-2DVDE-coloring ϕ using the color set C with $\phi(xv) = 2$ and $\phi(xw) = 1$. Assume that ux cannot be colored with any color in C. If $d_G(w_1) = d_G(w_2) = 2$, then we can color ux with a color in $\{3, 4, 5\} - \{\phi(vv_1), \phi(uu_1)\}$ such that x does not conflict with v_1. This is impossible. Thus, $d_G(w_2) = 3$. We discuss three possibilities depending on the color of uu_1.

(1) $\phi(uu_1) = 1$. Suppose that $2 \notin C(u_1)$, then $d_G(w_1) = 3$, otherwise we color ux with a color in $\{3, 4, 5\} - \{\phi(vv_1), \phi(ww_2)\}$ such that x does not conflict with v_1 or w_2. Without loss of generality, assume that $C(w_1) = \{1, 2, 3\}$, $C(w_2) = \{1, 2, 4\}$, and $C(v_1) = \{1, 2, 5\}$. We recolor vx with 4 and color ux with 3 or 5 such that x does not conflict with u_1. If $2 \in C(u_1)$, then we suppose by symmetry that $C(u_1) = \{1, 2, 3\}$. We first assume that $C(v_1) \notin \{\{1, 2, 4\}, \{1, 2, 5\}\}$, then $C(w_1) = \{1, 2, 4\}$ and $C(w_2) = \{1, 2, 5\}$. Since $d_{H'}(v_3) = d_{H'}(v_4) = 3$, if $C(v_1) \neq \{1, 3, 4\}$, say $3 \notin C(v_1)$, then recolor vx with 3 and color ux with 4. Otherwise, $C(v_1) = \{1, 3, 4\}$, recolor vx with 5 and color ux with 4. we recolor vx with a color $c \in \{3, 4, 5\} - \{\phi(vv_1)\}$, and color ux with a color in $\{3, 4, 5\} - \{c\}$. Now if $C(v_1) \in \{\{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(v_1) = \{1, 2, 4\}$, then $C(w_2) = \{1, 2, 5\}$, recolor vx with 5 and color ux with 3 or 4 such that x does not conflict with w_1.

(2) $\phi(uu_1) = 2$. If $1 \notin C(u_1)$, then $d_G(w_1) = 3$, otherwise we color ux with a color in $\{3, 4, 5\} - \{\phi(vv_1), \phi(ww_2)\}$ such that x does not conflict with v_1 or w_2. Without loss of generality, assume that $C(w_1) = \{1, 2, 3\}$, $C(w_2) = \{1, 2, 4\}$, and either $C(v_1) = \{1, 2, 5\}$ or $\phi(vv_1) = 5$. We recolor vx with 4 and color ux with 3. If $1 \in C(u_1)$, we proceed in a similar way as for the previous case when $2 \in C(u_1)$.

(3) $\phi(uu_1) \in \{3, 4, 5\}$, say $\phi(uu_1) = 3$ by symmetry. If $\phi(vv_1) \neq 3$, it suffices to recolor vx with 3 an obtain a situation similar to (2). Thus, suppose that $\phi(vv_1) = 3$, then $d_G(w_1) = 3$, otherwise we color ux with a color in $\{4, 5\} - \{\phi(ww_2)\}$ such that x does not conflict with w_2. Furthermore, $C(w_1) = \{1, 2, 4\}$ and $C(w_2) = \{1, 2, 5\}$. It suffices to recolor vx with 4 and color ux with 5.

Case 2. $d_H(u) = 3$ and $d_H(v) = 2$.

Set $N_H(u) = \{x, u_1, u_4\}$ with $d_H(u_4) = 1$. Let $G = H - u_4$, which admits a 5-2DVDE-coloring ϕ using the color set C such that $\phi(ux) = 1$ and $\phi(uu_1) = 2$. Assume that uu_4 cannot be colored with any color in C. By symmetry, we suppose that $C(w) = \{1, 2, 3\}$, $C(u_2) = \{1, 2, 4\}$, and $C(u_3) = \{1, 2, 5\}$. If $C(u_1) = \{2, 4, 5\}$, then we recolor uu_1 with 3 and color uu_4 with 4. Otherwise, we assume that $C(u_1) = \{1, 2, 5\}$ by symmetry. Since $\phi(xw) \in \{2, 3\}$, we need to consider two possibilities as follows.

- $\phi(xw) = 2$. Note that $\phi(vx) \in \{3, 4, 5\}$, say $\phi(vx) = 3$ (if $\phi(vx) \in \{4, 5\}$, we will have a similar proof). If we can legally recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 4, we are done. Otherwise, we may assume that $C(v_1) = \{2, 3, 5\}$ and $C(w_1) = \{2, 3, 4\}$. It suffice to recolor vx with 4, ux with 5 and color uu_4 with 3.
- $\phi(xw) = 3$. Then, $\phi(vx) \in \{2, 4, 5\}$. First suppose that $\phi(vx) = 2$. If we can legally recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 4, we are done. Otherwise, it is easy to see that at least one of w_1 and w_2 is of degree 3, say $d_G(w_1) = 3$, and $C(w_1) = \{2, 3, 4\}$ and $C(v_1) = \{2, 3, 5\}$. We recolor vx with 4, ux with 5 and color uu_4 with 3. If $\phi(vx) \in \{4, 5\}$, we assume that $\phi(vx) = 4$ by symmetry. If possible, we recolor ux with 5 and color uu_4 with 4. Otherwise, assume that $C(v_1) = \{3, 4, 5\}$, recolor vx with 1, ux with 4, and color uu_4 with 5.

Case 3. $d_H(u) = d_H(v) = 3$.

We continue to use notations in Case 2 and let $N_H(v) = \{x, v_1, v_2\}$ with $d_H(v_2) = 1$. Then, $G = H - u_4$ has a 5-2DVDE-coloring ϕ such that $\phi(ux) = 1$ and $\phi(uu_1) = 2$. Assume that uu_4 cannot be colored with any color in C. We discuss the following possibilities according to the color set of v.

(1) $C(v) \notin \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$. Assume that $C(u_2) = \{1, 2, 4\}$, $C(u_3) = \{1, 2, 5\}$, and $C(w) = \{1, 2, 3\}$. If $C(u_1) = \{2, 4, 5\}$, we recolor uu_1 with 3 and colors uu_4 with 4 or 5 such that u does not conflict with v. Otherwise, assume that $C(u_1) = \{1, 2, 5\}$ by symmetry. Noting that $\phi(xw) \in \{2, 3\}$, we discuss two subcases below.

(1.1) Assume that $\phi(xw) = 2$, then $\phi(vx) \in \{3, 4, 5\}$. First suppose that $\phi(vx) = 3$. If possible, we recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 4. Otherwise, we assume that $C(v_1) = \{2, 3, 4\}$ and $C(w_1) = \{2, 3, 5\}$. There are two possibilities to be considered.

- $\phi(vv_1) = 4$. Then, $\phi(vv_2) \in \{1, 2, 5\}$. If $\phi(vv_2) = 1$, then we recolor vx with 5, ux with 4, and color uu_4 with 3. If $\phi(vv_2) = 2$, then we recolor vv_2 with 5, vx with 1, ux with 3, and color uu_4 with 5. If $\phi(vv_2) = 5$, then we recolor vx with 1, ux with 3, and color uu_4 with 5.
- $\phi(vv_1) = 2$. Then, $\phi(vv_2) \in \{4, 5\}$. If $\phi(vv_2) = 4$, then we recolor vv_2 with 3, vx with 4, ux with 5, and color uu_4 with 3. If $\phi(vv_2) = 5$, then we recolor vv_2 with 3, vx with 5, ux with 4, and color uu_4 with 3. The cases $\phi(vx) = 4$ and $\phi(vx) = 5$ are symmetric and are solve in a similar way as the one of $\phi(vx) = 3$.

(1.2) Assume that $\phi(xw) = 3$. Since $\phi(vx) \in \{2, 4, 5\}$, we investigate two situations as follows.

(1.2.1) $\phi(vx) = 2$. If we can legally recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 4, we are done. Otherwise, it follows that $C(v_1) = \{2, 3, 5\}$ and $C(w_2) = \{2, 3, 4\}$, say. Note that $\phi(vv_1) \in \{3, 5\}$.

- $\phi(vv_1) = 5$. Then, $\phi(vv_2) \in \{1, 3, 4\}$. If $\phi(vv_2) = 1$, then we recolor vx with 4, ux with 5, and color uu_4 with 3. If $\phi(vv_2) = 3$, then we recolor vv_2 with 1, vx with 4, ux with 5, and color uu_4 with 3. If $\phi(vv_2) = 4$, then we recolor vv_2 with 2, vx with 4, ux with 5, and color uu_4 with 3.
- $\phi(vv_1) = 3$. Then, $\phi(vv_2) \in \{4, 5\}$. If $\phi(vv_2) = 4$, then we recolor vv_2 with 2, vx with 4, ux with 5, and color uu_4 with 4. If $\phi(vv_2) = 5$, then we recolor vv_2 with 2, vx with 5, ux with 4, and color uu_4 with 3.

(1.2.2) $\phi(vx) \in \{4, 5\}$, say $\phi(vx) = 4$. If we can legally recolor ux with 5 and color uu_4 with 3, we are done. Otherwise, assume that $C(v_1) = \{3, 4, 5\}$. Note that $\phi(vv_1) \in \{3, 5\}$. Suppose that $\phi(vv_1) = 3$, then $\phi(vv_2) \in \{1, 2, 5\}$. If $\phi(vv_2) = 1$, then we recolor vv_2 with 4, vx with 1, ux with 4, and color uu_4 with 5. If $\phi(vv_2) = 2$, then we recolor vx with 1, ux with 4, and color uu_4 with 5. If $\phi(vv_2) = 5$, then we recolor vv_2 with 2, vx with 1, ux with 4, and color uu_4 with 5. The case $\phi(vv_1) = 5$ is solved using a similar recoloring strategy.

(2) $C(v) \in \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(v) = \{1, 2, 3\}$. The proof is split into the following two subcases, depending on the color set of w.

(2.1) $C(w) \notin \{\{1, 2, 4\}, \{1, 2, 5\}\}$. Then, we can assume that $C(u_2) = \{1, 2, 4\}$ and $C(u_3) = \{1, 2, 5\}$. Since $\phi(vx) \in \{2, 3\}$, we have two possibilities.

(2.1.1) $\phi(vx) = 2$. It is straightforward to see that $\phi(wx) \in \{3, 4, 5\}$.

• $\phi(wx) = 3$. If we can legally recolor ux with 4 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 4, we are done. Otherwise, we have two possibilities as follows:

Suppose that $C(v_1) \in \{\{2, 3, 4\}, \{2, 3, 5\}\}$, say $C(v_1) = \{2, 3, 4\}$. Let $C(w_2) = \{2, 3, 5\}$. If $C(w_1) \neq \{3, 4, 5\}$, then we recolor vv_2 with 1, vx with 5, ux with 4, and color uu_4 with 5. If $C(w_1) = \{3, 4, 5\}$, then we recolor vv_2 with 5, vx with 1, ux with 5, and color uu_4 with 4.

Suppose that $C(v_1) \notin \{\{2, 3, 4\}, \{2, 3, 5\}\}$, then $C(w_1) = \{2, 3, 4\}$ and $C(w_2) = \{2, 3, 5\}$. If $C(v_1) \neq \{1, 3, 4\}$, then we recolor vv_2 with 2, vx with 1, ux with 4, and color uu_4 with 5. If $C(v_1) = \{1, 3, 4\}$, then we recolor vv_2 with 2, vx with 5, ux with 4, and color uu_4 with 3 or 5 such that u does not conflict with v.

• $\phi(wx) \in \{4, 5\}$, say $\phi(wx) = 4$. If we can legally recolor ux with 3 and color uu_4 with 5, or recolor ux with 5 and color uu_4 with 3, we are done. Otherwise, we have two possibilities: (i) $C(v_1) \neq \{2, 3, 4\}$. We may assume that $C(w_1) = \{2, 4, 5\}$ and $C(w_2) = \{2, 3, 4\}$. It suffices to exchange the colors of vv_2 and vx, recolor ux with 5, and color uu_4 with 3. (ii) $C(v_1) = \{2, 3, 4\}$, then $C(w_2) = \{2, 4, 5\}$. It suffices to exchange the colors of vv_2 and vx, recolor ux with 3 or 5 such that x does not conflict with w_1, and color uu_4 with 4.

(2.1.2) $\phi(vx) = 3$. Then, $\phi(wx) \in \{2, 4, 5\}$.

First assume that $\phi(wx) = 2$. If we can legally recolor ux with 5, and color uu_4 with 3 or 4 such that u does not conflict with w, or recolor ux with 4 and color uu_4 with 3 or 5 such that u does not conflict with w, we are done. Otherwise, we have two possibilities: If $C(v_1) \in \{\{2, 3, 4\}, \{2, 3, 5\}\}$, say $C(v_1) = \{2, 3, 4\}$, then we may assume that $C(w_1) = \{2, 3, 5\}$. Now, if $C(w_2) \neq \{2, 4, 5\}$, then we recolor vx with 5, ux with 4, and color uu_4 with 3 or 5 such that u does not conflict with w. If $C(w_2) = \{2, 4, 5\}$, then we exchange the colors of vv_2 and vx, recolor ux with 4, and color uu_4 with 3 or 5 such that u does not conflict with w. So suppose that $C(v_1) \notin \{\{2, 3, 4\}, \{2, 3, 5\}\}$. We may assume that $C(w_1) = \{2, 3, 5\}$ and $C(w_2) = \{2, 3, 4\}$. Recolor vv_2 with 3, vx with 1, ux with 3 and color uu_4 with 4 or 5 such that u does not conflict with w.

Next assume that $\phi(wx) \in \{4, 5\}$, say $\phi(wx) = 4$. If possible, we recolor ux with 5 and color uu_4 with 3. Otherwise, we may assume that $C(w_1) = \{3, 4, 5\}$, and furthermore $\phi(vv_2) = 1$ (if $\phi(vv_2) = 2$, we have a similar discussion). If $C(w_2) \neq \{1, 3, 4\}$, then we recolor vv_2 with 3, vx with 1, ux with 3, and color uu_4 with 5. If $C(w_2) = \{1, 3, 4\}$, then we recolor vv_2 with 3, vx with 1, ux with 5, and color uu_4 with 3.

(2.2) $C(w) \in \{\{1, 2, 4\}, \{1, 2, 5\}\}$, say $C(w) = \{1, 2, 4\}$. Then, we may suppose that $C(u_2) = \{1, 2, 5\}$. Note that $\phi(vx) \in \{2, 3\}$.

(2.2.1) Let $\phi(vx) = 2$. Then, $\phi(wx) = 4$. If we can recolor ux with 3 and color uu_4 with 4 or 5 such that u does not conflict with u_3, or recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3, we are done. Otherwise, we may assume that $C(v_1) = \{2, 3, 4\}$ and $C(w_2) = \{2, 4, 5\}$. When $C(w_1) = \{1, 4, 5\}$, we recolor vv_2 with 2, vx with 1, ux with 3, and color uu_4 with 4 or 5 such that u does not conflict with u_3. When $C(w_2) \neq \{1, 4, 5\}$, we recolor vv_2 with 2, vx with 1, ux with 5, and color uu_4 with 3 or 4 such that u does not conflict with u_3.

(2.2.2) Let $\phi(vx) = 3$. Then, $\phi(wx) \in \{2, 4\}$. If $\phi(wx) = 4$, then we recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3. So assume that $\phi(wx) = 2$. If we can legally recolor ux with 4 and color uu_4 with 3 or 5 such that u does not conflict with u_3, or recolor ux with 5 and color uu_4 with 3 or 4 such that u does not conflict with u_3, we are done. Otherwise, we may assume that $C(v_1) = \{2, 3, 5\}$ and $C(w_2) = \{2, 3, 4\}$. Recolor vv_2 with 3, vx with 1, ux with 3, and color uu_4 with 4 or 5 such that u does not conflict with u_3. □

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