Bull. Malays. Math. Sci. Soc. (2020) 43:3143–3158
https://doi.org/10.1007/s40840-019-00859-w

BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

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The Third-Order Hermitian Toeplitz Determinant for Classes of Functions Convex in One Direction

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Received: 7 May 2019 / Revised: 23 September 2019 / Published online: 19 November 2019

Abstract
In this paper, the sharp bounds for the third Hermitian Toeplitz determinant over classes of functions convex in the direction of the imaginary axis and convex in the direction of the positive real axis are computed.

Keywords Hermitian Toeplitz matrix · Univalent functions · Functions convex in the direction of the imaginary axis · Functions convex in the direction of the positive real axis · Carathéodory class

Mathematics Subject Classification 30C45 · 30C50

Communicated by V. Ravichandran.

The fourth author was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korea government (MSIP; Ministry of Science, ICT & Future Planning) (No. NRF-2017R1C1B5076778).

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the subclass normalized by $f(0) := 0$, $f'(0) := 1$, that is, functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \ z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions.

In this paper, we estimate the Hermitian Toeplitz determinants for functions convex in the direction of the imaginary axis and convex in the direction of the positive real axis. Hermitian Toeplitz matrices play an important role in the applied mathematics as well as in technical sciences, e.g., in the Szegö theory the stochastic filtering, the signal processing, the biological information processing and other engineering problems.

Given $q, n \in \mathbb{N}$, the Hermitian Toeplitz matrix $T_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1) is defined by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{bmatrix},$$

where $\overline{a}_k := \overline{a_k}$. Let $|T_{q,n}(f)|$ denote the determinant of $T_{q,n}(f)$.

Recently, Ali et al. [1] introduced the concept of the symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$ in the following way:

$$T_q(n)[f] := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix},$$

They found a number of estimates for $T_2(n)$, $T_3(1)$, $T_3(2)$ and $T_2(3)$ over selected subclasses of $\mathcal{A}$.

In recent years, a lot of papers have been devoted to the estimation of determinants built on coefficients of functions in the class $\mathcal{A}$ or its subclasses. Hankel matrices, i.e., square matrices which have constant entries along the reverse diagonal and the generalized Zalcman functional $J_{m,n}(f) := a_{m+n-1} - a_m a_n$, $m, n \in \mathbb{N}$, are of particular interest. From the large number of papers in this direction, we recall [2,3,5–7,15,16,19–21,26–28,30,33], where the second- and third-order Hankel determinants over selected subclasses of $\mathcal{A}$ have been studied. Some of these papers and many others concern also the generalized Zalcman functional, particularly the functional $J_{2,3}(f)$.

Being in interest in this research topic in [11], the study of the Hermitian Toeplitz determinants on classes of analytic normalized functions has been initiated. In this paper, we compute the second and third Toeplitz determinants over class of functions convex in the imaginary axis and the class of functions convex in the positive direction of the real axis.

Let us recall some properties of the Toeplitz determinant $|T_{q,1}(f)|$ (see [11]).

- For each $\theta \in \mathbb{R}$, $|T_{q,1}(f)| = |T_{q,1}(f_\theta)|$, where $f_\theta(z) := \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z)$, $z \in \mathbb{D}$, i.e., $|T_{q,1}(f)|$ is rotation invariant.
- Since $a_1 = 1$ is a real number, $T_{q,1}(f)$ is a Hermitian matrix, i.e., $T_{q,1}(f) = \overline{T^{\mathrm{T}}_{q,1}(f)} =: T^*$, where $\overline{T^{\mathrm{T}}_{q,1}(f)}$ is the conjugate transpose matrix of $T_{q,1}(f)$.
- Since $|T_{q,1}(f)|$ for $f \in \mathcal{A}$ is a determinant of Hermitian matrix, it is a real number.

Given a subclass $\mathcal{F}$ of $\mathcal{A}$, let $A_2(\mathcal{F}) := \max\{|a_2| : f \in \mathcal{F}\}$ if exists. Since for $f \in \mathcal{A}$,

$$|T_{2,1}(f)| = 1 - |a_2|^2,$$

we get the result below. The equality for the lower bound is attained by a function in $\mathcal{F}$ which is extremal for $A_2(\mathcal{F})$. The identity makes equality for the upper bound.

Theorem 1 *Let $\mathcal{F}$ be a subclass of $\mathcal{A}$ and $A_2(\mathcal{F})$ exists. If the identity is an element of $\mathcal{F}$, then*

$$1 - A_2^2(\mathcal{F}) \le |T_{2,1}(f)| \le 1.$$

Both inequalities are sharp.

By $\mathcal{CV}(\mathrm{i})$ and $\mathcal{CV}(1)$, we denote the subclasses of $\mathcal{A}$ of functions f which satisfy

$$\mathrm{Re}\{(1-z^2)f'(z)\} > 0, \quad z \in \mathbb{D}, \tag{2}$$

and

$$\mathrm{Re}\{(1-z)^2 f'(z)\} > 0, \quad z \in \mathbb{D}, \tag{3}$$

respectively. Both classes play an important role in the geometric function theory in view of their geometrical properties. Each function $f \in \mathcal{CV}(\mathrm{i})$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ convex in the direction of the imaginary axis, i.e., for $w_1, w_2 \in f(\mathbb{D})$ such that $\mathrm{Re}\, w_1 = \mathrm{Re}\, w_2$ the line segment $[w_1, w_2]$ lies in $f(\mathbb{D})$, with the additional property that there exist two points ω_1, ω_2 on the boundary of $f(\mathbb{D})$ for which $\{\omega_1 + \mathrm{i}t : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D})$ and $\{\omega_2 - \mathrm{i}t : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D})$ (see, e.g., [13, p. 199]). In fact, the class $\mathcal{CV}(\mathrm{i})$ is the subclass of the class $\mathcal{CV}$ of functions convex in the direction of the imaginary axis which was introduced by Robertson [31] in 1936. Robertson's analytic condition for the class $\mathcal{CV}$ was shown by him under some regularity of functions in $\mathcal{CV}$ on the unit circle. The proof of Robertson's conjecture for the whole class $\mathcal{CV}$ was finally completed by Hengartner and Schober [14] who divided the class $\mathcal{CV}$ into three subclasses with the class $\mathcal{CV}(\mathrm{i})$ as one of them (see also [13, pp. 193–206]).

Each function in the class $\mathcal{CV}(1)$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ called convex in the positive direction of the real axis, i.e., $\{w + it : t \geq 0\} \subset f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ [4,8–10,12,24,25].

The condition (3) was generalized by replacing the polynomial $1 - z^2$ by quadratic polynomials [22,23] and by any polynomials having their roots in $\mathbb{C} \setminus \mathbb{D}$ [17,18].

In this paper, we compute sharp lower and upper bounds for

$$|T_{3,1}(f)| = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 2\,\mathrm{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2 + 1. \tag{4}$$

over the classes $\mathcal{CV}(\mathrm{i})$ and $\mathcal{CV}(1)$.

Let $\mathcal{P}$ be the class of all $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{5}$$

having a positive real part in $\mathbb{D}$.

In the proof of the main result, we will use the following lemma which contains the well-known formula for c_2 (see, e.g., [29, p. 166]) and further remarks in [7]).

Lemma 1 *If $p \in \mathcal{P}$ is of the form (5), then*

$$c_1 = 2\zeta_1 \tag{6}$$

and

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{7}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (6), namely

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (6) and (7), namely

$$p(z) = \frac{1 + (\overline{\zeta}_1\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\overline{\zeta}_1\zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{8}$$

2 Functions Convex in the Direction of the Imaginary Axis

Since $A_2(\mathcal{CV}(\mathrm{i})) = 1$ ([14], see also [13, Vol. I, pp. 200–201]) with the extremal function

$$f(z) = \frac{z}{1-z}, \quad z \in \mathbb{D},$$

and since the identity belongs to $\mathcal{CV}(\mathrm{i})$, by Theorem 1, we have

Theorem 2 *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{CV}(\mathrm{i})$, *then*

$$0 \le |T_{2,1}(f)| \le 1.$$

Both inequalities are sharp.

Now, we will compute the bounds of $|T_{3,1}(f)|$.

Theorem 3 *If* $f \in \mathcal{CV}(\mathrm{i})$, *then*

$$|T_{3,1}(f)| \le 1. \tag{9}$$

The inequality is sharp.

Proof Let $f \in \mathcal{CV}(\mathrm{i})$ be the form (1). Since $|a_2| \le 1$, $|a_3| \le 1$ ([14], see also [13, Vol. I, pp. 200–201]) and $\operatorname{Re}\left(a_2^2\overline{a}_3\right) \le |a_2^2 a_3|$, from (4), we get

$$|T_{3,1}(f)| \le F(|a_2|, |a_3|), \tag{10}$$

where

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 1] \times [0, 1].$$

We have

$$\frac{\partial F}{\partial x} = -4x(1-y) \le 0, \quad (x, y) \in [0, 1] \times [0, 1].$$

Thus,

$$F(x, y) \le F(0, y) = 1 - y^2 \le 1, \quad (x, y) \in [0, 1] \times [0, 1],$$

which in view of (10) shows (9).

Clearly, the identity makes the inequality (9) sharp. □

Theorem 4 *If* $f \in \mathcal{CV}(\mathrm{i})$, *then*

$$|T_{3,1}(f)| \ge -\frac{1}{2}. \tag{11}$$

The inequality is sharp.

Proof By (2), there exists $p \in \mathcal{P}$ of the form (5) such that

$$(1-z^2)f'(z) = p(z), \quad z \in \mathbb{D}. \tag{12}$$

Putting the series (1) and (5) into (12) by equating the coefficients, we get

$$a_2 = \frac{1}{2}c_1, \qquad a_3 = \frac{1}{3}(c_2+1). \tag{13}$$

By (13), (6) and (7), we have

$$a_2 = \zeta_1, \qquad a_3 = \frac{1}{3}\left(1 + 2\zeta_1^2 + 2(1-|\zeta_1|^2)\zeta_2\right)$$

with $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2$. Therefore, from (4), we get

$$|T_{3,1}(f)| = \frac{1}{9}(\Psi_1 + \Psi_2), \tag{14}$$

where

$$\Psi_1 := 8 - 18|\zeta_1|^2 + 8|\zeta_1|^4 - 4(1-|\zeta_1|^2)^2|\zeta_2|^2$$

and

$$\Psi_2 := 2\operatorname{Re}\zeta_1^2 - 4(1-|\zeta_1|^2)\operatorname{Re}\zeta_2 + 4(1-|\zeta_1|^2)\operatorname{Re}(\zeta_1^2\overline{\zeta}_2).$$

A. When $\zeta_1 = 0$, then

$$|T_{3,1}(f)| = \frac{1}{9}\left(8 - 4|\zeta_2|^2 - 4\operatorname{Re}(\zeta_2)\right) \geq \frac{1}{9}\left(8 - 4|\zeta_2|^2 - 4|\zeta_2|\right) \geq 0.$$

When $\zeta_2 = 0$, then

$$\begin{aligned} |T_{3,1}(f)| &= \frac{1}{9}\left(8 - 18|\zeta_1|^2 + 8|\zeta_1|^4 + 2\operatorname{Re}(\zeta_1^2)\right) \\ &\geq \frac{4}{9}\left(2 - 5|\zeta_1|^2 + 2|\zeta_2|^4\right) \geq -\frac{4}{9}. \end{aligned}$$

B. Suppose that $\zeta_1, \zeta_2 \in \overline{\mathbb{D}} \setminus \{0\}$. Then, there exist unique θ and ψ in $[0, 2\pi)$ such that $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$, where $r := |\zeta_1| \in (0, 1]$ and $s := |\zeta_2| \in (0, 1]$. Thus,

$$\begin{aligned} \Psi_2 &= 2r^2\cos 2\theta - 4s(1-r^2)\cos\psi + 4r^2 s(1-r^2)\cos(2\theta - \psi) \\ &= 2r^2\sin(2\theta+\alpha)\sqrt{1 + 4s^2(1-r^2)^2 + 4s(1-r^2)\cos\psi} - 4s(1-r^2)\cos\psi, \end{aligned} \tag{15}$$

where $\alpha \in [0, 2\pi)$ is a unique quantity satisfying

$$\cos\alpha = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \qquad \sin\alpha = \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}} \tag{16}$$

with

$$\kappa_1 := 2s(1-r^2)\sin\psi, \qquad \kappa_2 := 1+2s(1-r^2)\cos\psi.$$

From (15), we have

$$\begin{aligned} -\Psi_2 &\le 2r^2\sqrt{1+4s^2(1-r^2)^2+4s(1-r^2)\cos\psi}+4s(1-r^2)\cos\psi \\ &\le 2r^2-4sr^4+4s. \end{aligned} \tag{17}$$

Therefore, by (14) and (17), we obtain

$$9|T_{3,1}(f)| = \Psi_1+\Psi_2 \ge F(r,s), \tag{18}$$

where

$$F(x,y) := 4(2-5x^2+2x^4)-4(1-x^4)y-4(1-x^2)^2y^2, \quad x,y \in (0,1].$$

Since

$$\frac{\partial F}{\partial y} = -4(1-x^2)\left(1+x^2+2(1-x^2)y\right) \le 0, \quad x,y \in (0,1],$$

we see that

$$F(x,y) \ge F(x,1) = 4x^2(-3+2x^2) \ge -\frac{9}{2}, \quad x,y \in (0,1].$$

Hence, by (18) and part A, it follows that the inequality (11) is true.

The inequality is sharp with the equality attained by the function

$$f(z) = \int_0^z \frac{1+t^2}{(1-t^2)(1-\mathrm{i}\sqrt{3}t-t^2)}\mathrm{d}t, \quad z \in \mathbb{D},$$

which belongs to $\mathcal{CV}(\mathrm{i})$ and for which $a_2 = \mathrm{i}\sqrt{3}/2$ and $a_3 = 0$. □

3 Functions Convex in the Positive Direction of the Real Axis

Since $A_2(\mathcal{CV}(1)) = 2$ [10] with the Koebe function

$$k(z) := \frac{z}{(1-z)^2}, \quad z \in \mathbb{D}, \tag{19}$$

as the extremal, and the identity belongs to $\mathcal{CV}(1)$, by Theorem 1, we have

Theorem 5 *If* $f \in \mathcal{CV}(1)$, *then*

$$-3 \leq |T_{2,1}(f)| \leq 1.$$

Both inequalities are sharp.

Now, we will compute the bounds of $|T_{3,1}(f)|$.

Theorem 6 *If* $f \in \mathcal{CV}(1)$, *then*

$$|T_{3,1}(f)| \leq 8. \tag{20}$$

The inequality is sharp.

Proof Let $f \in \mathcal{CV}(1)$ be the form (1). Since $|a_2| \leq 2$, $|a_3| \leq 3$ [10] and $\operatorname{Re}\left(a_2^2\overline{a}_3\right) \leq |a_2^2 a_3|$, from (4), we get

$$T_{3,1}(f) \leq F(|a_2|, |a_3|), \tag{21}$$

where

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 2] \times [0, 3].$$

Solving the system of equations $\partial F/\partial x = 0 = \partial F/\partial y$, we see that $(1, 1)$ is the unique critical point in $(0, 2) \times (0, 3)$. Since

$$\left(\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}\right)(1, 1) = -4 < 0,$$

F has a saddle point at $(1, 1)$. On the boundary of $[0, 2] \times [0, 3]$, we have

(1) $F(0, y) = 1 - y^2 \leq 1, \quad y \in [0, 3];$
(2) $F(2, y) = -7 + 8y - y^2 \leq 8, \quad y \in [0, 3];$
(3) $F(x, 0) = 1 - 2x^2 \leq 1, \quad x \in [0, 2];$
(4) $F(x, 3) = -8 + 4x^2 \leq 8, \quad x \in [0, 2].$

Hence and from (21), the inequality (20) follows.

Equality in (20) holds for the Koebe function k given by (19) for which $a_2 = 2$ and $a_3 = 3$. □

Theorem 7 *If* $f \in \mathcal{CV}(1)$, *then*

$$\begin{aligned} |T_{3,1}(f)| &\geq -\frac{1}{225}\left(-558 + 286\sqrt{5} + 19\sqrt{54 + 14\sqrt{5}} - 5\sqrt{270 + 70\sqrt{5}}\right) \\ &\approx -0.68328. \end{aligned} \tag{22}$$

The inequality is sharp.

Proof By (3), there exists $p \in \mathcal{P}$ of the form (5) such that

$$(1-z)^2 f'(z) = p(z), \quad z \in \mathbb{D}. \tag{23}$$

Putting the series (1) and (5) into (23) by equating the coefficients, we get

$$a_2 = \frac{1}{2}(2+c_1), \qquad a_3 = \frac{1}{3}(3+2c_1+c_2). \tag{24}$$

Substituting (6) and (7) into the equalities (24), we get

$$a_2 = 1+\zeta_1, \qquad a_3 = \frac{1}{3}(3+4\zeta_1+2\zeta_1^2+2(1-|\zeta_1|^2)\zeta_2)$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2$. Furthermore, from (4), we obtain

$$|T_{3,1}(f)| = \frac{1}{9}(\Psi_1+\Psi_2), \tag{25}$$

where

$$\Psi_1 := 14|\zeta_1|^2 + 8|\zeta_1|^4 - 4(1-|\zeta_1|^2)^2|\zeta_2|^2 \tag{26}$$

and

$$\begin{aligned}\Psi_2 := {} & 18\,\mathrm{Re}\,\zeta_1^2 + 32|\zeta_1|^2\,\mathrm{Re}\,\zeta_1 + 8(1-|\zeta_1|^2)\,\mathrm{Re}(\zeta_1\overline{\zeta}_2) \\ & + 4(1-|\zeta_1|^2)\,\mathrm{Re}(\zeta_1^2\overline{\zeta}_2).\end{aligned} \tag{27}$$

A. Suppose that $\zeta_1, \zeta_2 \in \overline{\mathbb{D}} \setminus \{0\}$. Then, there exist unique θ and ψ in $[0, 2\pi)$ such that $\zeta_1 = r\mathrm{e}^{\mathrm{i}\theta}$ and $\zeta_2 = s\mathrm{e}^{\mathrm{i}\psi}$, where $r := |\zeta_1| \in (0, 1]$ and $s := |\zeta_2| \in (0, 1]$. From (26) and (27), we, respectively, have

$$\Psi_1 \geq 14r^2 + 8r^4 - 4(1-r^2)^2 = 4r^4 + 22r^2 - 4 \tag{28}$$

and

$$\Psi_2 = 18r^2\cos 2\theta + 32r^3\cos\theta + 4(1-r^2)rs\sqrt{\kappa_1^2+\kappa_2^2}\sin(\psi+\alpha),$$

where $\alpha \in [0, 2\pi)$ is a unique quantity satisfying (16) with

$$\kappa_1 := 2\sin\theta + r\sin 2\theta, \qquad \kappa_2 := 2\cos\theta + r\cos 2\theta. \tag{29}$$

Hence,

$$\Psi_2 \geq 18r^2\cos 2\theta + 32r^3\cos\theta - 4(1-r^2)r\sqrt{4+4r\cos\theta+r^2}. \tag{30}$$

Let $\Omega := (0,1] \times [-1,1]$. From (25), (28) and (30), it follows that

$$|T_{3,1}(f)| \geq -\frac{4}{9} F(r, \cos\theta), \tag{31}$$

where

$$F(x,y) := 1 - x^2 - x^4 - 9x^2y^2 - 8x^3y + x(1-x^2)\sqrt{g(x,y)}, \quad (x,y) \in \Omega,$$

with

$$g(x,y) := 4 + 4xy + x^2, \quad (x,y) \in \Omega.$$

Let

$$\Theta := \frac{1}{100}\left(-558 + 286\sqrt{5} + 19\sqrt{54 + 14\sqrt{5}} - 5\sqrt{270 + 70\sqrt{5}}\right) = 1.53738\cdots.$$

Now, we will show that

$$\max\{F(x,y) : (x,y) \in \Omega\} = \Theta. \tag{32}$$

(A1) For this, we first find the critical points of F in the interior of Ω, i.e., in $(0,1) \times (-1,1)$. Note that in Int Ω, the equation

$$\frac{\partial F}{\partial y} = -18x^2y - 8x^3 + 2x^2(1-x^2)g(x,y)^{-1/2} = 0,$$

is equivalent to

$$g(x,y)^{-1/2} = \frac{9y + 4x}{1 - x^2}. \tag{33}$$

Furthermore, note that

$$9y + 4x \geq 0 \tag{34}$$

holds, since $g(x,y)^{1/2} \geq 0$ and $1 - x^2 > 0$. Under the condition (34), Eq. (33) can be written as

$$324xy^3 + (369x^2 + 324)y^2 + (136x^3 + 288x)y + 15x^4 + 66x^2 - 1 = 0. \tag{35}$$

The equation

$$\begin{aligned}\frac{\partial F}{\partial x} = &-2x - 4x^3 - 18xy^2 - 24x^2y + (1-3x^2)g(x,y)^{1/2} \\ &+ x(1-x^2)(2y+x)g(x,y)^{-1/2} = 0,\end{aligned}$$

is equivalent to

$$63x^2y^2 + 2(9x + 14x^3)y - (1 - 12x^2 + 3x^4) = 0. \tag{36}$$

Note that $\Delta = \Delta(x) := 144 - 504x^2 + 385x^4 \geq 0$ iff $x \in (0, x_1] \cup [x_2, 1)$, where

$$x_1 = \frac{2}{\sqrt{55}}\sqrt{9 - 6\sqrt{\frac{2}{7}}} \approx 0.64908, \qquad x_2 = \frac{2}{\sqrt{55}}\sqrt{9 + 6\sqrt{\frac{2}{7}}} \approx 0.94223.$$

Since $\Delta(x_1) = 0$, Eq. (36) has a unique root $y_0' = -(9+14x_1^2)/(63x_1) \approx -0.36433$. Analogously, $\Delta(x_2) = 0$, so Eq. (36) has a unique root $y_0'' = -(9 + 14x_2^2)/(63x_2) \approx -0.36100$. As easy to check, the polynomial in (35) does not vanish for $x = x_1,\ y = y_0'$ and for $x = x_2,\ y = y_0''$.

Let now $x \in (0, x_1) \cup (x_2, 1)$. Thus, there are two roots y_1 and y_2 of (36), namely

$$y_j = \frac{-(9 + 14x^2) + (-1)^j\sqrt{144 - 504x^2 + 385x^4}}{63x}, \quad j = 1, 2. \tag{37}$$

(1) Consider the case $y = y_1$. Note that $y_1 > -1$ is equivalent to

$$-9 + 63x - 14x^2 > \sqrt{144 - 504x^2 + 385x^4}. \tag{38}$$

We have $-9 + 63x - 14x^2 > 0$ iff $x \in (x_3, 1)$, where $x_3 = (63 - \sqrt{3465})/28 \approx 0.14771$. Thus, for $x \in (x_3, x_1) \cup (x_2, 1)$ by squaring the both sides of (38), we get the inequality

$$189x^4 + 1764x^3 - 4725x^2 + 1134x + 63 < 0$$

which is true for $x \in (x_4, x_1) \cup (x_2, 1)$, where $x_4 \approx 0.32137$. Moreover, $y_1 < 1$ is equivalent to the inequality

$$-9 - 63x - 14x^2 < \sqrt{144 - 504x^2 + 385x^4},$$

which is clearly true for $x \in (x_4, x_1) \cup (x_2, 1)$.

Substituting $y = y_1$ into Eq. (35), we get

$$Q_1(x)\sqrt{144 - 504x^2 + 385x^4} = Q_2(x), \tag{39}$$

where

$$\begin{aligned} Q_1(x) &:= 242028 - 234738x^2 - 204120x^4, \\ Q_2(x) &:= -3096792 + 9103752x^2 - 3419010x^4 - 3572100x^6. \end{aligned} \tag{40}$$

Since $Q_1(x) > 0$ for $x \in (x_4, x_1)$, $Q_1(x) < 0$ for $x \in (x_2, 1)$, $Q_2(x) < 0$ for $x \in (x_4, x_1)$ and $Q_2(x) > 0$ for $x \in (x_2, 1)$, Eq. (39) has no solution.

(2) Consider now the case $y = y_2$. Note that $y_2 > -1$ is equivalent to

$$\sqrt{144 - 504x^2 + 385x^4} > 9 - 63x + 14x^2. \tag{41}$$

Since $9 - 63x + 14x^2 < 0$ for $x \in (x_5, 1)$, where

$$x_5 = \frac{3}{4}\left(3 - \sqrt{\frac{55}{7}}\right) \approx 0.14771,$$

let us consider $x \in (0, x_5]$. By squaring both sides of (41) and grouping, we get the inequality

$$189x^4 + 1764x^3 - 4725x^2 + 1134x + 63 \geq 0,$$

which is true for $x \in (0, x_5]$. Thus, $y_2 > -1$ holds for all $x \in (0, x_1) \cup (x_2, 1)$. Moreover, $y_2 < 1$ is equivalent to the inequality

$$\sqrt{144 - 504x^2 + 385x^4} < 9 + 63x + 14x^2.$$

Since the right hand of the above inequality is positive, by squaring both sides and grouping, we get the inequality

$$189x^4 - 1764x^3 - 4725x^2 - 1134x + 63 > 0,$$

which is true for $x \in (x_6, x_1) \cup (x_2, 1)$, where $x_6 \approx 0.04624$. Thus, we now consider $x \in (x_6, x_1) \cup (x_2, 1)$.

Substituting $y = y_2$ into Eq. (35), we get

$$Q_1(x)\sqrt{144 - 504x^2 + 385x^4} = -Q_2(x), \tag{42}$$

where Q_1 and Q_2 are given by (40). Since $Q_1(x)Q_2(x) > 0$ for $x \in (x_6, x_1) \cup (x_2, 1)$, by squaring both sides of (42) and grouping, we equivalently get the equation

$$26248933872(x^2 - 2)(x^2 - 1)(5x^2 - 2)(25x^4 + 45x^2 - 11) = 0$$

which has two roots

$$\tilde{x}_1 = \sqrt{\frac{2}{5}} \approx 0.63246, \qquad \tilde{x}_2 = \sqrt{-\frac{9}{10} + \frac{\sqrt{5}}{2}} \approx 0.46694. \tag{43}$$

Substituting $x = \tilde{x}_1$ into y_1 given by (37), we get $\tilde{y}_1 = -1/\sqrt{10} \approx -0.31623$. But $9\tilde{y}_1 + 4\tilde{x}_1 < 0$ which contradicts (34). Therefore, $(\tilde{x}_1, \tilde{y}_1)$ is not a critical point of F in Int Ω.

Substituting $x = \tilde{x}_2$ into y_1 given by (37), we get

$$\tilde{y}_2 = -\frac{1}{220}\left(15\sqrt{2(-9+5\sqrt{5})} + \sqrt{10(-9+5\sqrt{5})}\right) = -0.163604\cdots \quad (44)$$

Since

$$9\tilde{y}_2 + 4\tilde{x}_2 = \frac{1}{220}(-135\sqrt{2} + 79\sqrt{10})\sqrt{-9+5\sqrt{5}} = 0.395333\cdots > 0,$$

$(\tilde{x}_2, \tilde{y}_2)$ satisfies (34), and thus, it is a unique critical point of F.

Denote

$$\lambda_1 := F_{xx}(\tilde{x}_2, \tilde{y}_2), \qquad \lambda_2 := F_{xy}(\tilde{x}_2, \tilde{y}_2), \qquad \lambda_3 := F_{yy}(\tilde{x}_2, \tilde{y}_2).$$

Numerical calculations yield

$$\lambda_1 = -6.74042\cdots < 0, \qquad \lambda_1\lambda_3 - \lambda_2^2 = 22.9037\cdots > 0.$$

Thus, F has a local maximum at $(\tilde{x}_2, \tilde{y}_2)$ with

$$F(\tilde{x}_2, \tilde{y}_2) = \Theta.$$

(A2) It remains to consider F in the boundary of Ω.

(1) On the side $x = 0$, we have $F(0, y) \equiv 1 < \Theta,\ y \in [-1, 1]$.
(2) On the side $x = 1$, we have

$$F(1, y) = -1 - 8y - 9y^2 \le F\left(1, -\frac{4}{9}\right) = \frac{7}{9} < \Theta, \quad y \in [-1, 1].$$

(3) On the side $y = -1$, we have

$$F(x, -1) \le F\left(\frac{1}{18}(11 - \sqrt{85}), -1\right) = \frac{1}{486}(-251 + 85\sqrt{85}) = 1.09601\cdots < \Theta, \quad x \in [0, 1].$$

(4) On the side $y = 1$, we have

$$F(x, 1) = 1 + 2x - 9x^2 - 10x^3 - 2x^4, \quad x \in [0, 1].$$

It is easy to see that

$$F(x, 1) \le \gamma(x) \le \gamma(x_7) = 1 + \frac{3}{2}x_7 = 1.43153\cdots < \Theta, \quad x \in [0, 1],$$

where $x_7 = 42^{-1/3} \approx 0.28769$ and

$$\gamma(x) := 1 + 2x - 21x^4, \quad x \in [0, 1].$$

B. When $\zeta_1 = 0$, then

$$T_{3,1}(f) = -\frac{4}{9}|\zeta_2|^2 \geq -\frac{4}{9}.$$

C. Let $\zeta_2 = 0$ and $\zeta_1 \neq 0$. Then,

$$\Psi_1 := 14|\zeta_1|^2 + 8|\zeta_1|^4, \quad \Psi_2 = 18\,\mathrm{Re}\,\zeta_1^2 + 32|\zeta_1|^2\,\mathrm{Re}\,\zeta_1.$$

Thus, taking $\zeta_1 = r\mathrm{e}^{\mathrm{i}\theta}$, where $r \in [0, 1]$ and $\theta \in [0, 2\pi)$, we have

$$\Psi_1 = 14r^2 + 8r^4, \quad \Psi_2 = 18r^2 \cos 2\theta + 32r^3 \cos\theta.$$

Since the inequalities (28) and (30) hold, further argumentation of part A remains valid.

Summarizing from parts A–C, it follows that $F(x, y) \leq \Theta$ holds for all $(x, y) \in [0, 1] \times [-1, 1]$. This together with (31) proves (22).

Now, we discuss the sharpness of (22). From (25), (28) and (30), that $|T_{3,1}(f)| = -(4/9)\Theta$ holds when the following conditions are satisfied:

$$x = \tilde{x}_2, \qquad \cos\theta = \tilde{y}_2, \qquad s = 1, \quad \sin(\psi + \alpha) = -1, \tag{45}$$

where $\tilde{x}_2$ and $\tilde{y}_2$ are given by (43) and (44), and where α is determined by the condition (16) with κ_1 and κ_2 given in (29). Set $\theta = \mathrm{Arccos}(\tilde{y}_2)$ so that it satisfies the second condition in (45). Then, we have $\kappa_1 = 1.82232\cdots > 0$ and $\kappa_2 = -0.76915\cdots < 0$. Thus, (16) is satisfied if we take α by

$$\alpha = -\,\mathrm{Arccos}\left(\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}\right) = -\,\mathrm{Arccos}\left(\frac{2\sin\theta + x_2 \sin 2\theta}{\sqrt{4 + 4x_2\cos\theta + x_2^2}}\right).$$

Thus, if we put

$$\psi = \frac{3\pi}{2} - \alpha = \frac{3\pi}{2} + \mathrm{Arccos}\left(\frac{2\sin\theta + \tilde{x}_2 \sin 2\theta}{\sqrt{4 + 4\tilde{x}_2\cos\theta + \tilde{x}_2^2}}\right) = 5.11178\cdots,$$

then ψ satisfies the fourth condition in (45).

Now, let us consider a function $\tilde{p}$ which has the form (8) with $\zeta_1 = \tilde{x}_2\mathrm{e}^{\mathrm{i}\theta}$ and $\zeta_2 = \mathrm{e}^{\mathrm{i}\psi}$. Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, in view of Lemma 1, we see that $\tilde{p}$ belongs to the class $\mathcal{P}$. Finally, let

$$\tilde{f}(z) := \int_0^z \frac{\tilde{p}(\zeta)}{(1-\zeta)^2} \mathrm{d}\zeta, \quad z \in \mathbb{D}.$$

Clearly, $\tilde{f} \in \mathcal{CV}(1)$ and $|T_{3,1}(\tilde{f})| = -(4/9)\Theta$. Thus, the proof of the theorem is completed. ◻

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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