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Some Remarks on Partial Metric Spaces

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Abstract

In this paper, we investigate some topological properties of partial metric spaces (in short PMS). We give some relationship between metric-like PMS, sequentially isosceles PMS and sequentially equilateral PMS. We also prove a type of Urysohn's lemma for metric-like PMS. By applying the construction of Hartman–Mycielski, we show that every bounded PMS can be isometrically embedded into a pathwise connected and locally pathwise connected PMS. In the end, we show that a partial metric space is compact iff it is totally bounded and complete.

Keywords Partial metric spaces · Urysohn's lemma · Isometric embedding · Compact space · Complete space

Mathematics Subject Classification Primary 54E35 · 54E50; Secondary 54D05 · 54D30

1 Introduction and Preliminaries

In 1992, in order to model computation over a metric space, G. Matthews proposed the concept of partial metric spaces on the basis of metric spaces [11,12]. This concept is not only widely used in many branches of mathematics, but also used to calcu-

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late domains and semantics. At present, the research on partial metric space mainly focuses on the topological properties [4,5,8,10], fixed point theory [1,2,9], theoretical computing science and other aspects [10,13,14] . Most of these researches are concentrated on the theory of fixed points. However, the research on the topological properties of partial metric space is rather lacking. Ge et al. [5] presented the existence and uniqueness of the completion of a partial metric space in classical sense. Han et al. [7] showed that some familiar topological properties and principles are still true in the partial metric space under certain conditions. Their approach includes axioms of separation, compactness, countability, completeness and Ekeland's variational principle. At the same time, they also posed some open questions. In this paper, some topological properties of partial metric spaces are further studied on the basis of [7].

The structure of this paper is arranged as follows: The first section is introduction and preliminaries. In the second section, we give some relationship between metric-like PMS, sequentially isosceles PMS and sequentially equilateral PMS. We also prove a type of Urysohn's lemma for metric-like PMS. In Section 3, we show that every bounded partial metric space can be isometrically embedded into a connected and locally connected partial metric space. In the fourth section, we show that a partial metric space is compact if and only if it is complete and totally bounded. We also show that the completion of a partial metric space X is in fact the complete reflection of X.

The following notion of partial metric space is given in [11,12].

Definition 1.1 ([11,12]) Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if the following conditions hold:

(PM1) $p(x, y) = p(y, x)$;
(PM2) if $0 \leq p(x, x) = p(x, y) = p(y, y)$, then $x = y$;
(PM3) $p(x, x) \leq p(x, y)$;
(PM4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

for any $x, y, z \in X$. The pair (X, p) is then called a partial metric space (in short PMS).

Let (X, p) be a PMS. Then, the function $d_p : X \times X \to [0, \infty)$ defined by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X. So, partial metric can degenerate into ordinary metric. But in general, partial metric spaces are not necessarily metric spaces. In fact, a partial metric on X is a metric if and only if $p(x, x) = 0, \forall x \in X$. For convenience, we write $\tilde{p}(x, y) = p(x, y) - p(x, x)$.

Example 1.2 (1) Let $p(a, b) = \max\{a, b\}$, $a, b \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a PMS.
(2) Let $\mathbb{I}$ be the set of all nonempty closed intervals of the real line $\mathbb{R}$, $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$, then $(\mathbb{I}, p)$ is a PMS.

Remark 1.3 (1) Let (X, p) be a PMS and let $x \in X$, $\varepsilon > 0$. The subsets $B_p(x, \varepsilon) = \{y \in X : \tilde{p}(x, y) < \varepsilon\}$ and $B_p(x, \varepsilon] = \{y \in X : \tilde{p}(x, y) \leq \varepsilon\}$ are, respectively, called the p-open and p-closed balls centred at x with radius ε with respect to the partial metric p.
(2) Each partial metric p on X induces a topology $\mathscr{T}_p$ on X with the base consisting of open p-balls $B_p(x, \varepsilon) = \{y \in X : \tilde{p}(x, y) < \varepsilon\}$. It is easily to verify that the topology $\mathscr{T}_p$ is T_0.

(3) The difference between partial metric space and metric space mainly lies in the following two aspects:

Firstly, $p(x, x)$ is not necessarily zero ($p(x, x) \geq 0$), and secondly, $y \in B_p(x, \varepsilon)$ does not imply $x \in B_p(y, \varepsilon)$. Thus, in general, $\mathscr{T}_p$ is an asymmetric topology on X.

Definition 1.4 ([11,12]) Let (X, p) be a PMS.

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $\lim_{n\to\infty} \tilde{p}(x, x_n) = 0$. So a sequence $\{x_n\}$ converges to x in (X, p) if and only if $\{x_n\}$ converges to x in $(X, \mathscr{T}_p)$. Sometimes, we abbreviate $\lim_{n\to\infty} \tilde{p}(x, x_n) = 0$ to $\lim x_n = x$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite). (That is, there is an $a \in [0, \infty)$, for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any positive integers $n, m \geq N_0$, we have that $|p(x_n, x_m) - a| < \varepsilon$.)
(3) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Definition 1.5 Let (X, p_1) and (Y, p_2) be PMS. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $p_2(f(x_0), f(x)) < p_2(f(x_0), f(x_0)) + \varepsilon$ whenever $x \in X$ and $p_1(x_0, x) < p_1(x_0, x_0) + \delta$. Equivalently, $\tilde{p}_2(f(x_0), f(x)) < \varepsilon$ whenever $x \in X$ and $\tilde{p}_1(x_0, x) < \delta$. If f is continuous at every point of X, then it is said to be continuous on X.

By Definition 1.5, it is clear that a function $f : (X, p_1) \to (Y, p_2)$ between partial metric spaces is continuous if and only if it is continuous with respect to the topologies $\mathscr{T}_{p_1}$ and $\mathscr{T}_{p_2}$ induced by the partial metrics p_1 and p_2, respectively. So the following result is straightforward.

Theorem 1.6 *Let (X, p_1) and (Y, p_2) be PMS. Let $f : X \to Y$ be a mapping, $x_0 \in X$. Then, the following are equivalent:*

(a) f is continuous at x_0;
(b) For every neighbourhood V of $f(x_0)$ in Y, there exists a neighbourhood U of x_0 in X such that $f(U) \subset V$;
(c) Let $\{x_n\}$ be any sequence in X. If $\lim_{n\to\infty} x_n = x_0$ in (X, p_1), then $\lim_{n\to\infty} f(x_n) = f(x_0)$ in (Y, p_2).

2 Convergence of Sequences and Urysohn's Lemma

In [7], the authors introduced the notions of sequentially isosceles PMS, sequentially equilateral PMS, sequentially symmetrical PMS and metric-like PMS and applied them to study separation properties of PMS.

Definition 2.1 ([7]) Let (X, p) be a PMS.

(1) If for each $x \in X$ and each sequence $\{x_n\}$ in X, $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$ implies $\forall y \in X$, $\lim_{n\to\infty} p(x_n, y) = p(x, y)$, then (X, p) is sequentially isosceles.

(2) If for each $x \in X$ and each pair of sequences $\{x_n\}$ and $\{y_n\}$ in X, $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0 = \lim_{n\to\infty} \widetilde{p}(y_n, x_n)$ implies $\lim_{n\to\infty} \widetilde{p}(x, y_n) = 0$, then (X, p) is sequentially equilateral.
(3) If for each $x \in X$ and each sequence $\{x_n\}$ in X, $\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0$ implies $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$, then (X, p) is sequentially symmetric.
(4) If for each $x \in X$ and each sequence $\{x_n\}$ in X, $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$ implies $\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0$ (or equivalently, implies $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$) , then (X, p) is metric-like.

Let $x \in X$, $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0$. Consider a constant sequence $\{y_n\}$ with $y_n = x$ for each n. Then, we have $\lim_{n\to\infty} \widetilde{p}(x, y_n) = 0 = \lim_{n\to\infty} \widetilde{p}(x_n, y_n)$. Hence, if X is sequentially equilateral, then X is sequentially symmetric. In [7], the authors asked whether we can give more connections between these special PMS defined above. We now show the following result.

Theorem 2.2 *Let (X, p) be a PMS.*

(1) If (X, p) is metric-like, then it is sequentially isosceles.
(2) If (X, p) is metric-like and sequentially symmetric, then (X, p) is sequentially equilateral.

Proof (1) Suppose that (X, p) is metric-like. Let $\{x_n\}$ be a sequence in X, and $x \in X$ such that $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$. For arbitrary $y \in X$, we have

$$\begin{aligned} p(x_n, y) &\le p(x_n, x) + p(x, y) - p(x, x), \\ p(x, y) &\le p(x, x_n) + p(x_n, y) - p(x_n, x_n). \end{aligned}$$

Thus,

$$p(x, y) - p(x, x_n) + p(x_n, x_n) \le p(x_n, y) \le p(x_n, x) + p(x, y) - p(x, x),$$

that is,

$$p(x, y) - \widetilde{p}(x_n, x) \le p(x_n, y) \le \widetilde{p}(x, x_n) + p(x, y).$$

Since (X, p) is metric-like, $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$ implies that $\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0$. We have that

$$p(x, y) \le \lim_{n\to\infty} p(x_n, y) \le p(x, y).$$

It follows that $\lim_{n\to\infty} p(x_n, y) = p(x, y)$. Thus, (X, p) is sequentially isosceles.

(2) Let $x \in X$, $\{x_n\}$ and $\{y_n\}$ be a pair of sequences in X such that $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0 = \lim_{n\to\infty} \widetilde{p}(y_n, x_n)$. Hence,

$$\lim_{n\to\infty} p(x, x_n) = p(x, x), \ \lim_{n\to\infty} p(y_n, x_n) = \lim_{n\to\infty} p(y_n, y_n).$$

Since (X, p) is metric-like, then

$$\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0, \text{ that is}$$
$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n).$$

Although we have

$$(x, y_n) \le p(x, x_n) + p(x_n, y_n) - p(x_n, x_n),$$
$$0 \le p(x, y_n) - p(y_n, y_n) \le p(x, x_n) - p(x_n, x_n) + p(x_n, y_n) - p(y_n, y_n).$$

Thus, $\lim_{n\to\infty} \widetilde{p}(y_n, x) = 0$. Therefore, $\lim_{n\to\infty} \widetilde{p}(x, y_n) = 0$ since (X, p) is metric-like. It follows that (X, p) is sequentially equilateral. □

Let (X, p) be a partial metric space and $A \subset X$ be a nonempty subset, $x \in X$. We define a distance from x to A as:

$$\tilde{p}(x, A) = \inf\{\tilde{p}(x, a) : a \in A\}.$$

Lemma 2.3 *Let (X, p) be a PMS, $A \subset X$, $x \in X$, $y \in X$. Then,* $|\tilde{p}(x, A) - \tilde{p}(y, A)| \le \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$.

Proof For every $a \in A$, we have

$$p(x, a) \le p(x, y) + p(y, a) - p(y, y), \text{ thus}$$
$$\tilde{p}(x, A) \le p(a, y) + p(x, y) - p(x, x) - p(y, y).$$

This implies that

$$\tilde{p}(x, A) \le \tilde{p}(y, A) + p(x, y) - p(x, x).$$

Hence,

$$\tilde{p}(x, A) - \tilde{p}(y, A) \le p(x, y) - p(x, x).$$

Similarly, we have

$$\tilde{p}(y, A) - \tilde{p}(x, A) \le p(x, y) - p(y, y).$$

Therefore, $|\tilde{p}(x, A) - \tilde{p}(y, A)| \le \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$. □

Lemma 2.4 *Suppose that (X, p) is a metric-like PMS and $A \subset X$. For any $x \in X$, let $g(x) = \tilde{p}(x, A)$, then $g : X \to \mathbb{R}$ is a continuous mapping.*

Proof Let $x \in X$ and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$; equivalently, $\lim_{n\to\infty} \widetilde{p}(x, x_n) = 0$. Since (X, p) is a metric-like PMS, we have $\lim_{n\to\infty} \widetilde{p}(x_n, x) = 0$. By lemma 2.3, we have

$$\begin{aligned}|g(x_n) - g(x)| &= |\tilde{p}(x, A) - \tilde{p}(x_n, A)| \\ &\le \max\{p(x, x_n) - p(x, x), p(x, x_n) - p(x_n, x_n)\} \\ &= \max\{\widetilde{p}(x, x_n), \widetilde{p}(x_n, x)\}.\end{aligned}$$

It implies that $\lim_{n\to\infty} g(x_n) = g(x)$. Thus, g is continuous at x. □

Theorem 2.5 (Urysohn's lemma) *Let (X, p) be a metric-like PMS, A, B be two disjoint closed sets in X. Then, there is a continuous map $f : X \longrightarrow [0, 1]$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.*

Proof If $A = \varnothing$ or $B = \varnothing$, then it is clear. Now assume that A, B are both nonempty. Define $f : X \longrightarrow [0, 1]$ as follows:

$$f(x) = \frac{\tilde{p}(x, A)}{\tilde{p}(x, A) + \tilde{p}(x, B)}, x \in X.$$

By Theorem 2.2, (X, p) is sequentially isosceles since it is metric-like. Hence, $(X, \mathscr{T}_p)$ is a Hausdorff space which follows from [7, Theorem 2.2].

For every $x \in X$, since $A \cap B = \varnothing$, without losing generality, we may assume that x does not belong to A. By [7, Theorem 2.1], there exists a positive real number r such that $\widetilde{p}(x, a) \ge r$ for each $a \in A$. Hence, $\tilde{p}(x, A) > 0$. This means that f is well defined. By Lemma 2.4, f is continuous. Clearly, we have $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. □

Corollary 2.6 *Every metric-like PMS is a normal space.*

Let (X, p) be a PMS and $A \subset X$. We write $B_p(A, r) = \{x \in X : \tilde{p}(x, A) < r\}$.

Corollary 2.7 *Let (X, p) be a metric-like PMS. For every compact set $A \subset X$ and any open set U containing A, there exists $r > 0$ such that $B_p(A, r) \subset U$.*

Proof The function $f : X \to R$, defined by $f(x) = \tilde{p}(x, X \setminus U)$, is positive on the set A. By Lemma 2.4, $f(x) = \tilde{p}(x, X \setminus U)$ is continuous. Since every metric-like PMS is a Hausdorff space, $p(x, y) - p(x, x) > 0$. Thus, $f(x) = \tilde{p}(x, X \setminus U) > 0$. Hence, by the compactness of A, there exists an $r > 0$ such that $f(x) \ge r$ for every $x \in A$, clearly $B_p(A, r) \subset U$. □

The following question is posed in the literature [7].

Question 2.8 *Let $Y \subset X$, p is a partial metric on Y, then there always exists a partial metric P on X, such that $P|_Y = p$.*

Let $Y = \{a\}$, P is defined as follows:

$$P(x, y) = \begin{cases} p(a, a), & \text{if } x = y; \\ p(a, a) + 1, & \text{if } x \ne y. \end{cases}$$

If $p(a, a) = 0$*, then* P *is a discrete metric.*
If $|Y| \geq 2$*, let* $\alpha = \sup\{p(y, y) : y \in Y\}$*,* P *is defined as follows:*

$$P(x, y) = \begin{cases} 0, & \text{if } x = y, x \notin Y; \\ p(x, y), & \text{if } x, y \in Y; \\ \alpha + 1, & \text{if } x \notin Y. \end{cases}$$

The question is whether topological properties hold in (Y, p) *still hold in* (X, P)*? If not, what can we do to make them hold?*

We note that if the cardinal of Y is infinite, then P is not necessary a partial metric on Y. For example, let $p(x, y) = \max\{x, y\}$, $Y = R^+$, then $\alpha = \sup\{p(y, y) : y \in Y\} = \infty$; thus, for any $x \notin Y$, $P(x, y) = \infty$.

Generally, properties of (Y, p) cannot be "extended" to (X, P). For example, the compactness of (Y, p) does not imply the compactness of (X, P) even when p is a metric on Y. It is also clear that for every $x \in X \setminus Y$, the single point set $\{x\}$ is open and closed in X; hence, X is always disconnected.

Proposition 2.9 *If* Y *is a finite set that contains at least two elements, then the properties of* T_0, T_1, T_2 *of* (Y, p) *imply, respectively,* T_0, T_1, T_2 *properties of* (X, P).

Proof We only consider T_0 property; the proof of the properties of T_1 and T_2 is similar.

For any $x, y \in X$ and $x \neq y$, consider the following two cases:

Case 1: $x, y \notin Y$. Then, $P(x, y) = P(x, x) = \alpha + 1$. Consider the open balls $B_P(x, \frac{\alpha+1}{2}) = \{y \in X : \tilde{P}(x, y) < \frac{\alpha+1}{2}\} = \{x\}$, and $B_P(y, \frac{\alpha+1}{2}) = \{z \in X : \tilde{P}(y, z) < \frac{\alpha+1}{2}\} = \{y\}$. Then, it is obvious that $\{x\} \bigcap \{y\} = \emptyset$.

Case 2: $y \in Y$, but $x \notin Y$. Then, $\{x\}$ is an open ball of x, $y \notin \{x\}$.

Thus, X is a T_0 space. □

3 Embedding into Connected, Locally Connected Partial Metric Spaces

In this section, we will show that every bounded partial metric space can be isometrically embedded into a pathwise connected and locally pathwise connected partial metric space. If (X, p) is a PMS, $x \in X$, we will write $\mathcal{N}(x)$ the neighbourhood system of x with respect to induced topology $\mathcal{T}_p$ on X.

Definition 3.1 Let (X, p) be a PMS. Each continuous map $f : I \to X$ from $I = [0, 1]$ to X is called a path in X. $f(0)$ and $f(1)$ are called the start and end of the path, respectively. When $f(0) = f(1)$, the path is called a closed path or a circle. If there is a path $f : I \to X$ for any two points x, y in X satisfying $f(0) = x$ and $f(1) = y$, then X is called a pathwise connected partial metric space.

Definition 3.2 Let (X, p) be a PMS, $x_0 \in X$. If for any $U \in \mathcal{N}(x_0)$, there exists a pathwise connected $V \in \mathcal{N}(x_0)$, such that $V \subset U$, then X is called locally pathwise connected at point x_0. If (X, p) is locally pathwise connected at every point, then X is called a locally pathwise connected partial metric space.

Definition 3.3 Let (X, p_1) and (Y, p_2) be PMS, $f : X \to Y$ be an injective mapping. If $p_1(x, y) = p_2(f(x), f(y))$ for any two elements $x, y \in X$, we call $f : X \to Y$ an isometric embedding.

If $f : X \to Y$ is an isometric embedding, then it is clear that f must be a topological embedding.

Definition 3.4 Let (X, p) be a PMS. Denote $X^\bullet$ the set of all functions f on $I = [0, 1)$ with values in X such that, for some sequence $0 = a_0 < a_1 < \cdots < a_n = 1$, the function f is constant on $[a_k, a_{k+1})$ for each $k = 0, \cdots, n-1$. The elements of $X^\bullet$ are called step functions.

Let $f, g \in X^\bullet$. Then, there exist two partitions $0 = a_0 < a_1 < \cdots < a_n = 1$ and $0 = b_0 < b_1 < \cdots < b_m = 1$, such that f is constant on $[a_k, a_{k+1})$ for each $k = 0, \cdots, n-1$ and g is constant on $[b_k, b_{k+1})$ for each $k = 0, \cdots, m-1$. We will call the partition of the common refinement of $\{a_i\}$ and $\{b_j\}$ the join partition.

Lemma 3.5 *For arbitrary $f, g \in X^\bullet$, let $0 = a_0 < a_1 < \cdots < a_n = 1$ be the join partition of the partitions with respect to f and g. Define*

$$p^\bullet(f, g) = \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(f(a_i), g(a_i)).$$

Then, $(X^\bullet, p^\bullet)$ is a PMS.

Proof Obviously, the function $p^\bullet$ satisfies the (PM1) and (PM3) of definition 1.1. Next we verify that $p^\bullet$ satisfies conditions (PM2) and (PM4) in Definition 1.1.

First, we verify that $p^\bullet$ satisfies condition (PM2). If $p^\bullet(f, g) = p^\bullet(f, f)$, then

$$\begin{aligned} &p^\bullet(f, g) - p^\bullet(f, f)\\ &= \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(f(a_i), g(a_i)) - \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(f(a_i), f(a_i))\\ &= \sum_{i=0}^{n-1}(a_{i+1} - a_i)[p(f(a_i), g(a_i)) - p(f(a_i), f(a_i))] = 0. \end{aligned}$$

Thus, $p(f(a_i), g(a_i)) = p(f(a_i), f(a_i))$. By the same reason, we have $p(f(a_i), g(a_i)) = p(g(a_i), g(a_i))$, since $p^\bullet(f, g) = p^\bullet(g, g)$. Because p is a partial metric, $f(a_i) = g(a_i)$. Then, $f = g$.

Second, we verify that $p^\bullet$ satisfies condition (PM4).

$\forall f, g, h \in X^\bullet$,

$p(f(a_i), g(a_i)) \le p(f(a_i), h(a_i)) + p(h(a_i), g(a_i)) - p(h(a_i), h(a_i))$. Thus,

$$\sum_{i=0}^{n-1}(a_{i+1} - a_i)p(f(a_i), g(a_i)) \le \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(f(a_i), h(a_i))$$

$$+\sum_{i=0}^{n-1}(a_{i+1}-a_i)p(h(a_i), g(a_i)) - \sum_{i=0}^{n-1}(a_{i+1}-a_i)p(h(a_i), h(a_i)),$$

that is, $p^\bullet(f, g) \leq p^\bullet(f, h) + p^\bullet(h, g) - p^\bullet(h, h)$.

Hence, $(p^\bullet, X^\bullet)$ is a PMS. □

Theorem 3.6 *If (X, p) is a bounded PMS, then $(X^\bullet, p^\bullet)$ is both pathwise connected and locally pathwise connected.*

Proof We first show that (X, p) is locally pathwise connected.

For every $f \in X^\bullet$, take $m \in \mathbb{N}^+$. Let $g \in B_p(f, \frac{1}{m})$. By the definition of step function, there exists a nonnegative integer n and a sequence $0 = a_0 < a_1 < \cdots < a_n = 1$, such that $f|_{[a_i, a_{i+1})} \equiv x_i$, $g|_{[a_i, a_{i+1})} \equiv y_i$, $i = 0, 1, 2, \cdots, n-1$.

For each $t \in [0, 1]$, let $b_{i,t} = a_i + t(a_{i+1} - a_i)$. If $t = 0$, then $b_{i,0} = a_i$; if $t = 1$, then $b_{i,1} = a_{i+1}$.

Define $f_t : [0, 1) \to X$ as follows:

$$f_t(r) = \begin{cases} y_i, & \text{if } r \in [a_i, b_{i,t}); \\ x_i, & \text{if } r \in [b_{i,t}, a_{i+1}). \end{cases}$$

Claim 1 $f_t \in B_p(f, \frac{1}{m})$

If $t = 0$, then $f_0 = f$; if $t = 1$, then $f_1 = g$

$\forall s, t \in [0, 1]$, without losing generality, we suppose $s \leq t$. Then, by the definition of partial metric space $X^\bullet$, we have

$$\begin{aligned} p^\bullet(f_t, f_s) &= \sum_{i=0}^{n-1}(b_{i,s} - a_i)p(f_s(a_i), f_t(a_i)) \\ &\quad + \sum_{i=0}^{n-1}(b_{i,t} - b_{i,s})p(f_s(b_{i,s}), f_t(b_{i,s})) \\ &\quad + \sum_{i=0}^{n-1}(a_{i+1} - b_{i,t})p(f_s(b_{i,t}), f_t(b_{i,t})) \\ &= \sum_{i=0}^{n-1} s(a_{i+1} - a_i)p(y_i, y_i) + \sum_{i=0}^{n-1}(t - s)(a_{i+1} - a_i)p(x_i, y_i) \\ &\quad + \sum_{i=0}^{n-1}(1 - t)(a_{i+1} - a_i)p(x_i, x_i) = sp^\bullet(g, g) \\ &\quad + (t - s)p^\bullet(g, f) + (1 - t)p^\bullet(f, f) \\ &\qquad \cdots\cdots\cdots(\star). \end{aligned}$$

Let $s = 0$, then $f_s = f_0 = f$. The above formula becomes:

$$p^\bullet(f, f_t) = p^\bullet(f_0, f_t) = tp^\bullet(g, f) + (1 - t)p^\bullet(f, f).$$

Thus,

$$p^{\bullet}(f, f_t) - p^{\bullet}(f, f) = t(p^{\bullet}(g, f) - p^{\bullet}(f, f)).$$

Since $g \in B_p(f, \frac{1}{m})$,

$$p^{\bullet}(g, f) - p^{\bullet}(f, f) < \frac{1}{m}.$$

It follows that

$$p^{\bullet}(f_t, f) - p^{\bullet}(f, f) < \frac{t}{m} \leq \frac{1}{m}, \text{ that is } f_t \in B_p(f, \frac{1}{m}).$$

Claim 2 Let $\varphi(t) = f_t$, then $\varphi : [0, 1] \to B_p(f, \frac{1}{m})$ is continuous, and $\varphi(0) = f_0 = f, \varphi(1) = f_1 = g$.

Clearly, we have $\varphi(0) = f_0 = f, \varphi(1) = f_1 = g$ by the definition of $\varphi(t)$ and f_t.

Let $t_0 \in [0, 1]$, $\varepsilon > 0$. Consider the neighbourhood $B_p(f_{t_0}, \varepsilon) \cap B_p(f, \frac{1}{m})$ of f_{t_0}. If $t_0 \geq s$, by ($\star$), we have

$$\begin{aligned}&p^{\bullet}(f_s, f_{t_0}) - p^{\bullet}(f_{t_0}, f_{t_0})\\ &\quad = sp^{\bullet}(g, g) + (t_0 - s)p^{\bullet}(g, f) + (1 - t_0)p^{\bullet}(f, f) - t_0 p^{\bullet}(g, g)\\ &\quad\quad -(1 - t_0)p^{\bullet}(f, f) = (t_0 - s)[p^{\bullet}(g, f) - p^{\bullet}(g, g)].\end{aligned}$$

If $t_0 \leq s$, by ($\star$), we have

$$p^{\bullet}(f_s, f_{t_0}) - p^{\bullet}(f_{t_0}, f_{t_0}) = (s - t_0)[p^{\bullet}(g, f) - p^{\bullet}(f, f)].$$

Since the partial metric $p^{\bullet}$ is bounded, we can suppose $p^{\bullet}(g, f) - p^{\bullet}(g, g) < M$.

If $t_0 \geq s$, by ($\star$), we have

$$\begin{aligned}p^{\bullet}(f_s, f_{t_0}) - p^{\bullet}(f_{t_0}, f_{t_0}) &= sp^{\bullet}(g, g) + (t_0 - s)p^{\bullet}(g, f) + (1 - t_0)p^{\bullet}(f, f)\\ &\quad - t_0 p^{\bullet}(g, g) - (1 - t_0)p^{\bullet}(f, f) = (t_0 - s)[p^{\bullet}(g, f) - p^{\bullet}(g, g)].\end{aligned}$$

If $t_0 \leq s$, by ($\star$), we have

$$p^{\bullet}(f_s, f_{t_0}) - p^{\bullet}(f_{t_0}, f_{t_0}) = (s - t_0)[p^{\bullet}(g, f) - p^{\bullet}(f, f)].$$

Since the partial metric $p^{\bullet}$ is bounded, we can suppose $\max\{\tilde{p^{\bullet}}(g, f), \tilde{p^{\bullet}}(f, g)\} < M$.

Take $\delta < \frac{\varepsilon}{M}$. When $s \in (t_0 - \delta, t_0 + \delta)$, we have $p^{\bullet}(f_s, f_{t_0}) - p^{\bullet}(f_{t_0}, f_{t_0}) < \frac{\varepsilon}{M} \times M = \varepsilon$. Thus, $\varphi(s) = f_s \in B_p(f_{t_0}, \varepsilon)$, that is, $\varphi((t_0 - \delta, t_0 + \delta)) \subset B_p(f_{t_0}, \varepsilon) \cap B_p(f, \frac{1}{m})$. It follows that $\varphi : [0, 1] \to B_p(f, \frac{1}{m})$ is continuous.

Claim 3 $B_p(f, \frac{1}{m})$ is pathwise connected.

Suppose $g, h \in B_p(f, \frac{1}{m})$. By Claim 2, we know that there are two continuous functions,

$\varphi_1 : [0, 1] \to B_p(f, \frac{1}{m}), \varphi_2 : [0, 1] \to B_p(f, \frac{1}{m})$
and $\varphi_1(0) = g, \varphi_1(1) = f; \varphi_2(0) = f, \varphi_2(1) = h$.
Let

$$\varphi(t) = \begin{cases} \varphi_1(2t), & \text{if } t \in [0, \frac{1}{2}]; \\ \varphi_2(2t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

It's easy to verify that $\varphi(t)$ is continuous, $\varphi(0) = g, \varphi(1) = h$.
Thus, $B_p(f, \frac{1}{m})$ is pathwise connected.
Since $B_p(f, \frac{1}{m})$ is a neighbourhood of f in $X^\bullet$, $X^\bullet$ is locally pathwise connected.
Similarly, we can show that $(X^\bullet, p^\bullet)$ is pathwise connected. □

Theorem 3.7 *Each bounded partial metric space (X, p) can be isometrically embedded into a pathwise connected and locally pathwise connected partial metric space $(X^\bullet, p^\bullet)$. If X is a Hausdorff space, then the embedding is a closed embedding.*

Proof We assign to each $x \in X$ the element $x^\bullet$ of $X^\bullet$ defined by $x^\bullet(r) = x$ for all $r \in [0, 1)$. Since

$$p^\bullet(x^\bullet, y^\bullet) = \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(x^\bullet(a_i), y^\bullet(a_i)) = \sum_{i=0}^{n-1}(a_{i+1} - a_i)p(x, y) = p(x, y),$$

the function $i : X \to X^\bullet$, where $i(x) = x^\bullet$ for each $x \in X$, is an injective mapping and $p^\bullet(i(x), i(y)) = p(x, y)$. Then, the function $i : X \to X^\bullet$ is an isometric embedding of X into $X^\bullet$, and the partial metric spaces $X^\bullet$ is pathwise connected and locally pathwise connected by Theorem 3.6.

If X is a Hausdorff space, then $\forall x, y \in X, x \neq y, \inf_{z\in X}\{\widetilde{p}(x, z) + \widetilde{p}(y, z)\} \neq 0$ ([7] Theorem 2.1). For convenience, write $b = \inf_{z\in X} \widetilde{p}(x, z) + \widetilde{p}(y, z)\}$, then $b > 0$.

If $f \in X^\bullet$ and $f \notin i(X)$, then $\exists k, j$, such that $f|_{[a_k,a_{k+1})} \equiv x, f|_{[a_j,a_{j+1})} \equiv y$, and $x \neq y$. Let $0 = a_0 \leq a_k < a_{k+1} \leq a_j < a_{j+1} \leq a_n = 1$. Suppose $a = \min\{a_{k+1} - a_k, a_{j+1} - a_j\}$. Since $ab > 0$, there exists a positive integer n, such that $0 < \frac{1}{n} < ab$. Take the neighbourhood $B_p(f, \frac{1}{n})$ of f, we will prove that $B_p(f, \frac{1}{n}) \cap i(X) = \emptyset$.

$\forall z \in X$, we have $i(z) = z^\bullet \in X^\bullet$. Hence,

$$\begin{aligned} &p^\bullet(f, z^\bullet) - p^\bullet(f, f) \\ &= \sum_{i=1}^{n}(a_{i+1} - a_i)(p(f(a_i), z^\bullet(a_i)) - \sum_{i=1}^{n}(a_{i+1} - a_i)(p(f(a_i), f(a_i)) \\ &= \sum_{i=1}^{n}(a_{i+1} - a_i)(p(f(a_i), z^\bullet(a_i)) - p(f(a_i), f(a_i)) \\ &\geq (a_{k+1} - a_k)(p(x, z) - p(x, x)) + (a_{j+1} - a_j)(p(y, z) - p(y, y)) \end{aligned}$$

$$\geq a(p(x,z)-p(x,x)+p(y,z)-p(y,y)) = a(\widetilde{p}(x,z)+\widetilde{p}(y,z)) \geq ab > \frac{1}{n}.$$

Thus, $z^\bullet \notin B_p(f, \frac{1}{n})$. Then, $B_p(f, \frac{1}{n}) \cap i(X) = \emptyset$. Therefore, $i(X)$ is a closed set in $X^\bullet$. □

Lemma 3.8 *[7] Every partial metric on X is equivalent to some bounded partial metric on X.*

Corollary 3.9 *Every partial metric space can be topologically embedded into a pathwise connected and locally pathwise connected partial metric space.*

4 Compactness and Completions in Partial Metric Spaces

Definition 4.1 Let (X, p_1) and (Y, p_1) be two PMS and $f : X \to Y$ be a mapping. If for any $\varepsilon > 0$, there is a $\delta > 0$, such that, for arbitrary $x, y \in X$, $\widetilde{p_1}(x, y) < \delta$ implies that $\widetilde{p_2}(f(x), f(y)) < \varepsilon$, we call f a uniform continuous mapping from (X, p_1) to (Y, p_1).

Definition 4.2 ([7]) If (X, p) is a PMS and $(X, \mathscr{T}_p)$ is compact, then we call (X, p) a compact partial metric spaces.

Theorem 4.3 ([7]) (Lebesgue covering lemma) *Let $\{U_i : 1 \leq i \leq k\}$ be a finite open cover of a compact partial metric spaces(X, p). Then, there is some $\delta > 0$ such that for any $A \subset X$ with $diam(A) < \delta$, we have $A \subset U_i$ for some $1 \leq i \leq k$.*

Theorem 4.4 *Let $f : (X, p_1) \to (Y, p_2)$ be a continuous mapping from a compact partial metric space(X, p_1) to a partial metric space (Y, p_2), then f is uniform continuous.*

Proof For arbitrary $\varepsilon > 0$, we consider the open cover $V = \{B_{p_2}(y, \frac{\varepsilon}{2}) : y \in Y\}$ of (Y, p_2). Then, $U = \{f^{-1}(B_{p_2}(y, \frac{\varepsilon}{2})) : y \in Y\}$ is an open cover of (X, p_1), by Lebesgue number lemma (note that the finiteness of the open cover plays no role in the proof of the Lebesgue covering lemma in [7]); U has a Lebesgue number $\delta > 0$. Thus, for any $A \subset X$, if $diam_{p_1} A < \delta$, then there is a $y \in Y$, such that $A \subset f^{-1}(B_{p_2}(y, \frac{\varepsilon}{2}))$, that is, $f(A) \subset B_{p_2}(y, \frac{\varepsilon}{2})$. Then, $diam_{p_2} f(A) \leq diam_{p_2} B_{p_2}(y, \frac{\varepsilon}{2}) \leq \varepsilon$. It implies that for any $x_1, x_2 \in X$, if $\widetilde{p_1}(x_1, x_2) < \delta$, then $diam_{p_1}\{x_1, x_2\} < \delta$. Thus, $\widetilde{p_2}(f(x_1), f(x_2)) = diam_{p_2} f(\{x_1, x_2\}) \leq \varepsilon$. This shows that $f : (X, p_1) \to (Y, p_2)$ is uniform continuous.

Theorem 4.5 *Any continuous function f from a compact partial metric space (X, p_1) to a partial metric space (Y, p_2) is bounded, that is, $f(X)$ is a bounded subset of Y.*

Proof Take $y \in Y$. For each $n \in \mathbb{N}$, the set $B_p(y, n)$ is an open set in Y, and hence, $U_n := f^{-1}(B_p(y, n)) = \{x \in X : \widetilde{p_2}(f(x), y) < n\}$ is open. Note also that $U_n \subset U_{n+1}$. The collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. By the compactness of X, there exists U_N such that $X = U_N$. It follows that $f(X) \subset B_p(y, N)$. Hence, $f(X)$ is bounded. □

Definition 4.6 Let (X, p) be a PMS, $\varepsilon > 0$, $A \subset X$ be a finite set. If $\bigcup_{a \in A} B_p(a, \varepsilon) = X$, we call A an ε-net of (X, p). If for any $\varepsilon > 0$, (X, p) has an $\varepsilon-$net, we call (X, p) totally bounded.

Proposition 4.7 *Let (X, p) be a PMS. If for some $\varepsilon > 0$, there is an ε-net of (X, p), then (X, p) is bounded.*

Proof Suppose that for $\varepsilon > 0$, there exists an $\varepsilon-$net for X, say A. Since A is a finite set, $p(A) = \sup\{p(a, b) : a, b \in A\} < \infty$. Now, let x_1 and x_2 be any two points of X. There exist points a and b in A such that $x_1 \in B_p(a, \varepsilon)$, $x_2 \in B_p(b, \varepsilon)$. It is clear that $\widetilde{p}(a, x_1) < \varepsilon$ and $\widetilde{p}(b, x_2) < \varepsilon$.

It follows, using the triangle inequality, that

$$p(x_1, b) \leq p(x_1, a) + p(a, b) - p(a, a).$$

Thus,

$$p(x_1, b) + p(b, x_2) - p(b, b) \leq p(x_1, a) + p(a, b) + p(b, x_2) - p(b, b) - p(a, a).$$

So,

$$\begin{aligned} p(x_1, x_2) &\leq p(x_1, b) + p(b, x_2) - p(b, b) \\ &\leq \widetilde{p}(a, x_1) + p(a, b) + \widetilde{p}(b, x_2) \leq p(A) + 2\varepsilon. \end{aligned}$$

It implies that $p(X) = \sup\{p(x_1, x_2) : x_1, x_2 \in X\} \leq p(A) + 2\varepsilon$. Hence, X is bounded. □

Corollary 4.8 *Every totally bounded PMS is bounded.*

Theorem 4.9 *A PMS (X, p) is compact if and only if it is complete and totally bounded.*

Proof Let us first prove that a compact PMS is totally bounded. Let (X, p) be a compact partial metric space . For any given $\varepsilon > 0$, the collection of all open balls $\{B_p(x, \varepsilon) : x \in X\}$ is an open cover of X. The compactness of X implies that it contains a finite subcover. Hence, X is covered by a finite number of open balls of radius ε, i.e. the centres of the balls in the finite subcover form a finite ε-net for X. So, X is totally bounded.

Now we prove that a compact PMS is complete.

Suppose that (X, p) is a compact partial metric space which is not complete. Then, there exists a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, p) that does not have a limit in X. Let $y \in X$, since $\{x_n\}_{n \in \mathbb{N}}$ does not converge to y, there exists an $\varepsilon_0 > 0$ such that $\widetilde{p}(y, x_n) \geq 3\varepsilon_0$ for infinitely many of n. Since $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, there is an $M \in [0, \infty)$, for $\varepsilon_0 > 0$ there exists $n_0 \in \mathbb{N}$ such that for any positive integers $n, m \geq n_0$, we have that $|p(x_n, x_m) - M| < \varepsilon_0$. Put $k > n_0$ such that $\widetilde{p}(y, x_k) \geq 3\varepsilon_0$. Then, $p(y, x_k) \leq p(y, x_m) + p(x_m, x_k) - p(x_m, x_m)$, which implies that

$$p(y, x_m) \geq p(y, x_k) + p(x_m, x_m) - p(x_m, x_k).$$

So

$$p(y, x_m) - p(y, y) \geq p(y, x_k) + p(x_m, x_m) - p(x_m, x_k) - p(y, y).$$

By $|p(x_m, x_k) - M| < \varepsilon_0$ and $|p(x_m, x_m) - M| < \varepsilon_0$, we conclude that

$$\begin{aligned} 0 &\leq p(x_m, x_k) - p(x_m, x_m) \leq |p(x_m, x_k) - M + M - p(x_m, x_m)| \\ &\leq |p(x_m, x_k) - M| + |p(x_m, x_m) - M| \leq 2\varepsilon_0. \end{aligned}$$

Hence,

$\widetilde{p}(y, x_m) \geq \widetilde{p}(y, x_k) - (p(x_m, x_k) - p(x_m, x_m)) \geq 3\varepsilon_0 - 2\varepsilon_0 = \varepsilon_0$ for all $m \geq n_0$. So, the open ball $B_p(y, \varepsilon_0)$ contains only finite numbers of x_n. In this manner, we can associate with each $y \in X$ an open ball $B_p(y, \varepsilon_0(y))$, where $\varepsilon_0(y)$ is a positive real number that depends on y, and the open ball $B_p(y, \varepsilon_0(y))$ contains only finite numbers of x_n. Observe that $X = \bigcup\{B_p(y, \varepsilon_0(y)) : y \in X\}$, since X is compact, there exists a finite subcover $\{B_p(y_i, \varepsilon_0(y_i)) : y_i \in X\}$ of X. Since each open ball contains only finite numbers of x_n, also X must contain only finite numbers of x_n. This, however, is impossible. Hence, (X, p) must be a complete partial metric space.

Finally, we show that a totally bounded and complete PMS is compact.

Suppose that (X, p) is totally bounded and complete which is not compact. Then, there exists an open covering $\{G_\lambda\}_{\lambda \in \Lambda}$ of X that does not admit a finite subcover.

Since (X, p) is totally bounded, it is bounded. Hence, for some real number $r > 0$ and some $x_0 \in X$, we have $X \subseteq B_p(x_0, r)$. Observe that $X \subseteq B_p(x_0, r)$ implies $X = B_p(x_0, r)$. Let $\varepsilon_n = \frac{r}{2^n}$. We know that X, being totally bounded, can be covered by finitely many open balls of radius ε_1. By our hypothesis, at least one of these balls, say $B_p(x_1, \varepsilon_1)$, cannot be covered by a finite number of open sets of $\{G_\lambda\}$. Since $B_p(x_1, \varepsilon_1)$ is itself totally bounded (any nonempty subset of a totally bounded set is totally bounded, as shown above), we can find an $x_2 \in B_p(x_1, \varepsilon_1)$ such that $B_p(x_2, \varepsilon_2)$ cannot be covered by a finite number of open sets of $\{G_\lambda\}$. In this way, a sequence $\{x_n\}_{n \in \mathbb{N}}$ can be defined with the property that for each n, $B_p(x_n, \varepsilon_n)$ cannot be covered by a finite number of open sets of $\{G_\lambda\}$ and $x_{n+1} \in B(x_n, \varepsilon_n)$. We next show that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent.

Since $x_{n+1} \in B_p(x_n, \varepsilon_n)$, it follows that $\widetilde{p}(x_n, x_{n+1}) < \varepsilon_n$, and hence,

$$\begin{aligned} \widetilde{p}(x_n, x_{n+m}) &\leq \widetilde{p}(x_n, x_{n+1}) + \widetilde{p}(x_{n+1}, x_{n+2}) + \ldots + \widetilde{p}(x_{n+m-1}, x_{n+m}) \\ &\leq \varepsilon_n + \varepsilon_{n+1} + \ldots + \varepsilon_{n+m-1} \leq \frac{r}{2^{n-1}}. \end{aligned}$$

This implies that $\lim_{n,n+m \to \infty} p(x_n, x_{n+m}) = \lim_{n \to \infty} p(x_n, x_n)$. Since (X, p) is bounded, we have

$$\lim_{n,n+m \to \infty} p(x_n, x_{n+m}) < \infty.$$

So $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X, and since X is complete, it converges to $y \in X$. Since $y \in X$, there exists $\lambda_0 \in \Lambda$ such that $y \in G_{\lambda_0}$. Because G_{λ_0} is open, it contains $B_p(y,\delta)$ for some $\delta > 0$. Choose n large enough such that $\widetilde{p}(y,x_n) < \frac{\delta}{2}$ and $\varepsilon_n < \frac{\delta}{2}$. Then, for any $x \in X$ such that $\widetilde{p}(x_n,x) < \frac{\delta}{2}$ (namely $x \in B_p(x_n,\varepsilon_n)$), we have

$$\widetilde{p}(y,x) \le \widetilde{p}(y,x_n) + \widetilde{p}(x_n,x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

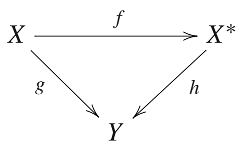
so that $x \in B_p(y,\delta)$, namely $B_p(x_n,\varepsilon_n) \subseteq B_p(y,\delta)$. Therefore, $B_p(x_n,\varepsilon_n)$ admits a finite subcover, namely by the open set G_{λ_0}. This contradicts the fact that each $B_p(x_n,\varepsilon_n)$ cannot be covered by finite many elements of $\{G_\lambda\}$). Hence, (X,p) is compact. □

In [5], the authors first introduce a stronger convergence of sequences in partial metric spaces, that is, a sequence $\{x_n\}$ in X converges in (X,p) if there is $x \in X$ such that $p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n\to\infty} p(x_n,x_n)$. Then, with this definition, the authors obtained for every partial metric space (X,p), a complete partial metric space (X^*,p^*) and an isometric embedding $f : X \to X^*$. The complete partial metric space X^* is called the completion of X. In the end of this section, we will show that the completions of partial metric spaces defined in [5] are in fact complete reflections of partial metric spaces in categorical sense.

Definition 4.10 ([5]) Let (X,p) be a PMS and Y be a subset of X. Y is called to be symmetrically dense in X if for any $x \in X$ and any $\varepsilon > 0$, there is $y \in Y$ such that $y \in B_p(x,\varepsilon)$ and $x \in B_p(y,\varepsilon)$.

Theorem 4.11 *The assignment $X \mapsto X^*$ defines a functor which is in fact a reflective functor from the category of partial metric spaces and uniform continuous mappings to the category of complete partial metric spaces and uniform continuous mappings.*

Proof Let (X,p) be a partial metric space, and let $f : X \to X^*$ be the completion of X. It suffices to show that for every complete partial metric space Y and a uniform continuous function $g : X \to Y$, there exists a unique uniform continuous mapping $h : X^* \to Y$ such that $g = h \circ f$.



Following [5], we know that $f(X)$ is symmetrically dense subset of X^*. By definition of symmetrically dense, for any $x^* \in X^*$, $\forall n \in \mathbb{N}$, there is $f(x_n) \in f(X)$, such that $f(x_n) \in B_p(x^*,\frac{1}{n})$ and $x^* \in B_p(f(x_n),\frac{1}{n})$. Thus, $\{f(x_n)\}$ is a Cauchy sequence in X^*, and $f(x_n) \to x^*$. Since f is an isometric embedding and g is uniform continuous, hence preserving Cauchy sequences, it follows that $\{g(x_n)\}$ is a Cauchy sequence in Y. By completeness of Y, we define $h(x^*) = \lim_{n\to\infty} g(x_n)$.

Claim 1 h is well defined.

If there is another sequence $\{x'_n\}$ in X such that $f(x'_n)$ is a Cauchy sequence, $f(x'_n) \to x^*$. To show the uniqueness of the limit, let $y' = \lim_{n\to\infty} g(x'_n)$. Since g is uniform continuous, then for any $\varepsilon > 0$, there exists an $\delta > 0$, such that $\widetilde{p_Y}(g(x_n), g(x'_n)) < \varepsilon$ whenever $\widetilde{p_X}(x_n, x'_n) < \delta$. For this $\delta > 0$, since $f(x_n) \to x^*$, $f(x'_n) \to x^*$, there is an $N \in \mathbb{N}$, such that $\widetilde{p_{X^*}}(f(x_n), f(x'_n)) < \delta$ when $n > N$. Thus, $\widetilde{p_X}(x_n, x'_n) < \delta$ since f is an isometric embedding mapping. Write $y = \lim_{n\to\infty} g(x_n)$. Then, $\widetilde{p_Y}(y, y') \leq \widetilde{p_Y}(y, g(x_n)) + \widetilde{p_Y}(g(x_n), g(x'_n)) + \widetilde{p_Y}(g(x'_n), y') \leq 3\varepsilon$. It follows that $\widetilde{p_Y}(y, y') = 0$; hence, $p_Y(y, y') = p(y, y)$. Similarly, we have $p_Y(y, y') = p(y', y')$. Therefore, $y = y'$.

Claim 2 h is uniform continuous.

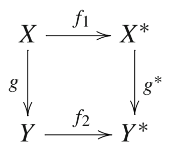
Let $a, b \in X^*$ and $\varepsilon > 0$. Since $g(x)$ is uniform continuous, then for the above ε, there is a $\delta > 0$, such that $\widetilde{p_Y}(g(x), g(y)) < \varepsilon$ whenever $\widetilde{p_X}(x, y) < \delta$. Let $f(x_n) \to a \in X^*$, $f(y_n) \to b \in X^*$. If $\widetilde{p_{X^*}}(a, b) < \delta$, then there is an $N \in \mathbb{N}$, such that $\widetilde{p_X}(x_n, y_n) = \widetilde{p_{X^*}}(f(x_n), f(y_n)) \leq \widetilde{p_{X^*}}(f(x_n), a) + \widetilde{p_{X^*}}(a, b) + \widetilde{p_{X^*}}(b, f(y_n)) < \delta$. Since $h(x^*) = \lim_{n\to\infty} g(x_n)$ and g is uniform continuous, we have

$$\begin{aligned}\widetilde{p_Y}(h(a), h(b)) &\leq \widetilde{p_Y}(h(a), g(x_n)) + \widetilde{p_Y}(g(x_n), g(y_n)) + \widetilde{p_Y}(g(y_n), h(b)) \\ &\leq \varepsilon + \varepsilon + \varepsilon \leq 3\varepsilon.\end{aligned}$$

Claim 3 $g = h \circ f$, and h is unique.

Otherwise, we suppose that there is an $h' : X^* \to Y$, such that $g = h \circ f = h' \circ f$. Then, for any $x^* \in X^*$, there is an $x_n \in X$, such that $f(x_n) \to x^*$. Thus, $\lim_{n\to\infty} g(x_n) = h(\lim_{n\to\infty} f(x_n)) = h'(\lim_{n\to\infty} f(x_n))$. Hence, $h(x^*) = h'(x^*)$. □

Corollary 4.12 *Let X, Y be two PMS, $f_1 : X \to X^*$ and $f_2 : Y \to Y^*$ be the completions of X and Y, respectively. If $g : X \to Y$ is an uniform continuous mapping, then f can be uniquely extended to an unform continuous mapping $g^* : X^* \to Y^*$ such that the following diagram commutes.*



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