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Gallagherian Prime Geodesic Theorem in Higher Dimensions

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Abstract
Using the Gallagher–Koyama approach, we reduce the exponent in the error term of the prime geodesic theorem for real hyperbolic manifolds with cusps.

Keywords Hyperbolic manifolds · Prime geodesic theorem · Selberg and Ruelle zeta functions

Mathematics Subject Classification 11M36 · 11F72 · 58J50

1 Introduction

Under the Riemann hypothesis, Gallagher [15] improved the error term in the prime number theorem from $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right)$ to $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log\log x)^2\right)$ outside a set of finite logarithmic measure, where ψ is the Chebyshev counting function over powers of primes $\psi(x) = \sum_{p^k \leq x} \log p$.

Having in mind that the Selberg zeta function for compact or generic hyperbolic surfaces satisfies an analogue of Riemann's hypothesis, Koyama [19] transferred Gallagher's method to that setting. A possible motivation for doing so comes from the fact that the best estimate up to now of the error term in the prime geodesic theorem on compact Riemann surfaces is still Randol's $O\left(\frac{x^{\frac{3}{4}}}{\log x}\right)$ (see [21]). Namely, Riemann's hypothesis gives a rise to the expectation that the exponent $\frac{3}{4}$ could be decreased to $\frac{1}{2}$.

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However, the abundance of zeros of the Selberg zeta function creates major obstacles in achieving such a result.

After indicating a possibility of stronger implications of the Gallagher–Koyama approach in [1], the first author was able to replace $\frac{3}{4}$ by $\frac{7}{10}$ in a Gallagherian prime geodesic theorem for Riemann surfaces (corresponding to a general case of a cofinite Fuchsian group of the first kind with finitely many inequivalent parabolic elements and a unitary multiplier system of an arbitrary weight on it) [4]. On the modular group $PSL(2,\mathbb{Z})$, the Soundararajan–Young [23] (see also [8]) unconditional exponent $\frac{25}{36}$ is reduced this way to $\frac{2}{3}$ in a Gallagherian PGT (see [2]). Moreover, under the generalized Lindelöf hypothesis, this method yields $\frac{5}{8}$ in the latter case, [5]. In dimension 3, our path substitutes the Sarnak unconditional exponent $\frac{5}{3}$ on cofinite Kleinian groups, resp. the Balkanova–Frolenkov [9] unconditional exponent $\frac{3}{2}+\frac{103}{1024}$ on the Picard group $PSL(2,\mathbb{Z}[i])$, by $\frac{21}{13}$, resp. $\frac{8}{5}$, outside a set of finite logarithmic measure (see [4]).

The approach via the Kuznetsov trace formula and the second moment technique brought a further progress in dimensions 2 and 3, to which we turn attention in the closing section.

In the present paper, we improve the exponent in the error term of the prime geodesic theorem for higher-dimensional real hyperbolic manifolds with cusps outside a set of finite logarithmic measure.

2 PGT in *d* Dimensions

Let Γ be a discrete cofinite torsion free subgroup of $G = \mathrm{SO}_0\,(d,1)$ satisfying the condition $\Gamma \cap P = \Gamma \cap N(P)$ for $P \in \mathfrak{P}_\Gamma$, where $\mathfrak{P}_\Gamma$ is the set of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups in G and $N(P)$ is the unipotent part of P. For K a maximal compact subgroup of G, the manifold $X_\Gamma = \Gamma\backslash G/K$ is a d-dimensional real hyperbolic manifold with cusps.

The Riemannian metric over X_Γ induced from the Killing form is normalized so that the sectional curvature of X_Γ equals -1. By $\pi_\Gamma(x)$, we denote the number of prime geodesics C_γ of length $l_\gamma \le \log x$ on X_Γ.

Park's paper [20] is devoted to a refinement of the prime geodesic theorem, due to Gangolli [16] and DeGeorge [13] in the compact case and to Gangolli–Warner [17] in the finite volume case. It is known that if the dimension d of a manifold is larger than 3, then the use of the Ruelle zeta is required instead of the Selberg zeta (see [20, p. 90]). As explained in [3], the correct form of PGT with an error term in d dimensions is given by the theorem below. This is the effect of additional term $O\left(x^{2d_0-1}h\right)$ that was missing in reduction from $\psi_{2d_0}(x)$ to $\psi_0(x)$ [20, relation (3.2) on p. 101].

Theorem 1 *Let X_Γ be as above. Then*

$$\pi_\Gamma\,(x) = \sum_{\frac{4d_0^2+d_0}{2d_0+1}<s_n(k)\le 2d_0} (-1)^k\, li\left(x^{s_n(k)}\right) + O\left(x^{\frac{4d_0^2+d_0}{2d_0+1}}\,(\log x)^{-1}\right)$$

as $x \to +\infty$, *where* $d_0 = \frac{d-1}{2}$, $(s_n(k) - k)(2d_0 - k - s_n(k))$ *is a small eigenvalue in* $\left[0, \frac{3}{4}d_0^2\right]$ *of* Δ_k *on* $\pi_{\sigma_k,\lambda_n(k)}$ *with* $s_n(k) = d_0 + i\lambda_n(k)$ *or* $s_n(k) = d_0 - i\lambda_n(k)$ *in* $\left(\frac{3}{2}d_0, 2d_0\right]$, Δ_k *is the Laplacian acting on the space of k-forms over* X_Γ *and* $\pi_{\sigma_k,\lambda_n(k)}$ *is the principal series representation.*

Notice that the error term in Theorem 1 coincides with Randol's $O\left(\frac{x^{\frac{3}{4}}}{\log x}\right)$ for compact Riemann surfaces ($d = 2$). For $d = 3$, the exponent is also in accordance with Sarnak's $\frac{5}{3}$ in PGT for groups of the form $\Gamma = \Gamma_D = PSL(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$ of class number one, [22].

Now, we proceed to our main result.

3 Gallagherian PGT in *d* Dimensions

Theorem 2 *Let* X_Γ *be a d-dimensional manifold with cusps as in Sect. 2. For* $\varepsilon > 0$, *there exists a set E of finite logarithmic measure such that*

$$\pi_\Gamma(x) = \sum_{\alpha_d < s_n(k) \le 2d_0} (-1)^k \, li\left(x^{s_n(k)}\right) + O\left(x^{\alpha_d} (\log x)^{\beta_d - 1} (\log\log x)^{\beta_d + \varepsilon}\right)$$

$$(x \to \infty, x \notin E),$$

where $\alpha_d = (d-1)\left(1 - \frac{2d+1}{4d^2+2}\right)$, $\beta_d = \frac{d-1}{2d^2+1}$.

Proof Let Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ denote the set of the Γ-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ. Set $\Lambda(\gamma) = l_{\gamma_0}$, where $\gamma = \gamma_0^{j(\gamma)}$, $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$, $j(\gamma) \in \mathbb{N}$.

Higher-order counting functions $\psi_n(x)$ are defined recursively by $\psi_{n,\Gamma}(x) = \int_0^x \psi_{n-1,\Gamma}(t)dt$ for $n \in \mathbb{N}$, where $\psi_{0,\Gamma}(x) = \sum_{\gamma \in \Gamma_{\mathrm{h}}, l_\gamma \le \log x} \Lambda(\gamma)$.

We shall derive the asymptotics of $\psi_{0,\Gamma}(x)$ from $\psi_{d,\Gamma}(x)$. For that purpose, one introduces the functions

$$\Delta_d^+ f(x) = \int_x^{x+h} \int_{x_{2d_0}}^{x_{2d_0}+h} \cdots \int_{x_1}^{x_1+h} f^{(d)}(x_0)dx_0 \ldots dx_{2d_0}$$

and

$$\Delta_d^- f(x) = \int_{x-h}^{x} \int_{x_{2d_0}-h}^{x_{2d_0}} \cdots \int_{x_1-h}^{x_1} f^{(d)}(x_0)dx_0 \ldots dx_{2d_0}$$

for some constant h to be specified later on.

Now, according to [6, relation (6) on p. 370], we have

$$\psi_{d,\Gamma}(x) = \sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\dots(s_n(k)+d)} + \sum_{s_n(0)=d_0\pm i\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\dots(s_n(0)+d)}. \tag{1}$$

We shall split the sum $\sum\limits_{s_n(0)=d_0\pm\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)}$ in explicit formula (1) for $\psi_{d,\Gamma}(x)$ into three parts:

$$\sum_{|\lambda_n(0)|\leq Y} + \sum_{Y<|\lambda_n(0)|\leq W} + \sum_{|\lambda_n(0)|>W}. \tag{2}$$

To prove that the exponent $\frac{4d_0^2+d_0}{2d_0+1}$ from Theorem 1 can be replaced by α_d outside a set of finite logarithmic measure, we direct our attention to a better control of the second sum.

Let E_j denote the set

$$\left\{x \in \left[e^j, e^{j+1}\right) : \left|\sum_{Y<|\lambda_n(0)|\leq W} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)}\right| > x^\alpha (\log x)^\beta (\log\log x)^{\beta+\varepsilon}\right\}.$$

Then

$$\mu^\times E_j = \int_{E_j} \frac{dx}{x} \ll \frac{e^{2(d+d_0-\alpha)j}}{j^{2\beta}(\log j)^{2\beta+2\varepsilon}} \times \int_{e^j}^{e^{j+1}} \left|\sum_{Y<|\lambda_n(0)|\leq W} \frac{x^{i\lambda_n(0)}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)}\right|^2 \frac{dx}{x}.$$

The Gallagher lemma [14] implies that the last integral is dominated by

$$\int_{-\infty}^{\infty} \left(\sum_{\substack{t\leq|\lambda_n(0)|<t+1 \\ Y<|\lambda_n(0)|\leq W}} \frac{1}{|s_n(0)|^{d+1}}\right)^2 dt. \tag{3}$$

By the Weyl law,

$$\sum_{\substack{t\le|\lambda_n(0)|<t+1\\ Y<|\lambda_n(0)|\le W}} \frac{1}{|s_n(0)|^{d+1}} = O\left(\frac{1}{t^2}\right).$$

Hence, the integral in (3) is $O\left(\frac{1}{Y^3}\right)$. Therefore,

$$\mu^\times E_j \ll \frac{e^{2(d+d_0-\alpha)j}}{Y^3 j^{2\beta} (\log j)^{2\beta+2\varepsilon}}.$$

Taking

$$Y \sim e^{\frac{2}{3}(d+d_0-\alpha)j} j^{\frac{1-2\beta}{3}} (\log j)^{\frac{1-2\beta}{3}}, \tag{4}$$

we get $\mu^\times E_j \ll \frac{1}{j(\log j)^{2\varepsilon}}$. Thus, the set $\cup E_j$ has a finite logarithmic measure.

In the process of deriving the asymptotics of $\psi_{0,\Gamma}$ through $\psi_{0,\Gamma}(x) \le h^{-d}\Delta_d^+ \psi_{d,\Gamma}(x)$, the second sum in (2) will lead to the term

$$O\left(\frac{x^\alpha (\log x)^\beta (\log\log x)^{\beta+\varepsilon}}{h^d}\right) \tag{5}$$

outside a set of finite logarithmic measure. The first sum in (2) will give us

$$O\left(x^{d_0} Y^{2d_0}\right) \tag{6}$$

because of

$$h^{-d}\Delta_d^+ \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)} = O\left(\min\left(x^{d_0}|s_n(0)|^{-1}, h^{-d}|s_n(0)|^{-(d+1)} x^{d+d_0}\right)\right)$$

(see [6, relation (8) on p. 370]).

The third sum in (2) yields

$$O\left(\frac{x^{d+d_0}}{h^d W}\right). \tag{7}$$

Relations (5), (6) and (7) together with

$$h^{-d}\Delta_d^+ \frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+d)} = \frac{x^{s_n(k)}}{s_n(k)} + O\left(x^{2d_0-1}h\right)$$

will give us

$$\psi_{0,\Gamma}(x) \leq \sum_{s_n(k)\in(d_0,2d]} \frac{x^{s_n(k)}}{s_n(k)} + O\left(x^{2d_0-1}h\right) + O\left(x^{d_0}Y^{2d_0}\right)$$
$$+O\left(\frac{x^{\alpha}(\log x)^{\beta}(\log\log x)^{\beta+\varepsilon}}{h^d}\right) + O\left(\frac{x^{d+d_0}}{h^d W}\right). \tag{8}$$

One will have $x^{2d_0-1}h = x^{d_0}Y^{2d_0}$ if

$$h = x^{1-d_0}Y^{2d_0}. \tag{9}$$

On the other hand, $x^{2d_0-1}h \leq \frac{x^{\alpha}(\log x)^{\beta}(\log\log x)^{\beta+\varepsilon}}{h^d}$ for

$$h^{d+1} = x^{\alpha-2d_0+1}(\log x)^{\beta}(\log\log x)^{\beta}. \tag{10}$$

We already have

$$Y^{2d_0} \sim x^{\frac{4d_0}{3}(d+d_0-\alpha)}(\log x)^{\frac{2d_0}{3}(1-2\beta)}(\log\log x)^{\frac{2d_0}{3}(1-2\beta)} \tag{11}$$

by (4).

Combining (9), (10) and (11) and looking at the exponents of $\log x$, we arrive at $\frac{\beta}{d+1} = \frac{2d_0}{3}(1-2\beta)$, i.e., $\beta = \frac{d^2-1}{2d^2+1}$. After insertion of h and Y in (8), the exponent of $\log x$ on the right hand side of (8) will be

$$\beta_d := \frac{\beta}{d+1} = \frac{d-1}{2d^2+1}.$$

Analogous calculations related to exponent of x in (9), (10) and (11) yield $\alpha = d + d_0 - \frac{dd_0}{1+\frac{4}{3}d_0(d+1)}$, what will give us $\alpha_d = (d-1)\left(1-\frac{2d+1}{4d^2+2}\right)$ in the asymptotics of $\psi_{0,\Gamma}(x)$.

Thus,

$$\psi_{0,\Gamma}(x) \leq \sum_{\alpha_d<s_n(k)\leq 2d_0} (-1)^k\, li\left(x^{s_n(k)}\right) + O\left(x^{\alpha_d}(\log x)^{\beta_d}(\log\log x)^{\beta_d+\varepsilon}\right)$$
$$(x\to\infty, x\notin E),$$

where $\alpha_d = (d-1)\left(1-\frac{2d+1}{4d^2+2}\right)$, $\beta_d = \frac{d-1}{2d^2+1}$.

In a similar way,

$$\psi_{0,\Gamma}(x) \geq h^{-d}\Delta_d^{-}\psi_{d,\Gamma}(x) = \sum_{\alpha_d<s_n(k)\leq 2d_0} (-1)^k\, li\left(x^{s_n(k)}\right)$$
$$+O\left(x^{\alpha_d}(\log x)^{\beta_d}(\log\log x)^{\beta_d+\varepsilon}\right)(x\to\infty, x\notin E).$$

It is well known that the proved relation for $\psi_{0,\Gamma}$ implies the assertion of Theorem 2 (see, e.g., [20, p. 102]). $\square$

4 Remark on Lower Dimensions

In the case of dimensions $d \geq 4$, the bound obtained in Theorem 2 is the best estimate at present. Here, we used the Ruelle zeta and the explicit formula for $\psi_{d,\Gamma}$.

In lower dimensions, the Selberg zeta approach and the explicit formula for $\psi_{d,\Gamma}$ would also yield $\alpha_2 = \frac{13}{18}$ and $\alpha_3 = \frac{31}{19}$ in accordance with Theorem 2. However, one is in position to derive the asymptotics of $\psi_{0,\Gamma}(x)$ from $\psi_{d-1,\Gamma}(x)$ instead of reaching for $\psi_{d,\Gamma}(x)$ in this setting. This gives the already mentioned exponents $\frac{7}{10}$ for Riemann surfaces and $\frac{21}{13}$ for hyperbolic 3-manifolds. The arithmetic features of $PSL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z}[i])$ allow us to work directly with the explicit formula for $\psi_{0,\Gamma}$ and get the exponents $\frac{2}{3}$ resp. $\frac{8}{5}$.

Now, more powerful tools are available in the latter cases. By strengthening the Cherubini–Guerreiro [12] result on the square mean of the PGT error term for $PSL(2,\mathbb{Z})$, Balog, Biró, Harcos and Maga [10] deduced the exponent $\frac{7}{12}$ outside a set of finite logarithmic measure. Analogously, building upon the work of Balkanova et al. [7], Chatzakos et al. obtained $\frac{3}{2}$ in [11] for $PSL(2,\mathbb{Z}[i])$ (see also Kaneko [18]). Further discussion on the reach of the Gallagher–Koyama method versus second moment approach can be found in [4, Sect. 4].

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