Bull. Malays. Math. Sci. Soc. (2020) 43:2799–2816
https://doi.org/10.1007/s40840-019-00838-1

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Characterizations of the Reflection Operators

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Received: 27 May 2019 / Revised: 8 September 2019 / Published online: 21 September 2019

Abstract
An operator P is said to be reflective if $P^* = P$ and $P^2 = I$. In this paper, we study the spectral properties of reflection operators and obtain a matrix representation of the reflection operator pair (P, Q). Some related properties of reflection operator pair (P, Q) are given.

Keywords Reflection operator · Spectrum · Matrix representation · Unitary equivalence

Mathematics Subject Classification 15A09 · 47A05

1 Introduction and Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional, complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. The set of all the unitary operators on $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, let A^*, $\sigma_P(A)$, $\sigma(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the point spectrum, the spectrum, the range and the null space of A, respectively. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is densely defined if the domain of A is a dense subset of $\mathcal{H}$ and the range of A is contained within $\mathcal{K}$. A is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. $I_{\mathcal{M}}$ denotes the identity onto $\mathcal{M}$ or I if there is no confusion. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a reflection operator if $P^* = P$ and $P^2 = I$. Let P and Q be two reflection operators. Throughout this paper, we assume that neither of P, Q is I or $-I$. The term "subspace" always means a closed linear manifold.

Communicated by Mohammad Sal Moslehian.

Supported by the National Natural Science Foundation of China under Grant 11671261.

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We say $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection if $P^2 = P = P^*$ and there are a plenty of researches about two orthogonal projections [1–5,12]. The most primitive representation of two orthogonal projections is usually referred to as Halmos' two projections theorem and sometimes also as the CS decomposition [9,10].

The reflection operators have much similarities with the orthogonal projections [6]. Furthermore, the reflection operators have some special properties different from the orthogonal projections, which attracts our attention. In [13], the authors establish an explicit characterization of the spectrum and spectral radius estimates for the reflection operator acting on L_p spaces on an infinite angle in two dimensions. The aim of this paper is to study the spectral properties and the matrix representations of the reflection operators on Hilbert spaces. Some related properties of reflection operator pair (P, Q) are also given.

As we know, if there exists a $U \in \mathcal{U}(\mathcal{H})$ such that $PU = UQ$ $(UP = QU)$, we say U is an inner (outer) intertwining operator of P respect to Q and all the inner (outer) intertwining operators of P respect to Q are denoted by $inn_Q(P)$ $(out_Q(P))$, respectively. If there exists a $U \in \mathcal{U}(\mathcal{H})$ such that $PU = UQ$ and $UP = QU$, we say U is an intertwining operator of P respect to Q and P, Q are unitary equivalence. The set of all the intertwining operators of P respect to Q is denoted by $int_Q(P)$ [4,8]. The related researches on unitary equivalence of two orthogonal projections can also be found in [3,5,11–17]. Based on the matrix representation of reflection operators, we study the unitary equivalence of two reflection operators. The general explicit descriptions for intertwining operators of two reflection operators are established. The paper mainly contains two parts. In Sect. 2, we investigate some spectral properties of reflection operators. In Sect. 3, we present the block operator matrix representations of the reflective operators and obtain some properties of the combinations of reflection operators by using their matrix representations.

2 The Spectral Properties of Reflection Operators

First, we characterize the spectrum of reflection operators. If $P^2 = I, \sigma(P) \subseteq \{e^{\frac{2k\pi}{2}i} : k = 0, 1\} = \{1, -1\}$ by the spectral mapping theorem [7]. Observing that if $\lambda \in \sigma(P)$, then $\lambda^2 = 1$. This shows that each $\lambda \in \sigma(P)$ is a simple root of the equation $\lambda^2 = 1$.

Theorem 2.1 *Let $P \in \mathcal{B}(\mathcal{H})$. Then P is a reflection operator ($P^* = P$ and $P^2 = I$) if and only if $P = I_{\mathcal{M}} \oplus -I_{\mathcal{M}^\perp}$, where $\mathcal{M} = \mathcal{R}(I + P)$.*

Theorem 2.2 *Let P and Q be the reflection operators.*

(i) *If $\lambda \in \mathbb{C}\backslash\{0, 2, -2\}$, then $\lambda \in \sigma(P - Q) \Longleftrightarrow 3 - \lambda^2 \in \sigma(P + Q + PQ)$.*
(ii) $\sigma[2(P - Q)^2] \cup \{0\} = \sigma[(I - P)(I + Q)(I - P)] \cup \sigma[(I + P)(I - Q)(I + P)] \cup \{0\}$.

Proof (i) If $\lambda \in \mathbb{C}\backslash\{0, 2, -2\}$, then $\lambda \pm 1 \neq 1$ or -1. By Theorem 2.1, $(\lambda - 1)I + P$ and $(\lambda + 1)I - Q$ are invertible. Note that

$$\begin{aligned}&[(\lambda - 1)I + P][\lambda I - (P - Q)][(\lambda + 1)I - Q]\\&\quad= [(\lambda^2 - \lambda - 1)I + P(I + Q) + (\lambda - 1)Q][(\lambda + 1)I - Q]\\&\quad= \lambda[(\lambda^2 - 3)I + P + Q + PQ].\end{aligned}$$

We get that $\lambda I - (P - Q)$ is invertible if and only if $(\lambda^2 - 3)I + P + Q + PQ$ is invertible. Hence, the result holds.

(ii) Since $2(P - Q)^2 = (I - P)(I + Q)(I - P) + (I + P)(I - Q)(I + P)$, we get that

$$\begin{aligned}&\lambda[\lambda I - 2(P - Q)^2]\\&\quad= \lambda^2 I - 2\lambda(P - Q)^2\\&\quad= \lambda^2 I - \lambda(I - P)(I + Q)(I - P) - \lambda(I + P)(I - Q)(I + P)\\&\quad= [\lambda I - (I - P)(I + Q)(I - P)][\lambda I - (I + P)(I - Q)(I + P)].\end{aligned}$$

Hence, the result holds. □

The following results are concerned with the commutator of two reflection operators.

Theorem 2.3 *Let P and Q be the reflection operators.*

(i) *There exists $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(PQ + QP)$ if and only if there exists $\mu \in \sigma(P - Q)$ such that $\lambda = 2 - \mu^2$.*
(ii) *There exists $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(PQ - QP)$ if and only if there exists $\mu \in \sigma(P - Q)$ such that $\lambda^2 = \mu^4 - 4\mu^2$.*

Proof Note that

$$PQ + QP = 2I - (P - Q)^2, \quad (PQ - QP)^2 = (P - Q)^4 - 4(P - Q)^2.$$

Then there exists $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(PQ+QP)$ if and only if $\lambda \in \sigma[2I-(P-Q)^2]$ if and only if there exists $\mu \in \sigma(P - Q)$ such that $\lambda = 2 - \mu^2$.

There exists $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(PQ - QP)$ if and only if $\lambda^2 \in \sigma[(P - Q)^4 - 4(P - Q)^2]$ if and only if there exists $\mu \in \sigma(P - Q)$ such that $\lambda^2 = \mu^4 - 4\mu^2$. □

Theorem 2.4 *Let P and Q be the reflection operators and $c_1, c_2 \in \mathbb{C}\backslash\{0\}$. Then $\lambda \in \sigma[(P - Q)^2]$ if and only if there exists $\mu \in \mathbb{C}$ such that $\mu \in \sigma(c_1P + c_2Q)$ and $\mu^2 = (c_1 + c_2)^2 - c_1c_2\lambda$. If there exist $\alpha, \beta \in \mathbb{R}$ such that $\sigma[(P - Q)^2] \subseteq [\alpha, \beta]$, then*

$$|\mu| \leq \max\left\{\sqrt{|(c_1 + c_2)^2 - c_1c_2\alpha|}, \sqrt{|(c_1 + c_2)^2 - c_1c_2\beta|}\right\}$$

for all $\mu \in \sigma(c_1P + c_2Q)$.

Proof Note that

$$\begin{aligned}\lambda I-(P-Q)^2 &= \lambda I+PQ+QP-2I\\ &= \tfrac{c_1c_2(PQ+QP)+2c_1c_2I}{c_1c_2}-4I+\lambda I\\ &= \tfrac{(c_1P+c_2Q)^2-[(c_1-c_2)^2-(\lambda-4)c_1c_2]I}{c_1c_2}\\ &= \tfrac{(c_1P+c_2Q)^2-[(c_1+c_2)^2-c_1c_2\lambda]I}{c_1c_2}.\end{aligned}$$

Hence, $\lambda\in\sigma[(P-Q)^2]$ if and only if there exists $\mu\in\mathbb{C}$ such that $\mu\in\sigma(c_1P+c_2Q)$ and $\mu^2=(c_1+c_2)^2-c_1c_2\lambda$.

If $\mu\in\sigma(c_1P+c_2Q)$ and $\lambda\in\sigma[(P-Q)^2]\subseteq[\alpha,\beta]$, then $\mu^2=(c_1+c_2)^2-c_1c_2\lambda$. We get $\lambda=\frac{(c_1+c_2)^2-\mu^2}{c_1c_2}\in[\alpha,\beta]$. Note that $f(x)=z_1x+z_2$, $\forall x\in\mathbb{R}$ is a line in complex plane for complex numbers z_1, z_2. We have μ^2 is a segment in $\mathbb{C}$ with boundary points $(c_1+c_2)^2-c_1c_2\alpha$ and $(c_1+c_2)^2-c_1c_2\beta$ if $\lambda\in[\alpha,\beta]$. Hence, for all $\mu\in\sigma(c_1P+c_2Q)$,

$$|\mu|\le\max\left\{\sqrt{|(c_1+c_2)^2-c_1c_2\alpha|},\sqrt{|(c_1+c_2)^2-c_1c_2\beta|}\right\}.$$

□

Observing that $\sigma[(P-Q)^2]\subseteq[0,\|P-Q\|^2]\subseteq[0,4]$ for reflection operators P and Q, we derive the following results.

Corollary 2.1 *Let P and Q be the reflection operators and $c_1, c_2\in\mathbb{C}\backslash\{0\}$. Then*

$$|\mu|\le max\left\{|c_1+c_2|,\sqrt{|(c_1+c_2)^2-c_1c_2\|P-Q\|^2|}\right\}$$

for every $\mu\in\sigma(c_1P+c_2Q)$.

Corollary 2.2 *Let P and Q be the reflection operators and $c_1, c_2\in\mathbb{C}\backslash\{0\}$. Then*

$$|\mu|\le max\{|c_1+c_2|,|c_1-c_2|\}$$

for every $\mu\in\sigma(c_1P+c_2Q)$.

For convenience, we define a subset Λ of $\mathbb{C}^2$ by

$$\Lambda=\left\{(c_1,c_2)\in\mathbb{C}^2 : c_1\neq 0, c_2\neq 0 \text{ and } c_1+c_2\neq 0\right\}.$$

Theorem 2.5 *Let P and Q be the reflection operators and let $(c_1,c_2)\in\Lambda$. Then*

$$\dim\mathcal{N}[(c_1+c_2)I+c_1P+c_2Q]=\dim\{\mathcal{N}[(I-P)(I+Q)]\cap\mathcal{N}(I+P)\}.$$

Proof If $x \in \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q]$, then

$$\begin{aligned}
&(I-P)(I+Q)(I-P)x \\
&= c_1^{-1}(I-P)(I+Q)[(c_1+c_2)I + c_1I + c_2Q - (c_1+c_2)I - c_1P - c_2Q]x \\
&= c_1^{-1}(I-P)(I+Q)[(c_1+c_2)I + c_1I + c_2Q]x \\
&= 2c_1^{-1}(I-P)[(c_1+c_2)I + (c_1+c_2)Q]x \\
&= 2(c_1+c_2)(c_1c_2)^{-1}(I-P)(c_2I + c_2Q)x \\
&= 2(c_1+c_2)(c_1c_2)^{-1}(I-P)[(c_1+c_2)I + c_1P + c_2Q]x \\
&= 0.
\end{aligned}$$

Since $(I-P)x \in \mathcal{N}(I+P)$, we conclude that

$$(I-P)\mathcal{N}[(c_1+c_2)I + c_1P + c_2Q] \subseteq \mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P).$$

If $x \in \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q]$ and $(I-P)x = 0$, then $x = Px$ and

$$(c_1+c_2)(I+Q)x = \frac{1}{2}(I+Q)[(c_1+c_2)I + c_1P + c_2Q]x = 0.$$

So $(I+Q)x = 0$. It follows that $[(c_1+c_2)I + c_1P + c_2Q]x = c_1(I+P)x = 0$. Hence, $x = 0$ since $x = -Px = Px$. We get that $I-P$ embeds $\mathcal{N}[(c_1+c_2)I + c_1P + c_2Q]$ injectively into $\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)$. Thus,

$$\dim \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q] \leq \dim \{\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)\}. \tag{1}$$

On the other hand, if $x \in \mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)$, then $x + Px = 0$ and $x + Qx = Px + PQx$. Note that

$$\begin{aligned}
&[(c_1+c_2)I + c_1P + c_2Q]\left(2c_1c_2^{-1}I + I - Q\right)x \\
&= [c_1(I+P) + c_2(I+Q)]\left(2c_1c_2^{-1}I + I - Q\right)x \\
&= 2c_1^2c_2^{-1}(I+P)x + 2c_1(I+Q)x + c_1(I+P)(I-Q)x \\
&= 2c_1(I+Q)x + 2c_1(P-Q)x \\
&= 2c_1(I+P)x \\
&= 0.
\end{aligned}$$

We get

$$\begin{aligned}
&\left(2c_1c_2^{-1}I + I - Q\right)\{\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)\} \\
&\subseteq \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q].
\end{aligned}$$

If $x \in \mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)$ and $(2c_1c_2^{-1}I+I-Q)x = 0$, then $(I+P)x = 0$, $(I-P)(I+Q)x = 0$ and $(I+Q)x = 2(c_1c_2^{-1}I+I)x$. From $(I-P)(I+Q)x = 0$, we get $(I+P)(I+Q)x = 2(I+Q)x$. Hence,

$$2\left(c_1c_2^{-1}I + I\right)x = (I+Q)x = \frac{1}{2}(I+P)(I+Q)x = \left(c_1c_2^{-1}I + I\right)(I+P)x = 0.$$

We get $x = 0$ and thus $2c_1c_2^{-1}I + I - Q$ embeds $\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)$ injectively into $\mathcal{N}[(c_1+c_2)I + c_1P + c_2Q]$. Hence,

$$\dim \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q] \geq \dim \{\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)\}. \tag{2}$$

By (1) and (2), we get that

$$\dim \mathcal{N}[(c_1+c_2)I + c_1P + c_2Q] = \dim \{\mathcal{N}[(I-P)(I+Q)] \cap \mathcal{N}(I+P)\}. \qquad \square$$

3 Representations of reflection operators

As we know, if $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ has the operator matrix form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then $A \geq 0$ if and only if $A_{ii} \geq 0$, $i = 1, 2$, $A_{21} = A_{12}^*$ and there exists a contraction operator D from $\mathcal{K}$ into $\mathcal{H}$ such that $A_{12} = A_{11}^{1/2} D A_{22}^{1/2}$, where $A_{ii}^{1/2}$ is the positive square root of A_{ii}, $i = 1, 2$ [9]. Recently, Moslehian, Kian and Xu used the Douglas theorem on equivalence of factorization, range inclusion and majorization of operators to characterize the positivity of 2×2 block operator matrices [14]. In this section, we will give detailed block operator matrices representations of reflection operators.

Let P and Q be two reflection operators. Since we assume that neither of P, Q is I or $-I$, by Theorem 2.1, $\mathcal{M} = \mathcal{R}(I+P)$, $\mathcal{M}^\perp = \mathcal{R}(I-P)$, $\mathcal{N} = \mathcal{R}(I+Q)$ and $\mathcal{N}^\perp = \mathcal{R}(I-Q)$ are non-degenerate subspaces. Therefore, the reflection operator P as an operator on $\mathcal{M} \oplus \mathcal{M}^\perp$, and the reflection operator Q as an operator on $\mathcal{N} \oplus \mathcal{N}^\perp$, have the diagonal matrix forms

$$P = I_{\mathcal{M}} \oplus -I_{\mathcal{M}^\perp} \text{ and } Q = I_{\mathcal{N}} \oplus -I_{\mathcal{N}^\perp}, \tag{3}$$

respectively. Denote

$$\begin{aligned} &\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}, \quad \mathcal{H}_3 = \mathcal{M}^\perp \cap \mathcal{N}, \quad \mathcal{H}_5 = \mathcal{M} \cap [\mathcal{H} \ominus (\oplus_{i=1}^4 \mathcal{H}_i)], \\ &\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^\perp, \quad \mathcal{H}_4 = \mathcal{M}^\perp \cap \mathcal{N}^\perp, \quad \mathcal{H}_6 = \mathcal{H} \ominus (\oplus_{i=1}^5 \mathcal{H}_i). \end{aligned} \tag{4}$$

It is clear that $\mathcal{H}_i \perp \mathcal{H}_j$, $j \neq i$ and $1 \leq i, j \leq 6$. The pair $(\mathcal{M}, \mathcal{N})$ of subspaces $\mathcal{M}$ and $\mathcal{N}$ is said to be regular if $\mathcal{H}_i = \{0\}$, $i = 1, 2, 3, 4$. Clearly, $(\mathcal{M}, \mathcal{N})$ is regular if and only if $(\mathcal{M}^\perp, \mathcal{N}^\perp)$ is. We say (P, Q) is a reflective regular pair whenever $(\mathcal{M}, \mathcal{N})$ is a non-trivial regular pair. First, we study the matrix structures of a reflective regular pair (P, Q).

Theorem 3.1 *Let (P, Q) be a reflective regular pair with $\mathcal{M} = \mathcal{R}(I + P)$ and $\mathcal{N} = \mathcal{R}(I + Q)$. Then there exist a selfadjoint contraction $A \in \mathcal{B}(\mathcal{M})$ with $1, -1 \notin \sigma_P(A)$ and a unitary operator $D \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$ such that P and Q have the operator matrix forms*

$$P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} A & (I - A^2)^{1/2} D \\ D^*(I - A^2)^{1/2} & -D^* A D \end{pmatrix} \tag{5}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, respectively.

Proof By Theorem 2.1, the reflection operator P, as an operator on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, has the operator matrix form $P = I \oplus -I$. Let Q have the corresponding matrix form

$$Q = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

with respect to the same space decomposition, where A and C are selfadjoint contractions on $\mathcal{M}$ and $\mathcal{M}^\perp$, respectively. Denote $\widetilde{P} = \frac{1}{2}(I + P)$ and $\widetilde{Q} = \frac{1}{2}(I + Q)$. It is easy to see that $\mathcal{R}(\widetilde{P}) = \mathcal{M}$, $\mathcal{N}(\widetilde{P}) = \mathcal{M}^\perp$, $\mathcal{R}(\widetilde{Q}) = \mathcal{N}$ and $\mathcal{N}(\widetilde{Q}) = \mathcal{N}^\perp$ by (3). From $\widetilde{Q}^2 = \frac{1}{2}(I + Q) = \widetilde{Q}$, one gets that

$$\frac{1}{4}\begin{pmatrix} (I + A)^2 + BB^* & (I + A)B + B(I + C) \\ B^*(I + A) + (I + C)B^* & (I + C)^2 + B^*B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I + A & B \\ B^* & I + C \end{pmatrix}.$$

Comparing the two sides of the above equation, we have

$$\begin{cases} (I + A)^2 + BB^* = 2(I + A), \\ (I + A)B + B(I + C) = 2B, \\ (I + C)^2 + B^*B = 2(I + C). \end{cases} \tag{6}$$

The regularity of $(\mathcal{M}, \mathcal{N})$ implies that $I \pm A$ and $I \pm C$ are injective. $\widetilde{Q} \geq 0$ implies that there exists a contraction D from $\mathcal{M}^\perp$ into $\mathcal{M}$ such that $B = (I + A)^{1/2} D (I + C)^{1/2}$. From the system of equations (6), we get

$$D(I + C)D^* = I - A, \quad AD = -DC, \quad D^*(I + A)D = I - C.$$

It follows that

$$C = -D^*AD, \quad B = (I - A^2)^{1/2} D, \quad D^*D = I_{\mathcal{M}^\perp} \text{ and } DD^* = I_{\mathcal{M}},$$

where $I \pm A$ are injective, i.e., 1 and $-1 \notin \sigma_P(A)$. □

Remark 3.1 If (P, Q) is a reflective regular pair, then

$$\dim \mathcal{M} = \dim \mathcal{M}^\perp = \frac{1}{2} \dim \mathcal{H} \text{ and } \dim \mathcal{N} = \dim \mathcal{N}^\perp = \frac{1}{2} \dim \mathcal{H}$$

since the operator D in Theorem 3.1 is a unitary operator from $\mathcal{M}^{\perp}$ onto $\mathcal{M}$.

It is well known that if $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and A is invertible, then the inverse of T is

$$T^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \tag{7}$$

whenever the Schur complement $S = D - CA^{-1}B$ of A in T is invertible. The expression (7) is called the Banachiewicz–Schur form of operator T and can be found in standard textbooks on linear algebra.

Corollary 3.1 *If (P, Q) is a reflective regular pair having the matrix representations (5), then the following statements hold:*

(i) *$\mathcal{R}(P + Q)$ is dense in $\mathcal{H}$;*
(ii) *$\|P + Q\| = \sqrt{2}\|I + A\|^{1/2}$;*
(iii) *$\mathcal{R}(P + Q)$ is closed if and only if $P + Q$ is invertible if and only if $-1 \notin \sigma(A)$. In this case,*

$$(P + Q)^{-1} = \frac{1}{2}\begin{pmatrix} I & (I + A)^{-1/2}(I - A)^{1/2}D \\ D^*(I + A)^{-1/2}(I - A)^{1/2} & I \end{pmatrix}.$$

Proof (i) $\mathcal{R}(P+Q)$ is not dense if and only if $0 \in \sigma_P(P+Q)$ since $P+Q$ is selfadjoint. Suppose that there exists a unit vector $x = (x_1, x_2) \in \mathcal{H}$ such that $(P + Q)x = 0$. By (5), one has

$$(P + Q)x = \begin{pmatrix} I + A & (I - A^2)^{1/2}D \\ D^*(I - A^2)^{1/2} & -D^*(I + A)D \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Then,

$$\begin{cases} (I + A)x_1 + (I - A^2)^{1/2}Dx_2 = 0, \\ D^*(I - A^2)^{1/2}x_1 - D^*(I + A)Dx_2 = 0. \end{cases}$$

Observing that $I \pm A$ are injective and D is unitary, we get

$$\begin{cases} (I + A)x_1 + (I - A^2)^{1/2}Dx_2 = 0, \\ (I - A)x_1 - (I - A^2)^{1/2}Dx_2 = 0. \end{cases}$$

Solving the above equations, we obtain that $x_1 = 0$ and $x_2 = 0$. Hence, $0 \notin \sigma_P(P+Q)$ and thus $\mathcal{R}(P + Q)$ is dense.

(ii) By (5), $(P+Q)^2 = 2(I+A)\oplus 2D^*(I+A)D$. So, $\|P+Q\| = \|(P+Q)^2\|^{1/2} = \sqrt{2}\|I + A\|^{1/2}$.

(iii) By items (i) and (ii), we know that $P+Q$ is invertible if and only if $\mathcal{R}(P+Q)$ is closed if and only if $-1 \notin \sigma(A)$. The inverse $(P + Q)^{-1}$ can be obtained by applying the representation (7). □

If P and Q are two general reflection operators, by Theorem 3.1, we obtain the following useful representations.

Corollary 3.2 *Let P and Q be two reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). Then P and Q have the following operator matrix forms*

$$\begin{aligned} P &= I_1 \oplus I_2 \oplus -I_3 \oplus -I_4 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & -I_6 \end{pmatrix}, \\ Q &= I_1 \oplus -I_2 \oplus I_3 \oplus -I_4 \oplus \begin{pmatrix} A & (I_5 - A^2)^{1/2} D \\ D^*(I_5 - A^2)^{1/2} & -D^* A D \end{pmatrix} \end{aligned} \tag{8}$$

with respect to the space decomposition $\mathcal{H} = \oplus_{i=1}^{6} \mathcal{H}_i$, respectively, where A is a selfadjoint contraction on $\mathcal{H}_5$ with that $1, -1 \notin \sigma_P(A)$, D is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$, and I_i is the identity onto $\mathcal{H}_i$, $i = 1, \ldots, 6$.

Note that $\dim \mathcal{H}_5 = \dim \mathcal{H}_6$ since D is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$.

Corollary 3.3 *Let P and Q be two reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). Then $PQ = QP$ if and only if $\dim \mathcal{H}_5 = \dim \mathcal{H}_6 = 0$.*

Proof By Corollary 3.2,

$$PQ - QP = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & 2(I_5 - A^2)^{1/2} D \\ -2D^*(I_5 - A^2)^{1/2} & 0 \end{pmatrix}. \tag{9}$$

Since $I_5 \pm A$ are injective and D is a unitary operator, $PQ = QP$ if and only if $\dim \mathcal{H}_5 = \dim \mathcal{H}_6 = 0$. □

Also by Corollary 3.2, we have $PQ + QP = 2I_1 \oplus -2I_2 \oplus -2I_3 \oplus 2I_4 \oplus 2A \oplus 2D^*AD$.

Theorem 3.2 *Let P and Q be two reflection operators on $\mathcal{H}$ having the matrix representations (8) and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). Then the following statements hold.*

(i) $\Gamma = c_1 P + c_2 Q$, *where $c_1, c_2 \in \mathbb{C} \backslash \{0\}$, is invertible if and only if $c_1 + c_2 \neq 0$ whenever $\mathcal{H}_1$ or $\mathcal{H}_4 \neq \{0\}$, $c_1 - c_2 \neq 0$ whenever $\mathcal{H}_2$ or $\mathcal{H}_3 \neq \{0\}$ and $-\frac{c_1^2 + c_2^2}{2c_1 c_2} \notin \sigma(A)$;*

(ii) *If $|c_1| \neq |c_2|$, then $\Gamma = c_1 P + c_2 Q$ is invertible for every $c_1, c_2 \in \mathbb{C}$.*

Proof (i) By Corollary 3.2, one has

$$\begin{aligned} \Gamma^2 &= (c_1 P + c_2 Q)^2 \\ &= \left(c_1^2 + c_2^2\right) I + c_1 c_2 (PQ + QP) \\ &= (c_1 + c_2)^2 I_1 \oplus (c_1 - c_2)^2 I_2 \oplus (c_1 - c_2)^2 I_3 \oplus (c_1 + c_2)^2 I_4 \\ &\quad \oplus \left[\left(c_1^2 + c_2^2\right) I_5 + 2c_1 c_2 A\right] \oplus D^* \left[(c_1^2 + c_2^2) I_5 + 2c_1 c_2 A\right] D. \end{aligned} \tag{10}$$

Γ is invertible if and only if Γ^2 is invertible if and only if $c_1 + c_2 \neq 0$ whenever $\mathcal{H}_1$ or $\mathcal{H}_4 \neq \{0\}$, $c_1 - c_2 \neq 0$ whenever $\mathcal{H}_2$ or $\mathcal{H}_3 \neq \{0\}$ and $-\frac{c_1^2 + c_2^2}{2c_1 c_2} \notin \sigma(A) \subseteq [-1, 1]$.

(ii) If $|c_1| \neq |c_2|$ and $c_1 c_2 = 0$, then $\Gamma = c_1 P + c_2 Q$ is obviously invertible by (10).

If $c_1c_2 \neq 0$, let $c_1 = r_1e^{i\theta_1} \neq 0$ and $c_2 = r_2e^{i\theta_2} \neq 0$, where r_i and θ_i denote the module and argument of complex numbers c_i, $i = 1, 2$, respectively. By direct computation,

$$\frac{c_1^2 + c_2^2}{2c_1c_2} = \frac{\left(r_1^2 + r_2^2\right)\cos(\theta_1 - \theta_2) + i\left(r_1^2 - r_2^2\right)\sin(\theta_1 - \theta_2)}{2r_1r_2}.$$

Since $|c_1| \neq |c_2|$, then $c_1 \pm c_2 \neq 0$ and $r_1 \neq r_2$, obviously.

(a) If $r_1 \neq r_2$, $\theta_1 = \theta_2 + k\pi$, $k \in \mathbb{Z}$, then $1 < |\frac{c_1^2+c_2^2}{2c_1c_2}| \notin \sigma(A) \subseteq [-1, 1]$.

(b) If $r_1 \neq r_2$, $\theta_1 \neq \theta_2 + k\pi$, $k \in \mathbb{Z}$, then $-\frac{c_1^2+c_2^2}{2c_1c_2} \notin \mathbb{R}$.

As a result, we get $c_1 \pm c_2 \neq 0$ and $-\frac{c_1^2+c_2^2}{2c_1c_2} \notin \sigma(A)$. By (10), we obtain that $\Gamma = c_1P + c_2Q$ is invertible for every $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ with $|c_1| \neq |c_2|$. □

We have discussed the invertibility of operator $P + Q$ if (P, Q) is a reflective regular pair having matrix representations (5) in Corollary 3.1. In the following, we will talk about the invertibility of $P - Q$ and $P + Q - I$ for the general reflection operators P and Q.

Theorem 3.3 *Let P and Q be two reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by* (4).

(i) *$P - Q$ is invertible if and only if* $\dim \mathcal{H}_1 = \dim \mathcal{H}_4 = 0$ *and* $1 \notin \sigma(A)$.
(ii) *$P + Q - I$ is invertible if and only if $3I - (P - Q)^2$ is invertible if and only if $-\frac{1}{2} \notin \sigma(A)$. In this case,*

$$\begin{aligned}
&(P + Q - I)^{-1}\\
&= \left[3I - (P - Q)^2\right]^{-1}(I + P + Q)\\
&= (I + P + Q)\left[3I - (P - Q)^2\right]^{-1}\\
&= I_1 \oplus -I_2 \oplus -I_3 \oplus -\frac{1}{3}I_4 \oplus \begin{pmatrix} (2I_5+A)(I_5+2A)^{-1} & (I_5+2A)^{-1}(I_5-A^2)^{1/2}D \\ D^*(I_5+2A)^{-1}(I_5-A^2)^{1/2} & -D^*A(I_5+2A)^{-1}D \end{pmatrix}.
\end{aligned}$$

Proof (i) By Corollary 3.2,

$$(P - Q)^2 = 0 \oplus 4I_2 \oplus 4I_3 \oplus 0 \oplus 2(I_5 - A) \oplus 2D^*(I_5 - A)D.$$

We get that $P - Q$ is invertible if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_4 = 0$ and $1 \notin \sigma(A)$.

(ii) By Corollary 3.2,

$$P + Q - I = I_1 \oplus -I_2 \oplus -I_3 \oplus -3I_4 \oplus \begin{pmatrix} A & (I_5-A^2)^{1/2}D \\ D^*(I_5-A^2)^{1/2} & -D^*(2I_5+A)D \end{pmatrix}.$$

By Theorem 3.1, $2I_5+A$ is invertible since A is a contraction operator. By (7), $P+Q-I$ is invertible if and only if the Schur complement $S = A + (I_5 - A^2)^{1/2}DD^*(2I_5$

$+ A)^{-1}DD^*(I_5 - A^2)^{1/2} = (2I_5 + A)^{-1}(I_5 + 2A)$ is invertible if and only if $I_5 + 2A$ is invertible if and only if $-\frac{1}{2} \notin \sigma(A)$. Applying the formula in (7), we get that

$$\begin{aligned}
&(P + Q - I)^{-1} \\
&= I_1 \oplus -I_2 \oplus -I_3 \oplus -\frac{1}{3}I_4 \oplus \begin{pmatrix} (2I_5+A)(I_5+2A)^{-1} & (I_5+2A)^{-1}(I_5-A^2)^{1/2}D \\ D^*(I_5+2A)^{-1}(I_5-A^2)^{1/2} & -D^*A(I_5+2A)^{-1}D \end{pmatrix}.
\end{aligned}$$

Note that $(P - Q)^2 = 0 \oplus 4I_2 \oplus 4I_3 \oplus 0 \oplus 2(I_5 - A) \oplus 2D^*(I_5 - A)D$,

$$3I - (P - Q)^2 = 3I_1 \oplus -I_2 \oplus -I_3 \oplus 3I_4 \oplus (I_5 + 2A) \oplus D^*(I_5 + 2A)D$$

and

$$3I - (P - Q)^2 = (P + Q - I)(I + P + Q) = (I + P + Q)(P + Q - I).$$

We also get $3I - (P - Q)^2$ is invertible if and only if $-\frac{1}{2} \notin \sigma(A)$ if and only if $P + Q - I$ is invertible. In this case,

$$\begin{aligned}
(P + Q - I)^{-1} &= \left[3I - (P - Q)^2\right]^{-1}(I + P + Q) \\
&= (I + P + Q)\left[3I - (P - Q)^2\right]^{-1}.
\end{aligned}$$

□

Let P and Q be the reflection operators on $\mathcal{H}$. According to (9), we observe that $\mathcal{N}(PQ - QP)$ and $\mathcal{N}(PQ - QP)^{\perp}$ are invariant subspaces of P and Q. Therefore, P and Q as operators on $\mathcal{H} = \mathcal{N}(PQ - QP) \oplus \mathcal{N}(PQ - QP)^{\perp}$ have the operator matrix representations

$$P = P_1 \oplus P_2 \text{ and } Q = Q_1 \oplus Q_2,$$

respectively, where the restrictions P_i and Q_i, $i = 1, 2$ have the following properties:

$$(1).\quad P_1Q_1 = Q_1P_1; \quad (2).\quad \mathcal{N}(P_2Q_2 - Q_2P_2) = \{0\}.$$

Besides, $\mathcal{N}(PQ - QP)$ admits the orthogonal decomposition $\mathcal{N}(PQ - QP) = \bigoplus_{i=1}^{4} \mathcal{H}_i$. Therefore, (P, Q) is a regular pair if and only if $\mathcal{N}(PQ - QP) = \{0\}$. For orthogonal projections cases, Halmos presented a nice and tractable characterization of the regular pair in [10, Theorem 2].

Theorem 3.4 *Let $\Gamma = c_1P + c_2Q$, where $c_1, c_2 \in \mathbb{R}\backslash\{0\}$ and P, Q are two reflection operators on $\mathcal{H}$ with $P \neq \pm Q$. Then Γ is a reflection operator if and only if the following conditions hold:*

(i) (P, Q) *is the reflective regular pair;*

(ii) $|\frac{1-c_1^2-c_2^2}{2c_1c_2}| < 1$*;*

(iii) *P and Q, as operators onto $\mathcal{H} = \mathcal{R}(I+P) \oplus \mathcal{R}(I-P)$, have the operator matrix forms*

$$P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } Q = \begin{pmatrix} A & (I-A^2)^{1/2}D \\ D^*(I-A^2)^{1/2} & -D^*AD \end{pmatrix},$$

where $A = \frac{1-c_1^2-c_2^2}{2c_1c_2} I$ and D is a unitary operator from $\mathcal{R}(I-P)$ onto $\mathcal{R}(I+P)$.

Proof The sufficiency is clear, so it is enough to prove the necessity. Let P and Q have the matrix representations in (8). Consequently, $PQ + QP = 2I_1 \oplus -2I_2 \oplus -2I_3 \oplus 2I_4 \oplus 2A \oplus 2D^*AD$. If Γ is a reflection operator, then

$$\Gamma^2 = (c_1P + c_2Q)^2 = \left(c_1^2 + c_2^2\right) I + c_1c_2(PQ + QP) = I.$$

Hence,

$$PQ + QP = 2I_1 \oplus -2I_2 \oplus -2I_3 \oplus 2I_4 \oplus 2A \oplus 2D^*AD = \frac{1 - c_1^2 - c_2^2}{c_1c_2} I.$$

Now, in view of the equations above, we consider the cases as follows:

If $\frac{1-c_1^2-c_2^2}{c_1c_2} = 2$, then $\dim \mathcal{H}_i = 0, i = 2, 3, 5, 6$ since A is a contraction with $1, -1 \notin \sigma_P(A)$. In this case,

$$P = I_1 \oplus -I_4 \text{ and } Q = I_1 \oplus -I_4,$$

which contradicts to $P \neq Q$. Similarly, if $\frac{1-c_1^2-c_2^2}{c_1c_2} = -2$, then $\dim \mathcal{H}_i = 0, i = 1, 4, 5, 6$. In this case,

$$P = I_2 \oplus -I_3 \text{ and } Q = -I_2 \oplus I_3,$$

which is a contradiction to $P \neq -Q$. Also, if $\frac{1-c_1^2-c_2^2}{c_1c_2} \neq \pm 2$, then $\dim \mathcal{H}_i = 0, i = 1, 2, 3, 4$. In this case,

$$P = I_5 \oplus -I_6 \text{ and } Q = \begin{pmatrix} A & (I_5-A^2)^{1/2}D \\ D^*(I_5-A^2)^{1/2} & -D^*AD \end{pmatrix},$$

where $A = \frac{1-c_1^2-c_2^2}{2c_1c_2} I_5$ and D is a unitary operator from $\mathcal{R}(I-P)$ onto $\mathcal{R}(I+P)$. Hence, (P, Q) is the reflective regular pair and $|\frac{1-c_1^2-c_2^2}{2c_1c_2}| < 1$. □

In the following, we will investigate the unitary equivalence of the reflection operators P and Q, which is similar to that of two orthogonal projections (see [4, Theorem 3.1]).

Let the reflective regular pair (P, Q) have the matrix representations (5). It is easy to check that the unitary operator

$$W_0 = \frac{\sqrt{2}}{2}\begin{pmatrix} (I+A)^{1/2} & (I-A)^{1/2}D \\ D^*(I-A)^{1/2} & -D^*(I+A)^{1/2}D \end{pmatrix} \tag{11}$$

satisfies $W_0P = QW_0$ and $PW_0 = W_0Q$. Moreover, we observe that the unitary $U \in inn_Q(P)$ if and only if $U^* \in out_Q(P)$. In the following, we obtain the general expressions for all the intertwining operators of P respect to Q when (P, Q) is a reflective regular pair.

Theorem 3.5 *Let (P, Q) be a reflective regular pair and W_0 be defined by (11). Then*

$$\begin{aligned} out_Q(P) &= \{W_0U : UP = PU, U \in \mathcal{U}(\mathcal{H})\}, \\ int_Q(P) &= \{W_0U : UP = PU, UQ = QU, U \in \mathcal{U}(\mathcal{H})\}. \end{aligned}$$

Moreover, if $U \in \mathcal{U}(\mathcal{H})$ such that $UP = PU$, then $W_0U = UW_0$ if and only if $QU = UQ$.

Proof Clearly, $\{W_0U : UP = PU, U \in \mathcal{U}(\mathcal{H})\} \subseteq out_Q(P)$. On the other hand, if there exists a unitary operator W such that $WP = QW$, then $U = W_0^*W$ is a unitary operator and $UP = W_0^*WP = W_0^*QW = PW_0^*W = PU$. Thus, $W = W_0U$, where $U \in \mathcal{U}(\mathcal{H})$ satisfies $UP = PU$, i.e., $out_Q(P) \subseteq \{W_0U : UP = PU, U \in \mathcal{U}(\mathcal{H})\}$. Therefore, we obtain that

$$out_Q(P) = \{W_0U : UP = PU, U \in \mathcal{U}(\mathcal{H})\}.$$

Clearly, $int_Q(P) \supseteq \{W_0U : UP = PU, UQ = QU, U \in \mathcal{U}(\mathcal{H})\}$. On the other hand, if $T \in int_Q(P) = out_Q(P) \cap inn_Q(P)$, then $T \in out_Q(P)$. Hence, there exists a $U \in \mathcal{U}(\mathcal{H})$ such that $UP = PU$ and $T = W_0U$. Since $T = W_0U \in inn_Q(P)$, we have $PW_0U = W_0UQ$. Since $PW_0 = W_0Q$ and W_0 is invertible, we get $UQ = QU$ and

$$int_Q(P) \subseteq \{W_0U : UP = PU, UQ = QU, U \in \mathcal{U}(\mathcal{H})\}.$$

Therefore, we obtain that

$$int_Q(P) = \{W_0U : UP = PU, UQ = QU, U \in \mathcal{U}(\mathcal{H})\}.$$

By (5), $P = I \oplus -I$. The relation $UP = PU$ implies that $U = U_{11} \oplus U_{22}$, where U_{11} and U_{22} are unitary operators. If $W_0U = UW_0$, then

$$\begin{pmatrix} (I+A)^{1/2}U_{11} & (I-A)^{1/2}DU_{22} \\ D^*(I-A)^{1/2}U_{11} & -D^*(I+A)^{1/2}DU_{22} \end{pmatrix} = \begin{pmatrix} U_{11}(I+A)^{1/2} & U_{11}(I-A)^{1/2}D \\ U_{22}D^*(I-A)^{1/2} & -U_{22}D^*(I+A)^{1/2}D \end{pmatrix}.$$

It follows that

$$\begin{cases} U_{11}(I+A)^{1/2} = (I+A)^{1/2}U_{11}, \\ U_{11}(I-A)^{1/2}D = (I-A)^{1/2}DU_{22}. \end{cases} \overset{\text{Theorem 3.1}}{\Longrightarrow} \begin{cases} U_{11}A = AU_{11}, \\ U_{22} = D^*U_{11}D. \end{cases} \Longrightarrow QU = UQ.$$

Similarly, if $QU = UQ$, we can derive that $W_0U = UW_0$. □

Remark 3.2 Let the reflective regular pair (P, Q) have the matrix representations (5) and W_0 be defined by (11). Let $B_0 = P - Q$ and $M_0 = I + PQ + QP$.

(i) Theorem 3.5 shows that if there exists a unitary operator $W \in \mathcal{U}(\mathcal{H})$ such that $PW = WQ$ and $WP = QW$, then

$$W = W_0U = \frac{\sqrt{2}}{2}\begin{pmatrix} (I+A)^{1/2} & (I-A)^{1/2}D \\ D^*(I-A)^{1/2} & -D^*(I+A)^{1/2}D \end{pmatrix}\begin{pmatrix} U_{11} & 0 \\ 0 & D^*U_{11}D \end{pmatrix}, \tag{12}$$

where $U_{11} \in \mathcal{U}(\mathcal{M})$ satisfies $U_{11}A = AU_{11}$.

(ii) Note that A is a contraction on $\mathcal{M}$ with that neither 1 nor -1 belongs to the point spectrum of A and D is a unitary operator from $\mathcal{M}^\perp$ onto $\mathcal{M}$. It is easy to get that

$$B_0 = \begin{pmatrix} I-A & -(I-A^2)^{1/2}D \\ -D^*(I-A^2)^{1/2} & -D^*(I-A)D \end{pmatrix},$$
$$2I - B_0 = \begin{pmatrix} I+A & (I-A^2)^{1/2}D \\ D^*(I-A^2)^{1/2} & D^*(3I-A)D \end{pmatrix}$$

and

$$2I + B_0 = \begin{pmatrix} 3I-A & -(I-A^2)^{1/2}D \\ -D^*(I-A^2)^{1/2} & D^*(I+A)D \end{pmatrix}$$

are three injective selfadjoint operators.

(iii) If $-\frac{1}{2}, \frac{1}{2} \notin \sigma_P(A)$, then $M_0 = (I + 2A) \oplus D^*(I + 2A)D$,

$$M_0 B_0 = \begin{pmatrix} (I - A)(I + 2A) & -(I - A^2)^{1/2}(I + 2A)D \\ -D^*(I - A^2)^{1/2}(I + 2A) & -D^*(I - A)(I + 2A)D \end{pmatrix},$$

$$2I - M_0 B_0 = \begin{pmatrix} I - A + 2A^2 & (I + 2A)(I - A^2)^{1/2} D \\ D^*(I + 2A)(I - A^2)^{1/2} & D^*(3I + A - 2A^2)D \end{pmatrix}$$

and

$$2I + M_0 B_0 = \begin{pmatrix} 3I + A - 2A^2 & -(I - A^2)^{1/2}(I + 2A)D \\ -D^*(I - A^2)^{1/2}(I + 2A) & D^*(I - A + 2A^2)D \end{pmatrix}$$

are injective selfadjoint operators.

(iv) As one example, we only prove that $M_0 B_0$ is injective if $-\frac{1}{2} \notin \sigma_P(A)$. In fact, if there exists $(x, y) \in \mathcal{H}$ such that $M_0 B_0 \begin{pmatrix} x \\ y \end{pmatrix} = 0$, then

$$\begin{pmatrix} (I - A)(I + 2A) & -(I - A^2)^{1/2}(I + 2A)D \\ -D^*(I - A^2)^{1/2}(I + 2A) & -D^*(I - A)(I + 2A)D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

That is,

$$\begin{cases} (I - A)(I + 2A)x - (I - A^2)^{1/2}(I + 2A)Dy = 0, \\ D^*(I - A^2)^{1/2}(I + 2A)x + D^*(I - A)(I + 2A)Dy = 0. \end{cases}$$

Then,

$$\begin{cases} (I - A)(I + 2A)x - (I - A^2)^{1/2}(I + 2A)Dy = 0, \\ (I + A)(I + 2A)x + (I - A^2)^{1/2}(I + 2A)Dy = 0. \end{cases}$$

Adding the two equations, we can get $2(I + 2A)x = 0$. Then $x = 0$ since $-\frac{1}{2} \notin \sigma_P(A)$ and $y = 0$ since D is a unitary operator. Hence, $M_0 B_0$ is injective.

Similarly, we can obtain that B_0, $2I - B_0$ and $2I + B_0$ are injective. Moreover, $2I - M_0 B_0$ and $2I + M_0 B_0$ are injective if $-\frac{1}{2}, \frac{1}{2} \notin \sigma_P(A)$.

Let (P, Q) be a pair of reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). If there exists a $U \in \mathcal{U}(\mathcal{H})$ such that $PU = UQ$ and $UP = QU$, then

$$U(P - Q) = -(P - Q)U.$$

Denote $B := P - Q = 0 \oplus 2I_2 \oplus -2I_3 \oplus 0 \oplus B_0$. Clearly, $\mathcal{N}(B) = \mathcal{H}_1 \oplus \mathcal{H}_4$, $\mathcal{N}(B - 2I) = \mathcal{H}_2$ and $\mathcal{N}(B + 2I) = \mathcal{H}_3$ are reduced subspaces of B. Let $\mathcal{H}_0 = \mathcal{H}_5 \oplus \mathcal{H}_6$, then B, as an operator on $\mathcal{H} = (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_0$, has the operator matrix form $B = 0 \oplus 2I_2 \oplus -2I_3 \oplus B_0$. Let $U = (U^0_{ij})_{1\le i,j\le 4}$. By $UB = -BU$, we have

$$\begin{pmatrix} U^0_{11} & E \\ F & U_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & B^0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & B^0 \end{pmatrix}\begin{pmatrix} U^0_{11} & E \\ F & U_2 \end{pmatrix}, \tag{13}$$

where $E = (U^0_{12}, U^0_{13}, U^0_{14})$, $F = (U^0_{21}, U^0_{31}, U^0_{41})^\top$, $B^0 = 2I_2 \oplus -2I_3 \oplus B_0$ is injective and $U_2 = (U^0_{ij})_{2\le i,j\le 4}$. The equations in (13) imply that $E = 0$, $F = 0$ and $U_2B^0 = -B^0U_2$. Thus, U^0_{12}, U^0_{13}, U^0_{14}, U^0_{21}, U^0_{31} and U^0_{41} are zero operators. By $U_2B^0 = -B^0U_2$, we have

$$\begin{pmatrix} 2U^0_{22} & -2U^0_{23} & U^0_{24}B_0 \\ 2U^0_{32} & -2U^0_{33} & U^0_{34}B_0 \\ 2U^0_{42} & -2U^0_{43} & U^0_{44}B_0 \end{pmatrix} = \begin{pmatrix} -2U^0_{22} & -2U^0_{23} & -2U^0_{24} \\ 2U^0_{32} & 2U^0_{33} & 2U^0_{34} \\ -B_0U^0_{42} & -B_0U^0_{43} & -B_0U^0_{44} \end{pmatrix}.$$

Since $2I - B_0$ and $2I + B_0$ are injective and dense, then U^0_{22}, U^0_{24}, U^0_{33}, U^0_{34}, U^0_{42} and U^0_{43} are zero operators. Hence,

$$U = U^0_{11} \oplus \begin{pmatrix} 0 & U^0_{23} \\ U^0_{32} & 0 \end{pmatrix} \oplus U^0_{44}.$$

Since $P|_{\mathcal{N}(B)} = Q|_{\mathcal{N}(B)} = I_1 \oplus -I_4$, the multi-commutativity of $P|_{\mathcal{N}(B)}, Q|_{\mathcal{N}(B)}$ and U^0_{11} implies that $U^0_{11} = U_{11} \oplus U_{44} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_4)$. Moreover, $\begin{pmatrix} 0 & U^0_{23} \\ U^0_{32} & 0 \end{pmatrix}$ is a unitary operator if and only if U^0_{23} and U^0_{32} are unitaries if and only if $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$. By Remark 3.2 (i), we get that U^0_{44} has the representation (12). As a consequence, we present the following corollary.

Corollary 3.4 *Let (P, Q) be a pair of reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). Then P and Q are unitary equivalence if and only if* $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$. *In addition, if P and Q are denoted by (8) and W_0 is denoted by (11), then $U \in \mathcal{U}(\mathcal{H})$ with $UP = QU$ and $PU = UQ$ has the representation*

$$U = U_{11} \oplus \begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix} \oplus U_{44} \oplus W_0 \begin{pmatrix} U_{55} & 0 \\ 0 & D^*U_{55}D \end{pmatrix},$$

where U_{ij} are unitary operators from $\mathcal{H}_j$ onto $\mathcal{H}_i$, $U_{55}A = AU_{55}$.

At last, we talk about the unitary equivalence of products PQP and QPQ.

Theorem 3.6 *Let (P, Q) be a pair of reflection operators and $\mathcal{H}_i$, $i = 1, \ldots, 6$ be defined by (4). If $\pm\frac{1}{2} \notin \sigma_P(A)$, then PQP and QPQ are unitary equivalence if and only if* $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$.

Proof Sufficiency. If $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$, then there is a $U \in \mathcal{U}(\mathcal{H})$ such that $PU = UQ$ and $UP = QU$ by Corollary 3.4. It follows that $PQPU = UQPQ$ and $UPQP = QPQU$.

Necessity. If there is a $U \in \mathcal{U}(\mathcal{H})$ such that $UPQP = QPQU$ and $PQPU = UQPQ$, then we have

$$U(PQP - QPQ) = -(PQP - QPQ)U.$$

Let $\widetilde{B} := PQP - QPQ$, $B := P - Q$ and $M := I + PQ + QP$. Clearly, $\widetilde{B} = MB$. Let $\mathcal{H}_0 = \mathcal{H}_5 \oplus \mathcal{H}_6$. Moreover, B and M as operators on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_0$ have the matrix forms

$$B = 0 \oplus 2I_2 \oplus -2I_3 \oplus 0 \oplus B_0 \ \text{ and } \ M = 3I_1 \oplus -I_2 \oplus -I_3 \oplus 3I_4 \oplus M_0,$$

respectively, where $B_0 = B|_{\mathcal{H}_0}$ and $M_0 = M|_{\mathcal{H}_0}$. Clearly, $\mathcal{N}(B) = \mathcal{H}_1 \oplus \mathcal{H}_4$. Furthermore, B and M as operators on $\mathcal{H} = (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_0$ have matrix forms

$$B = 0 \oplus 2I_2 \oplus -2I_3 \oplus B_0 \ \text{ and } \ M = 3I_{\mathcal{N}(B)} \oplus -I_2 \oplus -I_3 \oplus M_0.$$

Thus,

$$\widetilde{B} = MB = 0 \oplus -2I_2 \oplus 2I_3 \oplus M_0 B_0.$$

By Remark 3.2 (iii) and (iv), M_0B_0 and $2I \pm M_0B_0$ are injective. Let $U = (U_{ij})_{1 \le i,j \le 4}$. From $U\widetilde{B} = -\widetilde{B}U$, one has

$$\begin{pmatrix} 0 & -2U_{12} & 2U_{13} & U_{14}M_0B_0 \\ 0 & -2U_{22} & 2U_{23} & U_{24}M_0B_0 \\ 0 & -2U_{32} & 2U_{33} & U_{34}M_0B_0 \\ 0 & -2U_{42} & 2U_{43} & U_{44}M_0B_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2U_{21} & 2U_{22} & 2U_{23} & 2U_{24} \\ -2U_{31} & -2U_{32} & -2U_{33} & -2U_{34} \\ -M_0B_0U_{41} & -M_0B_0U_{42} & -M_0B_0U_{43} & -M_0B_0U_{44} \end{pmatrix}.$$

It is easy to see that U_{12}, U_{13}, U_{21}, U_{31}, U_{22} and U_{33} are zero operators. Besides, U_{14} and U_{41} are zero operators since M_0B_0 is injective and dense. U_{24} and U_{42} are zero operators since $2I - M_0B_0$ is injective and dense. U_{34} and U_{43} are zero operators since $2I + M_0B_0$ is injective and dense. Hence, $U = U_{11} \oplus \begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix} \oplus U_{44}$, where $U_{11} \in \mathcal{B}(\mathcal{N}(B))$, $U_{44} \in \mathcal{B}(\mathcal{H}_0)$, $U_{23} \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$ and $U_{32} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ are unitary operators. Therefore, $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$. □

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