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Uniqueness of the Boundary Value Problem of Harmonic Maps via Harmonic Boundary

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Abstract
We prove the uniqueness of solutions for the boundary value problem of harmonic maps in the setting: given any continuous data f on the harmonic boundary of a complete Riemannian manifold with image within a regular geodesic ball, there exists a unique harmonic map, which is a limit of a sequence of harmonic maps with finite total energy in the sense of the supremum norm, from the manifold into the ball taking the same boundary value at each harmonic boundary point as that of f.

Keywords Harmonic map · Harmonic boundary · Boundary value problem · Uniqueness

Mathematics Subject Classification 58E20 · 53C43

1 Introduction

In this paper, we study the uniqueness of harmonic maps on a complete Riemannian manifold with image within a regular geodesic ball taking the given boundary data on the harmonic boundary of the domain manifold. Let (M, g) and (N, h) be complete Riemannian manifolds of dimension m and n, respectively, with local expressions for their metrics $g = g_{ij}dx^i dx^j$ and $h = h_{\alpha\beta}dy^\alpha dy^\beta$, where (x^i) and (y^α) are local coordinates of M and N, respectively. Then the harmonic map equation can be written in local coordinates as follows: For each $\alpha = 1, 2, \ldots, n$,

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$$\Delta u^\alpha(x) + g^{ij}(x)\Gamma^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\gamma}{\partial x^j}(x) = 0,$$

where Δ is the Laplacian of M, the matrix (g^{ij}) is the inverse of (g_{ij}) and $\Gamma^\alpha_{\beta\gamma}$'s are the Christoffel symbols on N. Thus if the target manifold of a harmonic map is flat, then the harmonic map equation becomes the Laplace–Beltrami equation. On the other hand, given $f \in C^1(M, N)$, the energy density $e(f)$ of f at $x \in M$ is defined by

$$e(f)(x) = \frac{1}{2}\, g^{ij}(x)h_{\alpha\beta}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x).$$

The total energy $E(f)$ of f is defined by

$$E(f) = \int_M e(f)(x)\,\mathrm{d}x.$$

In the case that $E(f) < \infty$, we say that f has finite total energy. In particular, if a map $u \in C^1(M, N)$ is a critical point of the total energy functional E, then it satisfies the harmonic map equation.

In this paper, we prove the uniqueness of solutions for the boundary value problem of harmonic maps in terms of the harmonic boundary. Let us first introduce the following definition:

Definition 1 Let N be a complete Riemannian manifold with sectional curvature bounded above by $\kappa > 0$. Let $\mathcal{B}_r(p)$ be a geodesic ball centered at a point p of radius r in N, and denote by $C(p)$ the cut locus of its center. We call the ball $\mathcal{B}_r(p)$ a regular geodesic ball in N if $r < \pi/2\sqrt{\kappa}$ and $\mathcal{B}_r(p) \cap C(p)$ is empty.

Theorem 1 *Let M and N be complete Riemannian manifolds. Let Δ_M be the harmonic boundary of M and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then for any $f \in C(\Delta_M, \mathcal{B}_{r_0}(p))$, there exists a unique harmonic map $u \in C(M, \mathcal{B}_{r_0}(p))$, which is a limit of a sequence of harmonic maps with finite total energy in the sense of the supremum norm, such that for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{x \in M \to \mathbf{x}} u(x) = f(\mathbf{x}).$$

The existence of solutions for the above boundary value problem of harmonic maps is proven in [6]. Thus, in this paper, we focus on the uniqueness of solutions for the boundary value problem.

In particular, a complete Riemannian manifold M is parabolic if and only if the harmonic boundary Δ_M is empty. In the case that a complete Riemannian manifold is parabolic, every bounded harmonic function on the manifold is constant. On the other hand, Sung et al. [9] proved that if every bounded harmonic function on a complete Riemannian manifold is constant, then every bounded harmonic map is also constant. Therefore, the case of an empty harmonic boundary is a trivial one in our problem. So from now on, we assume that the harmonic boundary Δ_M of every manifold M is not empty, unless otherwise specified.

2 Harmonic Functions and Harmonic Boundary

We begin with introducing some notations and relevant results which we need in this paper. Let $\mathcal{BD}(M)$ be the set of all bounded continuous functions f on a complete Riemannian manifold M whose distributional gradient ∇f belongs to $L^2(M)$. We say that a sequence $\{f_n\}$ of functions in $\mathcal{BD}(M)$ converges to a function f in $\mathcal{BD}(M)$ if

(i) $\{f_n\}$ is uniformly bounded;
(ii) f_n converges uniformly to f on each compact subset of M;
(iii) $\lim_{n\to\infty} \int_M |\nabla(f_n - f)|^2 = 0$.

In particular, $\mathcal{BD}(M)$ is complete. Let $\mathcal{BD}_0(M)$ be the closure of the set of all compactly supported smooth functions in $\mathcal{BD}(M)$. We denote by $\mathcal{HBD}(M)$ the subset of all bounded harmonic functions with finite Dirichlet integral in $\mathcal{BD}(M)$. For each $f \in \mathcal{BD}(M)$, there exists a unique $h \in \mathcal{HBD}(M)$ such that $h - f \in \mathcal{BD}_0(M)$. We call it the Royden decomposition.

On the other hand, there exists a locally compact Hausdorff space $\hat{M}$, called the Royden compactification of M, which contains M as an open dense subset. (See [8].) In particular, every function $f \in \mathcal{BD}(M)$ can be extended to a continuous function, denoted again by f, on $\hat{M}$ and the class of such extended functions separates points in $\hat{M}$. The Royden boundary of $\hat{M}$ is the set $\hat{M} \setminus M$ and will be denoted by $\partial\hat{M}$. An important part of the Royden boundary $\partial\hat{M}$ is the harmonic boundary Δ_M defined by

$$\Delta_M = \{\mathbf{x} \in \partial\hat{M} : f(\mathbf{x}) = 0 \ \text{ for all } \ f \in \mathcal{BD}_0(M)\}.$$

In particular, the duality relation between $\mathcal{BD}_0(M)$ and the harmonic boundary Δ_M holds as follows:

$$\mathcal{BD}_0(M) = \{f \in \mathcal{BD}(M) : f(\mathbf{x}) = 0 \ \text{ for all } \ \mathbf{x} \in \Delta_M\}.$$

In the case of subharmonic functions, by using the Royden decomposition and the maximum principle via the harmonic boundary, we have a useful lemma as follows:

Lemma 1 *Let f be a subharmonic function in $\mathcal{BD}(M)$. Then there exists a harmonic function h in $\mathcal{HBD}(M)$ such that $h - f \in \mathcal{BD}_0(M)$ and $f \le h$ on M.*

In particular, the boundary value problem of harmonic functions via the harmonic boundary is solvable as follows: (See Theorem 1 in [6].)

Theorem 2 *Let M be a complete Riemannian manifold. Then for any continuous function f on the harmonic boundary Δ_M of M, there exists a unique harmonic function h on M, which is a limit of a sequence of bounded harmonic functions with finite Dirichlet integral in the sense of the supremum norm, such that for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{x\in M\to\mathbf{x}} h(x) = f(\mathbf{x}).$$

3 Harmonic Maps and Harmonic Boundary

By using the result of [4] and Lemma 1, we have the stability property of harmonic maps with finite total energy via the harmonic boundary as follows:

Proposition 1 *Let M and N be complete Riemannian manifolds. Let Δ_M be the harmonic boundary of M and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. If $u, \tilde{u} \in C(M, \mathcal{B}_{r_0}(p))$ are harmonic maps with finite total energy such that for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{x \in M \to \mathbf{x}} \rho(u(x), \tilde{u}(x)) \leq \epsilon, \tag{1}$$

then $\rho(u(x), \tilde{u}(x)) < \epsilon_1$ for all $x \in M$, where ρ is the distance function on N and $\epsilon_1 \to 0$ as $\epsilon \to 0$.

Proof Suppose that $\kappa \geq 0$ is an upper bound of the sectional curvature of N. For a fixed $\sigma > 0$, consider the function

$$\Phi_\sigma(y, z) = \frac{1 + \sigma - \cos\sqrt{\kappa}\rho(y, z)}{\kappa \cos\sqrt{\kappa}\rho(y, p) \cdot \cos\sqrt{\kappa}\rho(z, p)}$$

on any given compact subset of $\{(y, z) \in \mathcal{B}_{r_0}(p) \times \mathcal{B}_{r_0}(p) : y \neq z\}$. Then by Lemma 4.1 in [4], Φ_σ is a nonnegative bounded convex function on the subset. Let $U_\epsilon = \{(y, z) \in \mathcal{B}_{r_0}(p) \times \mathcal{B}_{r_0}(p) : \rho(y, z) > \epsilon\}$. Then for each $(y, z) \in \mathcal{B}_{r_0}(p) \times \mathcal{B}_{r_0}(p) \backslash U_\epsilon$,

$$\Phi_\sigma(y, z) \leq \frac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos\sqrt{\kappa}\rho(y, p) \cdot \cos\sqrt{\kappa}\rho(z, p)} \leq \frac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos^2\sqrt{\kappa}r_0}.$$

Define $\phi_\sigma(x) = \max\left\{\Phi_\sigma(u(x), \tilde{u}(x)), \dfrac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos^2\sqrt{\kappa}r_0}\right\}$ for each $x \in M$. Then ϕ_σ is nonnegative subharmonic on $\Omega_\epsilon = \{x \in M : \rho(u(x), \tilde{u}(x)) > \epsilon\}$ since both $\Phi_\sigma(u, \tilde{u})$ and $\dfrac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos^2\sqrt{\kappa}r_0}$ are nonnegative subharmonic on Ω_ϵ. On the other hand, for each $x \in M \setminus \Omega_\epsilon$,

$$\Phi_\sigma(u(x), \tilde{u}(x)) \leq \frac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos^2\sqrt{\kappa}r_0}.$$

Thus

$$\phi_\sigma = \frac{1 + \sigma - \cos\sqrt{\kappa}\epsilon}{\kappa \cos^2\sqrt{\kappa}r_0}$$

on $M \setminus \Omega_\epsilon$, hence it is nonnegative subharmonic on $M \setminus \Omega_\epsilon$. Therefore, ϕ_σ is a nonnegative bounded subharmonic function with finite Dirichlet integral on M since u and $\tilde{u}$ are harmonic maps with finite total energy.

By Lemma 1, there exists a harmonic function $h_\sigma \in \mathcal{HBD}(M)$ such that $h_\sigma - \phi_\sigma \in \mathcal{BD}_0(M)$ and $0 \le \phi_\sigma \le h_\sigma$ on M. Since $h_\sigma - \phi_\sigma \in \mathcal{BD}_0(M)$, from the assumption (1), we have

$$h_\sigma \le \frac{1+\sigma-\cos\sqrt{\kappa}\epsilon}{\kappa\cos^2\sqrt{\kappa}r_0}$$

on Δ_M. Then by the maximum principle via the harmonic boundary,

$$h_\sigma \le \frac{1+\sigma-\cos\sqrt{\kappa}\epsilon}{\kappa\cos^2\sqrt{\kappa}r_0}$$

on M. (See 2 H. Maximum Principle II in [8].) Since $\phi_\sigma \le h_\sigma$ on M,

$$\frac{1+\sigma-\cos\sqrt{\kappa}\rho(u(x),\tilde{u}(x))}{\kappa} \le \Phi_\sigma(u(x),\tilde{u}(x)) \le h_\sigma(x) \le \frac{1+\sigma-\cos\sqrt{\kappa}\epsilon}{\kappa\cos^2\sqrt{\kappa}r_0}$$

for each $x \in M$. Letting $\sigma \to 0$, we have

$$1-\cos\sqrt{\kappa}\rho(u(x),\tilde{u}(x)) \le \frac{1-\cos\sqrt{\kappa}\epsilon}{\cos^2\sqrt{\kappa}r_0}.$$

This implies that there exists $\epsilon_1 > 0$ such that for all $x \in M$,

$$\rho(u(x),\tilde{u}(x)) < \epsilon_1,$$

where $\epsilon_1 \to 0$ as $\epsilon \to 0$. □

In particular, the case that $\epsilon = 0$ in Proposition 1 implies that each bounded harmonic map with finite total energy is uniquely determined by the given data on the harmonic boundary.

Let M and N be complete Riemannian manifolds. Let $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then a map $f : M \to \mathcal{B}_{r_0}(p)$ can be regarded as an $\mathbf{R}^n$-valued map such that

$$f = (f^1, f^2, \ldots, f^n) : M \to \mathcal{B}_{r_0}(0) \subset \mathbf{R}^n.$$

Applying the program of Lemma 3.1 in [1], we have the following lemma:

Lemma 2 *Let M and N be complete Riemannian manifolds and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Let h be a continuous map such that*

$$h = (h^1, h^2, \ldots, h^n) : M \to \mathcal{B}_{r_0}(0) \subset \mathbf{R}^n,$$

where each h^α is a harmonic function on M for $\alpha = 1, 2, \ldots, n$. Then there exists a harmonic function v on M and a harmonic map $u \in C(M, \mathcal{B}_{r_0}(p))$ such that $|h|^2/2 \le v$ on M and

$$\rho(u(x), h(x))^2 \le C\Big(v(x) - \frac{1}{2}|h|^2(x)\Big)$$

for all $x \in M$.

In particular, if each $h^\alpha \in \mathcal{HBD}(M)$ for $\alpha = 1, 2, \ldots, n$, then $v \in \mathcal{HBD}(M)$, $v - |h|^2/2 \in \mathcal{BD}_0(M)$, u has finite total energy and

$$\lim_{x \in M \to \mathbf{x}} \rho(u(x), h(x)) = 0$$

for each $\mathbf{x} \in \Delta_M$.

Proof By constructing a weak solution and then proving its regularity or by constructing the Perron solutions, there exists a function v_R such that for each $R > 0$,

$$\begin{cases} \Delta v_R = 0 & \text{in } B_R(o); \\ v_R = \frac{1}{2}|h|^2 & \text{on } M \setminus B_R(o), \end{cases}$$

where o is a point in M. Since the function $|h|^2/2$ is subharmonic on M, the sequence $\{v_R\}$ is increasing; hence, it converges uniformly to a harmonic function v on any compact subset of M such that $|h|^2/2 \le v$ on M.

On the other hand, by Theorem 1 in [3], one can find a harmonic map $u_R : B_R(o) \to \overline{\mathcal{B}}_{r_0}(p)$ such that $u_R = h$ on $\partial B_R(o)$. The a priori estimates (Theorem 3) in [2] imply that for a sufficiently large $R_1 > 0$ and some $\lambda \in (0, 1)$, $|u_R|_{C^{2,\lambda}(B_{R_1}(o))}$ is bounded by a constant depending only on M, $\overline{\mathcal{B}}_{r_0}(p)$ and h, where $R \ge R_1$. Hence by the Arzela–Ascoli theorem, there exists a subsequence of $\{u_{R_j}\}$ of $\{u_R\}$ converging uniformly on any compact subset of M. In particular, the limit map $u : M \to \overline{\mathcal{B}}_{r_0}(p)$ is also harmonic. By Lemma 3.1 in [1], there exists a constant $C < \infty$ depending only on the geometry of $\overline{\mathcal{B}}_{r_0}(p)$ such that

$$\rho(u_R(x), h(x))^2 \le C\Big(v_R(x) - \frac{1}{2}|h|^2(x)\Big)$$

for all $x \in B_R(o)$. Since the sequence $\{v_R - |h|^2/2\}$ is increasing,

$$v_R(x) - \frac{1}{2}|h|^2(x) \le v(x) - \frac{1}{2}|h|^2(x)$$

for all $x \in B_R(o)$. By a diagonal sequence argument and the Arzela–Ascoli theorem,

$$\rho(u(x), h(x))^2 \le C\Big(v(x) - \frac{1}{2}|h|^2(x)\Big)$$

for all $x \in M$.

If each $h^\alpha \in \mathcal{HBD}(M)$ for $\alpha = 1, 2, \ldots, n$, then $E(u_R) \le E(h) < \infty$, $v_R \in \mathcal{BD}(M)$ and $v_R - |h|^2/2$ is compactly supported. Thus u has finite total energy, $v \in \mathcal{HBD}(M)$ and $v - |h|^2/2 \in \mathcal{BD}_0(M)$. Since $v - |h|^2/2 \equiv 0$ on Δ_M,

$$\lim_{x \in M \to \mathbf{x}} \rho(u(x), h(x)) = 0$$

for each $\mathbf{x} \in \Delta_M$. □

In line with the viewpoint of the duality relation between $\mathcal{BD}_0(M)$ and the harmonic boundary Δ_M, the Royden decomposition can be rephrased in such a way that for each $f \in \mathcal{BD}(M)$, there exists a unique $h \in \mathcal{HBD}(M)$ such that

$$h - f \equiv 0 \quad \text{on} \quad \Delta_M.$$

In [7], the present author proved the harmonic map version of the Royden decomposition for harmonic maps with finite total energy in the case when N is a Cartan–Hadamard manifold. Combining Proposition 1 and Lemma 2, we have a generalization of the harmonic map version of the Royden decomposition as follows:

Theorem 3 *Let M and N be complete Riemannian manifolds. Let Δ_M be the harmonic boundary of M and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then for any $f \in C(M, \mathcal{B}_{r_0}(p))$ with finite total energy, there exists a unique harmonic map $u \in C(M, \mathcal{B}_{r_0}(p))$ with finite total energy such that for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{x \in M \to \mathbf{x}} u(x) = \lim_{x \in M \to \mathbf{x}} f(x).$$

In order to prove Theorem 1, we will find a suitable sequence of harmonic maps with finite total energy converging uniformly to the harmonic map taking the same boundary value as that of the given data. Prior to this, we first prove that the desired sequence is a uniform Cauchy sequence as follows:

Lemma 3 *Let M and N be complete Riemannian manifolds. Let Δ_M be the harmonic boundary of M and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then for any $f \in C(\Delta_M, \mathcal{B}_{r_0}(p))$, there exists a sequence $\{u_k\}$ of harmonic maps with finite total energy from M into $\mathcal{B}_{r_0}(p)$ such that $\{u_k\}$ is a uniform Cauchy sequence and for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{k \to \infty} u_k(\mathbf{x}) = f(\mathbf{x}).$$

Proof By Theorem 2, there exists a sequence of maps $h_k = (h_k^1, h_k^2, \ldots, h_k^n) : M \to \mathcal{B}_{r_0}(0)$ and a unique map $h = (h^1, h^2, \ldots, h^n) : M \to \mathcal{B}_{r_0}(0)$ such that for each $\alpha = 1, 2, \ldots, n$ and each $\mathbf{x} \in \Delta_M$, $h_k^\alpha \in \mathcal{HBD}(M)$, h^α is a harmonic function on M,

$$\lim_{k \to \infty} \sup_M |h_k^\alpha - h^\alpha| = 0 \quad \text{and} \quad \lim_{x \in M \to \mathbf{x}} h^\alpha(x) = f^\alpha(\mathbf{x}).$$

Thus for any fixed $\epsilon > 0$, there exists a positive integer $N_0 \in \mathbf{N}$ such that for all $k \geq N_0$,

$$\rho(h_k, h) < \epsilon \quad \text{on} \quad M. \tag{2}$$

On the other hand, since each map $h_k \in C(M, \mathcal{B}_{r_0}(p))$ has finite total energy, by Theorem 3, there exists a unique harmonic map u_k with finite total energy such that for each $\mathbf{x} \in \Delta_M$,

$$\lim_{x \in M \to \mathbf{x}} u_k(x) = \lim_{x \in M \to \mathbf{x}} h_k(x). \tag{3}$$

For any $k, l \geq N_0$, from Eqs. (2) and (3),

$$\lim_{x \in M \to \mathbf{x}} \rho(u_k(x), u_l(x)) \leq 2\epsilon$$

for each $\mathbf{x} \in \Delta_M$. Then by Proposition 1, there exists $\epsilon_2 > 0$ such that

$$\rho(u_k, u_l) < \epsilon_2 \quad \text{on} \quad M,$$

where $\epsilon_2 \to 0$ as $\epsilon \to 0$. Therefore, the sequence $\{u_k\}$ of harmonic maps with finite total energy is a uniform Cauchy sequence and

$$\lim_{k \to \infty} u_k(\mathbf{x}) = \lim_{k \to \infty} h_k(\mathbf{x}) = h(\mathbf{x}) = f(\mathbf{x}).$$

□

We are now ready to prove our main result:

Theorem 4 *Let M and N be complete Riemannian manifolds. Let Δ_M be the harmonic boundary of M and $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then for any $f \in C(\Delta_M, \mathcal{B}_{r_0}(p))$, there exists a unique harmonic map $u \in C(M, \mathcal{B}_{r_0}(p))$, which is a limit of a sequence of harmonic maps with finite total energy in the sense of the supremum norm, such that for each $\mathbf{x} \in \Delta_M$,*

$$\lim_{x \in M \to \mathbf{x}} u(x) = f(\mathbf{x}). \tag{4}$$

Proof By Theorem 1 in [6], there exists a harmonic map $u \in C(M, \mathcal{B}_{r_0}(p))$ satisfying the limit (4). By Lemma 3, there exists a uniform Cauchy sequence $\{u_k\}$ of harmonic maps with finite total energy such that for each $\mathbf{x} \in \Delta_M$,

$$\lim_{k \to \infty} u_k(\mathbf{x}) = f(\mathbf{x}).$$

We first claim that the sequence $\{u_k\}$ converges uniformly to the harmonic map u. Since the sequence is a uniform Cauchy sequence, we have only to prove that the sequence converges pointwise to the harmonic map u.

Let x be a fixed point in M. Then there exists a sufficiently large $R_2 \geq R_1$ such that $x \in B_{R_2}(o)$. Let h and h_k be the maps satisfying (2) and (3), given in the proof of Lemma 3. Then by Theorem 1 in [3], there exist sequences $\{u_R\}$ and $\{u_{k_R}\}$ of harmonic maps on $B_R(o)$ such that $u_R = h$ on $\partial B_R(o)$ and $u_{k_R} = h_k$ on $\partial B_R(o)$. Applying the program in the proofs of Lemma 2 and of Theorem 1 in [6], there exists a subsequence $\{u_{R_j}\}$ of $\{u_R\}$ converging uniformly to the map u and a subsequence $\{u_{k_{R_j}}\}$ of $\{u_{k_R}\}$ converging uniformly to the map u_k, respectively, on $B_{R_2}(o)$, where $R \geq R_2$.

For any fixed $\epsilon > 0$, there exists a positive integer $J_0 \in \mathbf{N}$ such that for all $j \geq J_0$,

$$\rho(u_{R_j}(x), u(x)) < \epsilon \quad \text{and} \quad \rho(u_{k_{R_j}}(x), u_k(x)) < \epsilon. \tag{5}$$

On the other hand, since $u_{R_j} = h$ and $u_{k_{R_j}} = h_k$ on $\partial B_{R_j}(o)$, by Eq. (2),

$$\rho(u_{R_j}, u_{k_{R_j}}) < \epsilon \quad \text{on} \quad \partial B_{R_j}(o). \tag{6}$$

Define a function $\phi_{\sigma,R_j,k}(x) = \max\left\{\Phi_\sigma(u_{R_j}(x), u_{k_{R_j}}(x)), \dfrac{1+\sigma-\cos\sqrt{\kappa}\epsilon}{\kappa\cos^2\sqrt{\kappa}r_0}\right\}$ for each $x \in M$ as given in Proposition 1. Then $\phi_{\sigma,R_j,k}$ is a subharmonic function on $B_{R_j}(o)$. By the boundary condition (6),

$$\phi_{\sigma,R_j,k} \leq \frac{1+\sigma-\cos\sqrt{\kappa}\epsilon}{\kappa\cos^2\sqrt{\kappa}r_0}$$

on $B_{R_j}(o)$. Letting $\sigma \to 0$, we have

$$1-\cos\sqrt{\kappa}\rho(u_{R_j}(x), u_{k_{R_j}}(x)) \leq \frac{1-\cos\sqrt{\kappa}\epsilon}{\cos^2\sqrt{\kappa}r_0}.$$

Thus there exists $\epsilon_1 > 0$ such that

$$\rho(u_{R_j}(x), u_{k_{R_j}}(x)) < \epsilon_1, \tag{7}$$

where $\epsilon_1 \to 0$ as $\epsilon \to 0$. Combining (5) and (7),

$$\rho(u(x), u_k(x)) < 2\epsilon + \epsilon_1.$$

Hence, we have the claim.

Suppose that $\{\tilde{u}_k\}$ is a sequence of harmonic maps with finite total energy and $\tilde{u}$ is another harmonic map, which is the limit of the sequence $\{\tilde{u}_k\}$ in the sense of the supremum norm, satisfying Eq. (4).

For any fixed $\epsilon > 0$, there exists a positive integer $N_1 \geq N_0$ such that for all $k \geq N_1$,

$$\rho(u_k, u) < \epsilon \quad \text{and} \quad \rho(\tilde{u}_k, \tilde{u}) < \epsilon \quad \text{on} \quad M. \tag{8}$$

Since u and $\tilde{u}$ satisfy Eqs. (2) and (3) , we have

$$\rho\left(\lim_{x\in M\to\mathbf{x}} u_k(x),\, f(\mathbf{x})\right) \le \epsilon \quad \text{and} \quad \rho\left(\lim_{x\in M\to\mathbf{x}} \tilde{u}_k(x),\, f(\mathbf{x})\right) \le \epsilon$$

for each $\mathbf{x} \in \Delta_M$. Hence

$$\lim_{x\in M\to\mathbf{x}} \rho(u_k(x), \tilde{u}_k(x)) \le 2\epsilon$$

for each $\mathbf{x} \in \Delta_M$. Since u_k and $\tilde{u}_k$ are harmonic maps with finite total energy, by Proposition 1, there exists $\epsilon_3 > 0$ such that

$$\rho(u_k, \tilde{u}_k) < \epsilon_3 \quad \text{on} \quad M, \tag{9}$$

where $\epsilon_3 \to 0$ as $\epsilon \to 0$. Combining (8) and (9), we have

$$\rho(u, \tilde{u}) < 2\epsilon + \epsilon_3 \quad \text{on} \quad M.$$

Since $\epsilon > 0$ is arbitrarily chosen, $u \equiv \tilde{u}$ on M. Thus we have the uniqueness. □

On the other hand, if the harmonic boundary of a complete Riemannian manifold has finite cardinality, then every continuous function on the harmonic boundary can be extended to a bounded continuous harmonic function with finite Dirichlet integral on the manifold. (See 3 G. Space HD of Finite Dimension in [8].) In this case, by Theorem 3, there exists a unique bounded harmonic map with finite total energy as the solution of the boundary value problem of harmonic maps as follows:

Corollary 1 *Let M and N be complete Riemannian manifolds. Let $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Suppose that the cardinality of the harmonic boundary Δ_M of M is l. Then for any $p_1, p_2, \ldots, p_l \in \mathcal{B}_{r_0}(p)$, there exists a unique bounded harmonic map u with finite total energy from M into $\mathcal{B}_{r_0}(p)$ such that for each $i = 1, 2, \ldots, l$,*

$$\lim_{x\in M\to\mathbf{x}_i} u(x) = p_i,$$

where $\Delta_M = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l\}$.

Sung et al. [9] proved that if a complete Riemannian manifold M has finitely many nonparabolic ends $E_1, E_2, \ldots, E_l$ with respect to some compact set and every bounded harmonic function on M is asymptotically constant at infinity of each nonparabolic end, then for any $p_1, p_2, \ldots, p_l$ in a regular geodesic ball in a complete Riemannian manifold N, there exists a unique harmonic map $u : M \to N$ with finite total energy and image within the regular geodesic ball such that for each $i = 1, 2, \ldots, l$,

$$\lim_{x\to\infty, x\in E_i} u(x) = p_i.$$

In the case when every bounded harmonic function on M is asymptotically constant at infinity of each nonparabolic end of M, each nonparabolic end contains only one point of the harmonic boundary Δ_M. (See Lemma 15 in [5].) Furthermore, parabolic ends of M have no points of the harmonic boundary Δ_M. (See Lemma 14 in [5].) Thus we can prove the result of [9] as a corollary of our result.

Corollary 2 *Let $E_1, E_2, \ldots, E_l$ be the nonparabolic ends of M with respect to some compact set. Suppose that every bounded harmonic function on M is asymptotically constant at infinity of each E_i. Let $\mathcal{B}_{r_0}(p)$ be a regular geodesic ball centered at a point p of radius r_0 in N. Then for any $p_1, p_2, \ldots, p_l \in \mathcal{B}_{r_0}(p)$, there exists a unique bounded harmonic map u with finite total energy from M into $\mathcal{B}_{r_0}(p)$ such that for each $i = 1, 2, \ldots, l$,*

$$\lim_{x \to \infty, x \in E_i} u(x) = p_i.$$

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