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On the Paranormed Space of Bounded Variation Double Sequences

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Abstract

In this study, as the domain of four-dimensional backward difference matrix in the space $\mathcal{L}_u(t)$, we introduce the complete paranormed space $\mathcal{BV}(t)$ of bounded variation double sequences and examine some properties of that space. Also, we determine the γ-dual and $\beta(\vartheta)$-dual of the space $\mathcal{BV}(t)$. Finally, we characterize the classes $(\mathcal{BV}(t) : \mathcal{M}_u)$, $(\mathcal{BV}(t) : \mathcal{C}_\vartheta)$ and $(\mathcal{L}_u(t) : \mu)$ with $\mu \in \{\mathcal{BS}, \mathcal{CS}_\vartheta, \mathcal{M}_u(\Delta), \mathcal{C}_\vartheta(\Delta)\}$, where $\mathcal{M}_u(\Delta)$ and $\mathcal{C}_\vartheta(\Delta)$ denote the spaces of all double sequences whose Δ-transforms are in the spaces $\mathcal{M}_u$ and $\mathcal{C}_\vartheta$, respectively.

Keywords Summability theory · Double sequences · Double series · Alpha, beta and gamma duals · Matrix domain of four-dimensional matrices · Matrix transformations

Mathematics Subject Classification 46A45 · 40C05

1 Introduction

We denote the set of all real- or complex-valued double sequences by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any subspace of Ω is called as *a double sequence space*. We write $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp}$ and $\mathcal{C}_r$ for the spaces of bounded, convergent in the Pringsheim's sense, both convergent in the Pringsheim's

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sense and bounded, and regularly convergent double sequences, respectively. Here and after, unless stated otherwise we assume that ϑ denotes any of the symbols p, bp or r.

Let λ be a space of double sequences, converging with respect to some linear convergence rule $\vartheta-\lim:\lambda\to\mathbb{C}$. The sum of a double series $\sum_{k,l}x_{kl}$ with respect to this rule is defined by $\vartheta-\sum_{k,l}x_{kl}=\vartheta-\lim_{m,n\to\infty}\sum_{k,l=0}^{m,n}x_{kl}$. In short, throughout the text the summations without limits run from 0 to ∞, for example $\sum_{k,l}x_{kl}$ means that $\sum_{k,l=0}^{\infty}x_{kl}$. The α*-dual* λ^{α}, the $\beta(\vartheta)$*-dual* $\lambda^{\beta(\vartheta)}$ with respect to the ϑ-convergence and the γ*-dual* λ^{γ} of a double sequence space λ are, respectively, defined by

$$\lambda^{\alpha}:=\left\{a=(a_{kl})\in\Omega:\sum_{k,l}|a_{kl}x_{kl}|<\infty\text{ for all }x=(x_{kl})\in\lambda\right\},$$

$$\lambda^{\beta(\vartheta)}:=\left\{a=(a_{kl})\in\Omega:\left(\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\right)_{m,n\in\mathbb{N}}\in\mathcal{C}_{\vartheta}\text{ for all }x=(x_{kl})\in\lambda\right\},$$

$$\lambda^{\gamma}:=\left\{a=(a_{kl})\in\Omega:\sup_{m,n\in\mathbb{N}}\left|\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\right|<\infty\text{ for all }x=(x_{kl})\in\lambda\right\}.$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^{\zeta}\subset\lambda^{\zeta}$ whenever $\lambda\subset\mu$. Also, $\lambda^{\alpha}\subset\lambda^{\beta(\vartheta)}$ and $\lambda^{\alpha}\subset\lambda^{\gamma}$. Furthermore, $\lambda^{\beta(\vartheta)}\subset\lambda^{\gamma}$ for $\vartheta\in\{bp,r\}$.

Let λ and μ be two double sequence spaces, and $A=(a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix mapping* from λ into μ and we write $A:\lambda\to\mu$, if for every sequence $x=(x_{kl})\in\lambda$ the A-transform $Ax=\{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and is in μ, where $\mathbb{N}=\{0,1,2,\ldots\}$ and

$$(Ax)_{mn}=\vartheta-\sum_{k,l}a_{mnkl}x_{kl}\quad\text{for each }m,n\in\mathbb{N}.\tag{1.1}$$

We define the ϑ*-summability domain* $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences by

$$\lambda_A^{(\vartheta)}:=\left\{x=(x_{kl})\in\Omega:Ax=\left(\vartheta-\sum_{k,l}a_{mnkl}x_{kl}\right)_{m,n\in\mathbb{N}}\text{ exists and is in }\lambda\right\}.$$

We say with the notation (1.1) that A maps the space λ into the space μ if $\lambda\subset\mu_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices, transforming the space λ into the space μ, by $(\lambda:\mu)$. Thus, $A=(a_{mnkl})\in(\lambda:\mu)$ if and only if the double series on the right side of (1.1) converges in the sense of ϑ for each $m,n\in\mathbb{N}$, i.e., $A_{mn}\in\lambda^{\beta(\vartheta)}$ for all $m,n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax\in\mu$ for all $x\in\lambda$, where $A_{mn}=(a_{mnkl})_{k,l\in\mathbb{N}}$ for all $m,n\in\mathbb{N}$. If in λ and μ there is some notion of limit or sum, then we write $(\lambda:\mu;P)$ to denote the subset of $(\lambda:\mu)$ which preserves the limit or sum with boundedness. That is, if $\{(Ax)_{mn}\}$ converges to the same limit

L whenever $x = (x_{kl})$ is convergent in the Pringsheim's sense with the limit L and bounded, then the transformation is said to be bounded regular or RH-regular. In this paper, we only consider bp-summability domain.

For all $m, n, k, l \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both [1]. Following Adams [1], we also say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$.

Türkmenoğlu (Gökhan) [34] characterized the classes $(\mathcal{C}_{bp}(t) : \mathcal{C}_p(t); P)$ and $(\mathcal{L}_u(t) : \mathcal{M}_u)$ of four-dimensional matrices. Zeltser [44] essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences in her PhD thesis. Gökhan and Çolak [12–14] defined the spaces $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{C}_{bp}(t)$ and $\mathcal{L}_u(t)$, where $t = (t_{kl})$ is a double sequence of strictly positive real numbers t_{kl} and

$$\mathcal{L}_u(t) := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^{t_{kl}} < \infty \right\}.$$

Altay and Başar [2] defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_\vartheta$ and $\mathcal{BV}$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_\vartheta$ and $\mathcal{L}_u$, respectively, and also examined some properties of those spaces. Here and after, by $t = (t_{kl})$ and $t' = (t'_{kl})$, we denote the double sequence of strictly positive real numbers and any bounded sequence of strictly positive real numbers, respectively, such that $t_{kl}^{-1} + t_{kl}'^{-1} = 1$ for all $k, l \in \mathbb{N}$. Başar and Sever [4] introduced the Banach space $\mathcal{L}_q$ of absolutely q-summable double sequences corresponding to the well-known space ℓ_q of absolutely q-summable single sequences and studied some properties of the space $\mathcal{L}_q$ with $1 \leq q < \infty$. Gökhan et al. [15] characterized the matrix classes $(\mathcal{C}_{bp}(t) : \mathcal{C}_{bp})$, $(\mathcal{C}_{bp}(t) : \mathcal{C}_{bp}; P)$, $(\mathcal{C}_{bp0}(t) : \mathcal{C}_{p0}(t'))$, $(\mathcal{C}_{bp}(t) : \mathcal{C}_{p0}(t'))$, $(\mathcal{M}_u(t) : \mathcal{C}_{p0}(t'))$. Mursaleen and Başar [16] have introduced the spaces $\widetilde{\mathcal{M}_u}$, $\widetilde{\mathcal{C}_\vartheta}$ and $\widetilde{\mathcal{L}_s}$ of double sequences whose Cesàro transforms are in the spaces $\mathcal{M}_u$, $\mathcal{C}_\vartheta$ and $\mathcal{L}_s$, respectively. Quite recently, Demiriz and Duyar [11] have introduced the spaces $\mathcal{M}_u(\Delta)$, $\mathcal{C}_p(\Delta)$, $\mathcal{C}_{0p}(\Delta)$, $\mathcal{C}_{bp}(\Delta)$, $\mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ of double sequences whose difference transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, and regularly convergent and absolutely q-summable, respectively, and also, they have examined some inclusion relations concerning with those sequence spaces. Quite recently, Yeşilkayagil and Başar [35–43], Başar and Çapan [5,6], Çapan and Başar [7–10] and Tuğ [31–33] have worked on the normed/paranormed spaces of double sequences and domain of triangle matrices in these spaces, and matrix transformations. Patterson has studied the characterization of the classes of four-dimensional matrices, in [18–25]. We should note that by using functional analysis techniques Talebi has recently obtained various properties of linear operators represented by four-dimensional triangle matrices between certain spaces of double sequences, in [26–30]. The reader can refer to the textbooks Başar [3] and Mursaleen and Mohiuddine [17] for relevant terminology and required details on the double sequences and related topics.

The four-dimensional backward difference matrix $\Delta = (d_{mnkl})$ is defined by

$$d_{mnkl} := \begin{cases} (-1)^{m+n-(k+l)}, & m-1 \le k \le m,\ n-1 \le l \le n, \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$. Therefore, the Δ-transform of a double sequence $x = (x_{mn})$ is given by

$$y_{mn} = (\Delta x)_{mn} := \begin{cases} x_{00}, & m, n = 0, \\ x_{0n} - x_{0,n-1}, & m = 0,\ n \ge 1, \\ x_{m0} - x_{m-1,0}, & m \ge 1,\ n = 0, \\ x_{m-1,n-1} - x_{m-1,n} - x_{m,n-1} + x_{mn}, & m, n \ge 1 \end{cases} \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Additionally, a direct calculation gives the inverse $\Delta^{-1} = S = (s_{mnkl})$ of the triangle matrix Δ as follows:

$$s_{mnkl} := \begin{cases} 1, & 0 \le k \le m,\ 0 \le l \le n, \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

for all $k, l, m, n \in \mathbb{N}$. Here, we can redefine the relation between the double sequences $x = (x_{mn})$ and $y = (y_{kl})$ by summation matrix S as follows:

$$x_{mn} = (Sy)_{mn} = \sum_{k,l=0}^{m,n} y_{kl} \tag{1.4}$$

for all $m, n \in \mathbb{N}$. Throughout the paper, we suppose that the terms of the double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are connected with the relation (1.2), and we use the convention that any term with negative subscript is equal to zero.

In the present paper, we introduce the new paranormed space $\mathcal{BV}(t)$ of bounded variation double sequences; that is, $\mathcal{BV}(t)$ is defined by

$$\mathcal{BV}(t) := \{x = (x_{mn}) \in \Omega : \Delta x \in \mathcal{L}_u(t)\}.$$

2 The Space $\mathcal{BV}(t)$ of Double Sequences

This section is devoted to certain algebraic and topological properties of the paranormed space $\mathcal{BV}(t)$ of bounded variation double sequences.

Theorem 2.1 *Let* $0 < t_{mn} \le H = \sup_{m,n\in\mathbb{N}} t_{mn} < \infty$ *and* $M = \max\{1, H\}$. *Then, the set* $\mathcal{BV}(t)$ *is a linear space with the coordinatewise addition and scalar multiplication and is a complete linear metric space paranormed by g defined by*

$$g(x) = \left[\sum_{m,n\in\mathbb{N}} |(\Delta x)_{mn}|^{t_{mn}} \right]^{1/M}$$

which is linearly isomorphic to the spaces $\mathcal{L}_u(t)$.

Proof One can easily show the first part of the theorem by a routine verification. So, we omit the detail.

Consider the transformation T defined from $\mathcal{BV}(t)$ to $\mathcal{L}_u(t)$ by $x \mapsto y = Tx = \{(\Delta x)_{mn}\}$, where the sequences x and y are connected with the relation (1.2). It is trivial that T is linear and injective.

Let $y = (y_{mn}) \in \mathcal{L}_u(t)$ and define the sequence $x = (x_{kl})$ via y by (1.4). Therefore, by taking into account the hypothesis $y \in \mathcal{L}_u(t)$ one can derive the following equality

$$\sum_{m,n} \left|(\Delta x)_{mn}\right|^{t_{mn}} = \sum_{m,n} |y_{mn}|^{t_{mn}} < \infty$$

which gives that $x \in \mathcal{BV}(t)$. Therefore, T is surjective.

This step concludes the proof. □

Theorem 2.2 *Neither of the spaces $\mathcal{L}_u(t)$ and $\mathcal{BV}(t)$ includes the other one.*

Proof Define the double sequences $t = (t_{mn})$ and $x = (x_{mn})$ by

$$t_{mn} := \begin{cases} 1, & m \in \mathbb{N} \text{ and } n \text{ is even,} \\ 2, & m \in \mathbb{N} \text{ and } n \text{ is odd} \end{cases} \quad \text{and} \quad x_{mn} := \begin{cases} 1/n, & m = 1 \text{ and } n \text{ is odd,} \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$, respectively. Then, it is obvious that

$$\sum_{m,n} |x_{mn}|^{t_{mn}} = \sum_{k} \frac{1}{(2k+1)^2} < \infty,$$

that is, $x \in \mathcal{L}_u(t)$. But

$$\sum_{m,n} |(\Delta x)_{mn}|^{t_{mn}} \geq \sum_{n=0}^{\infty} |(\Delta x)_{1n}|^{t_{1n}} \geq \sum_{k=0}^{\infty} |(\Delta x)_{1,2k+2}|^{t_{1,2k+2}} = \sum_{k=0}^{\infty} \frac{1}{2k+1} = \infty,$$

i.e., $x \notin \mathcal{BV}(t)$.

Now, choose the sequence $y = (y_{mn})$ defined by $y_{mn} = 1$ for all $m, n \in \mathbb{N}$. Obviously, $y \in \mathcal{BV}(t)$. Nevertheless, since

$$\sum_{m,n} |y_{mn}|^{t_{mn}} = \infty$$

for all $m, n \in \mathbb{N}$, $y \notin \mathcal{L}_u(t)$.

This completes the proof. □

A double sequence space λ is said to be *solid* (cf. [4, p. 153]) if and only if

$$\widetilde{\lambda} := \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N}\} \subset \lambda.$$

A double sequence space λ is said to be *monotone* if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0, 1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0, 1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all double sequences consisting of 0s and 1s. If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta(\vartheta)}$ [44, p. 36], and λ is monotone whenever λ is solid.

Theorem 2.3 *The double sequence space $\mathcal{BV}(t)$ is not monotone.*

Proof Let us consider the double sequence $x = (x_{mn})$ defined by

$$x_{mn} := \begin{cases} 1, & m = 0, n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Thus, we have

$$\sum_{m,n} |(\Delta x)_{mn}|^{t_{mn}} = |x_{00}|^{t_{00}} + |x_{00}|^{t_{10}} = 2,$$

that is, $x \in \mathcal{BV}(t)$. Now, we consider the double sequence $u = (u_{mn}) \in \{0, 1\}^{\mathbb{N}\times\mathbb{N}}$ defined by

$$u_{mn} := \begin{cases} 1, & m + n \text{ is even}, \\ 0, & \text{otherwise} \end{cases}$$

and let $z = (z_{mn}) = (x_{mn}u_{mn})$. Hence,

$$\sum_{m,n} |(\Delta z)_{mn}|^{t_{mn}} = \sum_{n} |-z_{0,n-1}+z_{0n}|^{t_{0n}} + \sum_{n} |z_{0,n-1}-z_{0n}|^{t_{1n}} = \sum_{n} 1 + \sum_{n} 1 = \infty,$$

that is, $z \notin \mathcal{BV}(t)$.

This step completes the proof. □

As a natural consequence of Theorem 2.3, we have.

Corollary 2.4 *The space $\mathcal{BV}(t)$ is not solid.*

3 The Gamma and Beta Duals of the New Space of Double Sequences

In this section, we determine the γ-dual and $\beta(\vartheta)$-*dual* of the sequence space $\mathcal{BV}(t)$.

Lemma 3.1 [34, Theorem 2] *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{M}_u)$ if and only if*

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}|^{t_{kl}} < \infty. \tag{3.1}$$

(ii) *Let* $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{M}_u)$ *if and only if there exists an integer* $B > 1$ *such that*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| a_{mnkl} B^{-1} \right|^{t'_{kl}} < \infty. \tag{3.2}$$

Lemma 3.2 [8, Theorem 3.1] *Let* $A = (a_{mnkl})$ *be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let* $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{C}_\vartheta)$ *if and only if the condition* (3.1) *holds and*

$$\exists \alpha_{kl} \in \mathbb{C} \text{ such that } \vartheta - \lim_{m,n\to\infty} a_{mnkl} = \alpha_{kl} \text{ exists for all } k, l \in \mathbb{N}. \tag{3.3}$$

(ii) *Let* $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{C}_\vartheta)$ *if and only if the conditions* (3.1)–(3.3) *hold.*

Lemma 3.3 [14, Theorem 10] *Let* $1 < t_{kl}$ *for all* $k, l \in \mathbb{N}$. *Define the set* $M_2(t)$ *by*

$$M_2(t) := \bigcup_{B>1} \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^{t'_{kl}} B^{-t'_{kl}/t_{kl}} < \infty \right\}.$$

Then, $\{\mathcal{L}_u(t)\}^\alpha = \{\mathcal{L}_u(t)\}^{\beta(\vartheta)} = \{\mathcal{L}_u(t)\}^\gamma = M_2(t)$.

Lemma 3.4 [14, Theorem 11] *Let* $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ *for all* $k, l \in \mathbb{N}$. *Then,* $\{\mathcal{L}_u(t)\}^\alpha = \{\mathcal{L}_u(t)\}^{\beta(\vartheta)} = \{\mathcal{L}_u(t)\}^\gamma = M_u(t)$.

Theorem 3.5 *Define the sets* D_1, D_2, D_3 *and* D_4, *as follows:*

$$D_1 := \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n,k,l\in\mathbb{N}} \left| \sum_{i,j=k,l}^{m,n} a_{ij} \right|^{t_{kl}} < \infty \right\},$$

$$D_2 := \left\{ a = (a_{kl}) \in \Omega : \exists (\alpha_{kl}) \in \Omega \ni bp - \lim_{m,n\to\infty} \sum_{i,j=k,l}^{m,n} a_{ij} = \alpha_{kl} \right\},$$

$$D_3 := \bigcup_{B>1} \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| \sum_{i,j=k,l}^{m,n} a_{ij} B^{-1} \right|^{t'_{kl}} < \infty \right\}.$$

Then, the following statements hold:

(i) *If* $0 < \inf_{k,l\in\mathbb{N}} t_{kl} < t_{kl} \leq 1$ *for all* $k, l \in \mathbb{N}$, *then* $\{\mathcal{BV}(t)\}^\gamma = D_1$ *and* $\{\mathcal{BV}(t)\}^{\beta(bp)} = D_1 \cap D_2$.

(ii) *If* $1 < t_{kl} \leq H = \sup_{k,l\in\mathbb{N}} < \infty$ *for all* $k, l \in \mathbb{N}$, *then* $\{\mathcal{BV}(t)\}^\gamma = D_3$ *and* $\{\mathcal{BV}(t)\}^{\beta(bp)} = D_1 \cap D_2 \cap D_3$.

Proof (i) Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} < t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$ and $x = (x_{kl}) \in \mathcal{BV}(t)$. Then, from Theorem 2.1 there exists a double sequence $y = (y_{kl}) \in \mathcal{L}_u(t)$ via x by (1.2). Define the double sequence $z = (z_{mn})$ with respect to the double sequence $a = (a_{kl}) \in \{\mathcal{BV}(t)\}^{\gamma}$ such that $z_{mn} = \sum_{k,l=0}^{m,n} a_{kl}x_{kl}$. Then, we easily derive with (1.4) that

$$z_{mn} = \sum_{k,l=0}^{m,n} a_{kl}x_{kl} = \sum_{k,l=0}^{m,n} \left(\sum_{i,j=k,l}^{m,n} a_{ij} \right) y_{kl} = (By)_{mn} \tag{3.4}$$

for all $m, n \in \mathbb{N}$, where the four-dimensional matrix $B = (b_{mnkl})$ is defined by

$$b_{mnkl} = \begin{cases} \sum_{i,j=k,l}^{m,n} a_{ij}, & 0 \leq k \leq m,\ 0 \leq l \leq n \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

for all $m, n, k, l \in \mathbb{N}$. Thus, we see that $ax = (a_{mn}x_{mn}) \in \mathcal{BS}$ whenever $x = (x_{mn}) \in \mathcal{BV}(t)$ if and only if $z = (z_{mn}) \in \mathcal{M}_u$ whenever $y = (y_{mn}) \in \mathcal{L}_u(t)$. This leads us to the fact that $B \in (\mathcal{L}_u(t) : \mathcal{M}_u)$. Therefore, we obtain from Part (i) of Lemma 3.1 that $\{\mathcal{BV}(t)\}^{\gamma} = D_1$.

For determining the $\beta(bp)$-dual of the space $\mathcal{BV}(t)$, take any $a = (a_{kl}) \in \{\mathcal{BV}(t)\}^{\beta(bp)}$. It is easily seen from (3.4) that $ax = (a_{mn}x_{mn}) \in \mathcal{CS}_{bp}$ whenever $x = (x_{mn}) \in \mathcal{BV}(t)$ if and only if $z = (z_{mn}) \in \mathcal{C}_{bp}$ whenever $y = (y_{mn}) \in \mathcal{L}_u(t)$. Therefore, we derive from Part (i) of Lemma 3.2 that $\{\mathcal{BV}(t)\}^{\beta(bp)} = D_1 \cap D_2$.

(ii) This is easily obtained by proceeding as in the proof of Part (i), above by using Parts (ii) of Lemmas 3.1 and 3.2, respectively. So, we omit the detail. □

Corollary 3.6 *Define the sets D_4 and D_5 as*

$$D_4 = \left\{ a = (a_{kl}) \in \Omega : \exists(\alpha_{kl}) \in \Omega \ni p - \lim_{m,n\to\infty} \sum_{i,j=k,l}^{m,n} a_{ij} = \alpha_{kl} \right\},$$
$$D_5 = \left\{ a = (a_{kl}) \in \Omega : \exists(\alpha_{kl}) \in \Omega \ni r - \lim_{m,n\to\infty} \sum_{i,j=k,l}^{m,n} a_{ij} = \alpha_{kl} \right\}.$$

Then, we have the following statements:

(i) *If $0 < \inf_{k,l\in\mathbb{N}} t_{kl} < t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$, then $\{\mathcal{BV}(t)\}^{\beta(p)} = D_1 \cap D_4$ and $\{\mathcal{BV}(t)\}^{\beta(r)} = D_1 \cap D_5$.*

(ii) *If $1 < t_{kl} \leq H = \sup_{k,l\in\mathbb{N}} < \infty$ for all $k, l \in \mathbb{N}$, then $\{\mathcal{BV}(t)\}^{\beta(p)} = D_1 \cap D_3 \cap D_4$ and $\{\mathcal{BV}(t)\}^{\beta(r)} = D_1 \cap D_3 \cap D_5$.*

Remark 3.7 Since the class $(\mathcal{L}_u(t) : \mathcal{L}_u)$ of four-dimensional matrices is not yet characterized, the α-dual of the new space was not given.

4 Characterization of Some Matrix Classes

In this section, we characterize the classes of four-dimensional matrix transformations from the spaces $\mathcal{L}_u(t)$ and $\mathcal{BV}(t)$ to any of the spaces $\mathcal{C}_\vartheta, \mathcal{CS}_\vartheta, \mathcal{BS}, \mathcal{M}_u(\Delta)$ and $\mathcal{C}_\vartheta(\Delta)$.

Let $A = (a_{mnkl})$ be any four-dimensional matrix and $F = (f_{mnkl})$ be a four-dimensional triangle matrix such that $f_{mnkl} = 0$ for $k > m$ and $l > n$ for all $k, l, m, n \in \mathbb{N}$ with the inverse $F^{-1} = (h_{mnkl})$. We define the four-dimensional matrices $E = (e_{mnkl})$ and $G = (g_{mnkl})$ via A and F by

$$E = AF^{-1} \quad \text{and} \quad G = FA, \tag{4.1}$$

where A and E are the four-dimensional usual dual summability matrices introduced by Yeşilkayagil and Başar [35]. It is immediate that (4.1) is equivalent to the following relations

$$e_{mnkl} = \sum_{i,j=k,l}^{\infty} a_{mnij} h_{ijkl} \quad \text{and} \quad g_{mnkl} = \sum_{i,j=0}^{m,n} f_{mnij} a_{ijkl} \tag{4.2}$$

for all $k, l, m, n \in \mathbb{N}$. In the rest of the text, we suppose that the elements of the matrices A, E and G are connected with the relations given in (4.2). For simplicity in the notation, we write that

$$a(m,n,k,l) = \sum_{i,j=0}^{m,n} a_{ijkl}, \quad d(m,n,k,l) = \sum_{i,j=m-1,n-1}^{m,n} (-1)^{m+n-(i+j)} a_{ijkl},$$
$$b(m,n,k,l) = \sum_{i,j=k,l}^{\infty} a_{mnij}$$

for all $k, l, m, n \in \mathbb{N}$.

Theorem 4.1 *Let λ and μ be any given two double sequence spaces. Then, $A \in (\lambda_F : \mu)$ if and only if $A_{mn} \in \lambda_F^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and $E \in (\lambda : \mu)$.*

Proof Suppose that (4.2) holds and let $A \in (\lambda_F : \mu)$, and take any $u \in \lambda$, where $u = Fv$ for $v \in \lambda_F$. Since $EF = AF^{-1}F = A$, EF exists and $A_{mn} \in \lambda_F^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$. Hence, $Eu = EFv = Av$ which leads us to the consequence that $E \in (\lambda : \mu)$.

Conversely, assume that $A_{mn} \in \lambda_F^{\beta(\vartheta)}$ for each fixed $m, n \in \mathbb{N}$ and $E \in (\lambda : \mu)$, let $v \in \lambda_F$ and let $Fv = u \in \lambda$. Since $Eu = EFv = Av$ and by the hypothesis, Av exists and is in μ. This shows that $A \in (\lambda_F : \mu)$.

This completes the proof. □

Now, we can give the following lemma.

Lemma 4.2 [42, Theorem 4.7] *Let λ and μ be any two given double sequence spaces. Then, $A \in (\lambda : \mu_F)$ if and only if $G \in (\lambda : \mu)$.*

It is trivial that Theorem 4.1 and Lemma 4.2 enable the characterization of the classes of linear transformations on/into the domain of several four-dimensional triangle matrices in certain spaces of double sequences. In other words, since the double sequence spaces μ and μ_F are linearly isomorphic, Theorem 4.1 and Lemma 4.2 have several consequences depending on the choice of the spaces λ and μ, and a four-dimensional triangle matrix $F = (f_{mnkl})$. Indeed, if we choose $\lambda = \mathcal{L}_u(t)$, $\mu \in \{\mathcal{M}_u, \mathcal{C}_{bp}\}$ and $F \in \{S, \Delta\}$, then we obtain the following results from Theorem 4.1 and Lemmas 3.1, 3.2 and 4.2, respectively.

Corollary 4.3 *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}(t) : \mathcal{M}_u)$ if and only if (3.1) holds with $b(m, n, k, l)$ instead of a_{mnkl} and $A_{mn} \in \{\mathcal{BV}(t)\}^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$.*
(ii) *Let $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}(t) : \mathcal{M}_u)$ if and only if (3.2) holds with $b(m, n, k, l)$ instead of a_{mnkl}, and $A_{mn} \in \{\mathcal{BV}(t)\}^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$.*

Corollary 4.4 *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}(t) : \mathcal{C}_\vartheta)$ if and only if (3.1) and (3.3) hold with $b(m, n, k, l)$ instead of a_{mnkl}, and $A_{mn} \in \{\mathcal{BV}(t)\}^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$.*
(ii) *Let $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}(t) : \mathcal{C}_\vartheta)$ if and only if (3.1)–(3.3) hold with $b(m, n, k, l)$ instead of a_{mnkl}, and $A_{mn} \in \{\mathcal{BV}(t)\}^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$.*

Corollary 4.5 *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{BS})$ if and only if (3.1) holds with $a(m, n, k, l)$ instead of a_{mnkl}.*
(ii) *Let $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{BS})$ if and only if (3.2) holds with $a(m, n, k, l)$ instead of a_{mnkl}.*

Corollary 4.6 *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{CS}_\vartheta)$ if and only if the conditions (3.1) and (3.3) hold with $a(m, n, k, l)$ instead of a_{mnkl}.*
(ii) *Let $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ for all $k, l \in \mathbb{N}$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{CS}_\vartheta)$ if and only if the conditions (3.1)–(3.3) hold with $a(m, n, k, l)$ instead of a_{mnkl}.*

Corollary 4.7 *Let $A = (a_{mnkl})$ be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let* $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{M}_u(\Delta))$ *if and only if* (3.1) *holds with* $d(m, n, k, l)$ *instead of* a_{mnkl}.

(ii) *Let* $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{M}_u(\Delta))$ *if and only if* (3.2) *holds with* $d(m, n, k, l)$ *instead of* a_{mnkl}.

Corollary 4.8 *Let* $A = (a_{mnkl})$ *be any four-dimensional matrix. Then, the following statements hold:*

(i) *Let* $0 < \inf_{k,l\in\mathbb{N}} t_{kl} \leq t_{kl} \leq 1$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{C}_\vartheta(\Delta))$ *if and only if the conditions* (3.1) *and* (3.3) *hold with* $d(m, n, k, l)$ *instead of* a_{mnkl}.

(ii) *Let* $1 < t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} = H < \infty$ *for all* $k, l \in \mathbb{N}$. *Then,* $A = (a_{mnkl}) \in (\mathcal{L}_u(t) : \mathcal{C}_\vartheta(\Delta))$ *if and only if the conditions* (3.1)–(3.3) *hold with* $d(m, n, k, l)$ *instead of* a_{mnkl}.

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