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Caristi-Type Fixed Point Theorems and Some Generalizations on *M*-Metric Space

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Abstract
In this paper, taking into account Caristi's fixed point results on both metric spaces and partial metric spaces, we present their some extensions and generalizations on M-metric spaces. First, by providing a counter example, we noticed that a recent result on Caristi-type fixed point theorem on M-metric space is not suitable. Then we propounded two versions of Caristi's inequality and proved some related fixed point results on M -metric space.

Keywords M-metric space · Fixed point · Single valued mapping

Mathematics Subject Classification Primary 54H25; Secondary 47H10

1 Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ be a mapping. If there exists a lower semicontinuous function $\phi : X \to [0, \infty)$ satisfying

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$$d(x, Tx) \leq \phi(x) - \phi(Tx),$$

for all $x \in X$, then T is said to be a Caristi mapping on the metric space (X, d). Caristi [7] proved that if (X, d) is complete metric space, then every Caristi mapping on X has a fixed point. This result is known as Caristi's fixed point theorem in the literature. Later, Kirk [8] presented a characterization of completeness of a metric space via Caristi mappings as follows: If every Caristi mapping on a metric space (X, d) has a fixed point, then (X, d) is complete. Since Caristi's fixed point theorem is equivalent to Ekeland variational principle, it has many applications to great number of branches in mathematics such as nonlinear analysis, differential geometry, dynamical systems, optimization and mathematical programming [10]. Because of its importance, Caristi's fixed point theorem has been extended and generalized by many authors (see [6,13]). Romaguera [12] presented the partial metric version of Caristi's fixed point theorem and also obtained Kirk-type characterization of partial metric spaces. Then Acar et al. [3] modified the result of Romaguera to be more suitable. To state their results, we will remember the partial metric space and its some properties.

Matthews [9] introduced the notion of partial metric which is more general than ordinary metric as follows: A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$,

- $p(x, y) = p(x, x) = p(y, y) \Leftrightarrow x = y$,
- $p(x, x) \leq p(x, y)$,
- $p(x, y) = p(y, x)$,
- $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

If we take a partial metric p on a nonempty set X, then the couple (X, p) is called a partial metric space.

Let (X, p) be a partial metric space and $U \subseteq X$. Then, U is called open if there exists a real number $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ for each point x in U, where $B(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$. Then, the collection of all open subsets, say τ_p, of X is a topology on X.

Let (X, p) be a partial metric space, $\{x_n\} \subset X$ be a sequence and $x \in X$. Then,

- $\{x_n\}$ is said to be converged to x if and only if

$$\lim_{n\to\infty} p(x_n, x) = p(x, x).$$

- $\{x_n\}$ is called Cauchy sequence if

$$\lim_{n,m\to\infty} p(x_n, x_m)$$

  exist and are finite.
- (X, p) is said to be complete partial metric space if and only if every Cauchy sequence on this space converges to a point $x \in X$ and

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

Remark 1 ([9]) Let (X, p) be a partial metric space. If we define a mapping $d_p : X \times X \to [0, \infty)$ as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

then d_p is an ordinary metric on X.

Lemma 1 ([9,11]) *Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. In this case, we get*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
(b) (X, p) is complete partial metric space if and only if (X, d_p) is complete metric space. Moreover,

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \text{ if and only if } p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Let (X, p) be a partial metric space, $T : X \to X$ be a mapping and $\phi : X \to [0, \infty)$ be a function which is lower semicontinuous with respect to τ_{d_p}. Then Romaguera said that T is a Caristi mapping on the partial metric space (X, p), if it is satisfied

$$p(x, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$. However, although every identity mapping is a Caristi mapping on metric space, it may not be a Caristi mapping (in the sense of Romaguera) on partial metric space. By considering this fact, Acar et al. [3] stated the following theorem which is a more suitable version of Caristi's fixed point theorem on partial metric space.

Theorem 1 ([3]) *Let (X, p) be a complete partial metric space and T be a self mapping on X. If there exists $\phi : X \to [0, \infty)$ which is lower semicontinuous with respect to τ_{d_p}, satisfying*

$$p(x, Tx) \leq p(x, x) + \phi(x) - \phi(Tx)$$

for all $x \in X$, then T has a fixed point in X.

Recently, the concept of M-metric has been introduced by Asadi et al [4]. Then they proved M-metric version of Banach contraction principle. When we look at properties of M-metric, it can be easily seen that every ordinary metric and every partial metric are M-metric. Now, we recall the definition and some properties of it.

Definition 1 Let X be a nonempty set. The mapping $m : X \times X \to [0, \infty)$ is called M-metric on X such that for all $x, y, z \in X$,

m1) $m(x, y) = m(x, x) = m(y, y) \Leftrightarrow x = y$,
m2) $m_{x,y} = \min\{m(x, x), m(y, y)\} \leq m(x, y)$,
m3) $m(x, y) = m(y, x)$,

m4) $m(x,y) - m_{x,y} \leq \left(m(x,z) - m_{x,z}\right) + \left(m(z,y) - m_{z,y}\right)$.

Then the couple (X, m) is called M-metric space.

Let (X, m) be a M-metric space and $x \in X$. Then open ball with centered $x \in X$ and radius $\varepsilon > 0$ in the M-metric space is defined as

$$B(x, \varepsilon) = \{y \in X : m(x, y) < m_{xy} + \varepsilon\}.$$

We call a subset U of X is open if and only if there is a real number $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ for every $x \in U$. Then, the family τ_m of all open subsets of X is a topology on X.

Definition 2 Let (X, m) be an M-metric space, $\{x_n\} \subset X$ be a sequence and $x \in X$. Then,

(1) $\{x_n\}$ is said to be M-converged to x if and only if

$$\lim_{n\to\infty} m(x_n, x) - m_{x_n x} = 0.$$

(2) $\{x_n\}$ is called M-Cauchy sequence if

$$\lim_{n,k\to\infty} m(x_n, x_k)$$

exist and are finite.

(3) (X, m) is said to be M-complete if and only if every M-Cauchy sequence on this space M-converges to a point $x \in X$ and

$$\lim_{n,k\to\infty} m(x_n, x_k) = m(x, x).$$

Remark 2 ([5]) Let (X, m) be an M-metric space. In this case, the function $p_m : X \times X \to [0, \infty)$ defined by

$$p_m(x, y) = m(x, y) - m_{xy} + M_{xy}$$

is a partial metric on X, where

$$M_{xy} = \max\{m(x, x), m(y, y)\}.$$

By taking into account Remark 1 and Remark 2, we can obtain the following: Let (X, m) be a M-metric space. Then the function d_{p_m} defined by

$$\begin{aligned} d_{p_m}(x, y) &= 2p_m(x, y) - p_m(x, x) - p_m(y, y) \\ &= 2(m(x, y) - m_{xy}) + (M_{xy} - m_{xy}) \end{aligned}$$

for all $x, y \in X$ is an ordinary metric on X.

Lemma 2 *Let (X, m) be a M-metric space and $\{x_n\}$ be a sequence in X. In this case, we get*

(a) $\{x_n\}$ is a M-Cauchy sequence in (X, m) if and only if it is a Cauchy sequence in the metric space (X, d_{p_m}).
(b) (X, m) is M-complete if and only if (X, d_{p_m}) is complete. Moreover,

$$\lim_{n\to\infty} d_{p_m}(x_n, x) = 0 \text{ if and only if } m(x, x) = \lim_{n\to\infty} m(x_n, x) = \lim_{n,k\to\infty} m(x_n, x_k).$$

We state the following conclusion by using Lemma 1 and Lemma 2 because it plays an important role in our results.

Lemma 3 *Let (X, m) be a M-metric space and $\{x_n\}$ be a sequence in X. Then, we have*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p_m) if and only if it is a M-Cauchy sequence in the M-metric space (X, m).
(b) (X, p_m) is complete if and only if (X, m) is M-complete.

To present the Caristi's fixed point theorem on M-metric space, Abodayeh et al. [1] considered the inequality

$$m(x, Tx) \leq m_{xTx} + \phi(x) - \phi(Tx) \tag{1.1}$$

as a M-metric version of Caristi's inequality and then they presented the following result:

Theorem 2 *Let (X, m) be a M-complete M-metric space and $\phi : X \to [0, \infty)$ be a lower semicontinuous function with respect to τ_m. Assume that $T : X \to X$ is a self mapping of X satisfying the inequality (1.1) for all $x \in X$. Then, T has a fixed point.*

However, the following example shows this is not a suitable extension. In this example, T has no fixed point, but it satisfies all conditions of this theorem.

Example 1 Let $X = [0, \infty)$ and define the mapping $m : X \times X \to [0, \infty)$ by $m(x, y) = \min\{x, y\}$. In this case, (X, m) is a M-complete M-metric space. (Note that every sequence $\{x_n\}$,M-converges to each point in X) If we define the self mapping $T : X \to X$ by $Tx = x + 1$ and $\phi : X \to [0, \infty)$ by $\phi(x) = c$ (c is a constant), then the mapping ϕ is lower semicontinuous with respect to τ_m and the mapping T satisfies condition (1.1). But, T has no fixed point.

The aim of this paper is to overcome this problem. Here we proposed two new inequalities for M-metric version of Caristi's inequality so that by taking into account them we extend the Caristi's fixed point theorem to M -metric space as appropriately.

2 Main Result

Definition 3 Let (X, m) be an M-metric space, T be a self mapping of X and $\phi : X \to [0, \infty)$ be a lower semicontinuous function with respect to $\tau_{d_{p_m}}$. Then T is said to be

- Caristi mapping of type (I), if it satisfies

$$m(x, Tx) - m_{xTx} + M_{xTx} \leq m(x, x) + \phi(x) - \phi(Tx) \tag{2.1}$$

  for all $x \in X$,
- Caristi mapping of type (II), if it satisfies

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + \phi(x) - \phi(Tx) \tag{2.2}$$

  for all $x \in X$.

Theorem 3 *Let (X, m) be an M-complete M-metric space and $T : X \to X$ be a Caristi mapping of type (I), then T has a fixed point.*

Proof Since the mapping T is a Caristi mapping of type (I), then there exists a lower semicontinuous function (with respect to $\tau_{d_{pm}}$) $\phi : X \to [0, \infty)$ such that

$$m(x, Tx) - m_{xTx} + M_{xTx} \leq m(x, x) + \phi(x) - \phi(Tx)$$

for all $x \in X$. Therefore we have

$$p_m(x, Tx) \leq p_m(x, x) + \phi(x) - \phi(Tx)$$

for all $x \in X$. On the other hand, from Lemma 3, (X, p_m) is a complete partial metric space. So by Theorem 1, T has fixed point. □

Theorem 4 *Let (X, m) be an M-complete M-metric space and $T : X \to X$ be Caristi mapping of type (II), then T has a fixed point.*

Proof Since (X, m) is M-complete M-metric space, then from Lemma 3, (X, p_m) is complete partial metric space. Furthermore, because the mapping T is a Caristi mapping of type (II), then there exists a lower semicontinuous function (with respect to $\tau_{d_{pm}}$) $\phi : X \to [0, \infty)$ such that

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + \phi(x) - \phi(Tx) \tag{2.3}$$

for all $x \in X$. Now we have two cases:

Case 1. If $M_{xTx} = m(Tx, Tx)$, then from (2.3) we get both

$$m(x, Tx) \leq m_{xTx} + \phi(x) - \phi(Tx)$$

and

$$M_{xTx} = m(Tx, Tx) \leq m_{xTx} + \phi(x) - \phi(Tx).$$

By summing the last inequalities, we have

$$m(x, Tx) + M_{xTx} \leq 2m_{xTx} + 2\phi(x) - 2\phi(Tx)$$

$$= m_{xTx} + m(x,x) + 2\phi(x) - 2\phi(Tx)$$

and so

$$m(x,Tx) - m_{xTx} + M_{xTx} \le m(x,x) + 2\phi(x) - 2\phi(Tx).$$

Therefore we get

$$p_m(x,Tx) \le p_m(x,x) + 2\phi(x) - 2\phi(Tx).$$

Case 2. If $M_{xTx} = m(x,x)$, then from (2.2) we have (note that $\phi(x) - \phi(Tx) \ge 0$)

$$\begin{aligned} m(x,Tx) - m_{xTx} + M_{xTx} &\le \phi(x) - \phi(Tx) + M_{xTx} \\ &= m(x,x) + \phi(x) - \phi(Tx) \\ &\le m(x,x) + 2\phi(x) - 2\phi(Tx). \end{aligned}$$

Therefore we get

$$p_m(x,Tx) \le p_m(x,x) + 2\phi(x) - 2\phi(Tx).$$

Now, let $\beta : X \to [0,\infty)$, $\beta(x) = 2\phi(x)$ for all $x \in X$. Then the mapping β is also lower semicontinuous with respect to $\tau_{d_{p_m}}$. Thus from both Case 1 and Case 2, we get

$$p_m(x,Tx) \le p_m(x,x) + \beta(x) - \beta(Tx)$$

for all $x \in X$, so by Theorem 1, T has fixed point. □

The following examples give us the difference between Theorem 4 and Theorem 3.

Example 2 Let $X = [0,1] \cup \{2\}$ and $m : X \times X \to [0,\infty)$, $m(x,y) = \frac{x+y}{2}$ for all $x, y \in X$. Then (X, m) is a M-complete M-metric space. (Note that d_{p_m} is usual metric on X) Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2\ , & x = 1 \\ x\ , & \text{otherwise} \end{cases}$$

and $\phi : X \to [0,\infty)$ by

$$\phi(x) = \begin{cases} 0 & , \quad x = 0 \\ \frac{7}{4} & , \quad x = 2 \\ 2 + \frac{1}{x} & , \ \text{otherwise} \end{cases}.$$

In this case, ϕ is a lower semicontinuous w.r.t. $\tau_{d_{pm}}$ and then T is a Caristi mapping of type (II) but not of type (I). Indeed, since $Tx = x$ for $x \neq 1$, then (2.2) holds. Also, for $x = 1$, we have

$$\max\{m(x, Tx), m(Tx, Tx)\} = 2 \leq \frac{9}{4} = m_{xTx} + \phi(x) - \phi(Tx)$$

and so (2.2) holds. However, for $x = 1$, we have

$$m(x, Tx) - m_{xTx} + M_{xTx} = \frac{5}{2}$$

and

$$m(x, x) + \phi(x) - \phi(Tx) = \frac{9}{4}.$$

Thus T is not Caristi mapping of type (I).

Example 3 Let $X = [0, \infty)$ and $m : X \times X \to [0, \infty)$, $m(x, y) = |x - y| + \min\{x, y\}$ for all $x, y \in X$. Then (X, m) is a M-complete M-metric space. (Note that $\tau_{d_{pm}}$ is usual topology on X) Define the mapping $T : X \to X$ by $Tx = \sqrt{x}$ and $\phi : X \to [0, \infty)$ by

$$\phi(x) = \begin{cases} \frac{1}{x} \text{ , } 0 < x < 1 \\ x \text{ , otherwise} \end{cases}.$$

In this case, ϕ is a lower semicontinuous w.r.t. $\tau_{d_{pm}}$. Then it is clear that T is a Caristi mapping of type (II). However, by taking $x \in \left(\frac{1}{\sqrt[3]{4}}, 1\right)$ we can see that T is not Caristi mapping of type (I).

In the following, inspired by Bae [6], Suzuki [13], Acar and Altun [2] we provide some generalizations of Caristi's fixed point theorem on M-metric space.

Theorem 5 *Let (X, m) be an M-complete M-metric space, $\phi : X \to [0, \infty)$ be a function which is lower semicontinuous with respect to $\tau_{d_{pm}}$ satisfying*

$$m(x, y) = m(y, y) \text{ implies } \phi(y) \leq \phi(x) \tag{2.4}$$

and $\psi : X \to [0, \infty)$ be a function such that

$$\sup\{\psi(x) : x \in X, \phi(x) \leq \inf_{y \in X} \phi(y) + \mu\} < \infty$$

for some $\mu > 0$. If $T : X \to X$ be a mapping satisfying

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + \psi(x)\{\phi(x) - \phi(Tx)\} \tag{2.5}$$

for all $x \in X$, then T has a fixed point.

Proof Let $x \in X$. If $\psi(x) > 0$, then from (2.5), we get $\phi(Tx) \leq \phi(x)$. If $\psi(x) = 0$, then we have $m(x, Tx) = m(Tx, Tx)$ and thus $\phi(Tx) \leq \phi(x)$. Therefore, we can say $\phi(Tx) \leq \phi(x)$ for all $x \in X$. Now, we define the set Y and the real number γ for $\mu > 0$ as follows:

$$Y = \{x \in X : \phi(x) \leq \inf_{y \in X} \phi(y) + \mu\}$$

and

$$\gamma = \sup_{w \in Y} \psi(w).$$

Since (X, m) is M-complete M-metric space and from Lemma 2, (X, d_{p_m}) is complete metric space. Besides, since we have ϕ be a lower semicontinuous function, then Y is closed in (X, d_{p_m}). Therefore (Y, d_{p_m}) is a complete metric space and so (Y, m) is M -complete M-metric space. On the other hand, from the definition of infimum, the set Y is nonempty. Since $\phi(Tx) \leq \phi(x)$ for all $x \in X$, we get $T(Y) \subset Y$. Also, we have

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + \psi(x)\{\phi(x) - \phi(Tx)\}$$

and so

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + \gamma\{\phi(x) - \phi(Tx)\}$$

for all $x \in Y$. Now, if we define the function $\varphi : X \to [0, \infty)$ as $\varphi(x) = \gamma\phi(x)$ for all $x \in X$, then φ is lower semicontinuous with respect to $\tau_{d_{p_m}}$. So, by using Theorem 4, we can see that T has a fixed point. □

Remark 3 If the M-metric m is an ordinary metric, then the condition (2.4) is satisfied so that Theorem 5 turns to Theorem 2 of [13].

Theorem 6 *Let (X, m) be an M-complete M-metric space, $\phi : X \to [0, \infty)$ be a function which is lower semicontinuous with respect to $\tau_{d_{p_m}}$ satisfying condition (2.4) and $c : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} &\max\{m(x, Tx), m(Tx, Tx)\} \\ &\leq m_{xTx} + \max\{c(\phi(x), c(\phi(Tx))\}\{\phi(x) - \phi(Tx)\} \end{aligned}$$

for all $x \in X$. Then T has a fixed point.

Proof Fix $\gamma > c(t_0)$, where $t_0 = \inf_{w \in X} \phi(w)$. Then, since the function c is upper semicontinuous, there exists $\mu > 0$ such that $c(t) \leq \gamma$ for $t \in [t_0, t_0 + \mu]$. As in proof Theorem 5, we can show $\phi(Tx) \leq \phi(x)$ for all $x \in X$. On the other hand, if we define a function $\psi : X \to [0, \infty)$ by $\psi(x) = \max\{c(\phi(x)), c(\phi(Tx))\}$, then we have $\phi(Tx) \leq t_0 + \mu$ for all $x \in X$ with $\phi(x) \leq t_0 + \mu$ and so $\psi(x) \leq \gamma$. In this case, it can be easily seen that $\sup\{\psi(x) : x \in X, \phi(x) \leq \inf_{y\in X}\phi(y) + \mu\} \leq \gamma < \infty$. So, by using Theorem 5, we can see that T has a fixed point. □

Theorem 7 *Let (X, m) be an M-complete M-metric space, $\phi : X \to [0, \infty)$ be a function which is lower semicontinuous with respect to $\tau_{d_{pm}}$ satisfying condition (2.4) and $c : [0, \infty) \to [0, \infty)$ be a nondecreasing function. If $T : X \to X$ be a mapping satisfying either*

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + c(\phi(x))\{\phi(x) - \phi(Tx)\} \tag{2.6}$$

or

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + c(\phi(Tx))\{\phi(x) - \phi(Tx)\} \tag{2.7}$$

for all $x \in X$, then T has a fixed point.

Proof As in proof Theorem 5, we can prove that $\phi(Tx) \leq \phi(x)$ for all $x \in X$. Since c is a nondecreasing function, we have $c(\phi(Tx)) \leq c(\phi(x))$. So, it is enough to investigate just (2.6) inequality. If we define function $\psi : X \to [0, \infty)$ such that $\psi(x) = c(\phi(x))$ for all $x \in X$, then we have

$$\sup\{\psi(x) : x \in X, \phi(x) \leq \inf_{y\in X}\phi(y) + \mu\} \leq c\left(\inf_{y\in X}\phi(y) + \mu\right) < \infty.$$

Therefore, by Theorem 5, we can see that T has a fixed point. □

Theorem 8 *Let (X, m) be an M-complete M-metric space, $\phi : X \to [0, \infty)$ be a function which is lower semicontinuous with respect to $\tau_{d_{pm}}$ satisfying condition (2.4) and $c : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function. If $T : X \to X$ be a mapping satisfying $m(x, Tx) \leq \phi(x)$ and*

$$\max\{m(x, Tx), m(Tx, Tx)\} \leq m_{xTx} + c(m(x, Tx))\{\phi(x) - \phi(Tx)\}$$

for all $x \in X$, then T has a fixed point.

Proof Let define $\psi : X \to [0, \infty)$ as $\psi(x) = c(m(x, Tx))$ for all $x \in X$. Then for all $x \in X$ with $\phi(x) \leq \inf_{y\in X}\phi(y) + 1$, we have

$$\begin{aligned}\psi(x) &\leq \sup\{c(t) : 0 \leq t \leq m(x, Tx)\}\\ &\leq \sup\{c(t) : 0 \leq t \leq \phi(x)\}\\ &\leq \sup\{c(t) : 0 \leq t \leq \inf_{y\in X}\phi(y) + 1\}.\end{aligned}$$

In this case, we have

$$\sup\{\psi(x) : x \in X, \phi(x) \leq \inf_{y\in X}\phi(y) + 1\} \leq \max\{c(t) : 0 \leq t \leq \inf_{y\in X}\phi(y) + 1\} < \infty.$$

Hence, we obtain the desired result by Theorem 5. □

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