Bull. Malays. Math. Sci. Soc. (2020) 43:2509–2522
https://doi.org/10.1007/s40840-019-00820-x

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Eccentric Adjacency Index of Graphs with a Given Number of Cut Edges

Shehnaz Akhter[1] · Rashid Farooq[1]

Received: 1 January 2019 / Revised: 20 July 2019 / Published online: 3 August 2019

Abstract

The eccentric adjacency index of an n-vertex-connected and simple graph G is defined as $\xi^{ad}(G) = \sum_{x \in V_G} \frac{S_G(x)}{ec_G(x)}$, where $S_G(x)$ is the sum of degrees of neighbors of x and $ec_G(x)$ is the eccentricity of x in G. Denote by $\mathcal{G}(n, k)$ the set of n-vertex-connected graphs with k cut edges, where $0 \le k \le n-1 (k \ne n-2)$. In this paper, we determine the graph with largest eccentric adjacency index and characterize the extremal graph among all graphs in $\mathcal{G}(n, k)$.

Keywords Eccentric adjacency index · Extremal values · Cut edge

Mathematics Subject Classification 05C07 · 05C12 · 05C35

1 Introduction

Let G be an n-vertex-connected and simple graph with vertex set V_G and edge set E_G. The degree of a vertex x in G, denoted by $\deg_G(x)$, is $\deg_G(x) = |\Gamma_G(x)|$, where $\Gamma_G(x)$ is the set of neighbors of x in G. A vertex x is called a pendent vertex if $\deg_G(x) = 1$. The length of a shortest path connecting the vertices x and y in G is called the distance between x and y and is denoted by $d_G(x, y)$. For a vertex $x \in V_G$, the eccentricity of x in G is $ec_G(x) = \max_{y \in V_G} d_G(x, y)$. The radius r_G and the diameter d_G of G are defined by $r_G = \min_{x \in V_G} ec_G(x)$ and $d_G = \max_{x \in V_G} ec_G(x)$,

Communicated by Sandi Klavžar.

✉ Shehnaz Akhter
shehnazakhter36@yahoo.com

Rashid Farooq
farook.ra@gmail.com

1 School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad, Pakistan

respectively. For a vertex $x \in V_G$, the sum of degrees of neighbors of the vertex x in G is $S_G(x) = \sum_{y \in \Gamma_G(x)} \deg_G(y)$.

A path from x_1 to x_n consisting of vertices $x_1, x_2, \ldots, x_n$ is written by $x_1x_2 \ldots x_n$ and is called x_1, x_n-path. The vertices x_1 and x_n are the end vertices, and $x_2, x_3, \ldots, x_{n-1}$ are the internal vertices of the path $x_1x_2 \ldots x_n$. A path with length d_G in G is said to be a diametrical path in G. An n-vertex tree with $n-1$ pendant vertices and a vertex of degree $n-1$ is said to be a star and is denoted by S_n. An n-vertex simple graph is called a complete graph if every pair of its vertices is linked by an edge and is denoted by K_n. A complete subgraph of an n-vertex graph G is called a clique in G.

For $S \subseteq V_G$ and $F \subseteq E_G$, the graphs $G - S$ and $G - F$ are the subgraphs induced by $V_G - S$ and $E_G - F$, respectively. A vertex u (respectively, an edge e) is called a cut vertex (respectively, cut edge) of an n-vertex-connected graph G, if $G - v$ (respectively, $G - e$) has at least two components. A cut edge is called an internal cut edge if it is not a pendent edge. An n-vertex graph G is called a 2-connected (respectively, 2-edge-connected), if $G - v$ (respectively, $G - e$) is connected, for every $v \in V_G$ (respectively, $e \in E_G$). An n-vertex-connected graph is said to be a block if it does not have any cut vertex. The cyclomatic number of an n-vertex-connected graph G is $c(G) = m - n + 1$, where m is the size of G. In particular, if $c(G) = 0$ then G is a tree. If $c(G) = 1$ then G is a unicyclic graph, and if $c(G) = 2$ then G is a bicyclic graph. Every tree has at most $n-1$ cut edges and an n-vertex-connected graph having cyclomatic number at least one has at most $n-3$ cut edges. Thus, it is obvious from the above statement that for any n-vertex-connected graph with k cut edges, we always have $0 \leq k \leq n-1$ and $k \neq n-2$.

A molecular graph G is a representation of the structural formula of a chemical compound in terms of graph theory. A topological index is a number which characterizes properties of G. Recently, the development of computational chemistry owes much to the topological index of a molecular graph. The topological indices have mainly described the non-empirical molecular structure quantitatively and analyzed the structure and performance of molecules. There are many classes of topological indices; some of them are distance-based, degree-based, degree-distance-based and eccentricity-based indices of graphs. The one of the most used topological indexes, Wiener index, is defined as the sum of all distances between unordered pairs of vertices

$$W(G) = \sum_{\{x,y\} \subseteq V_G} d_G(x, y).$$

Recently, many eccentricity-based topological indices have been defined, e.g., eccentric connectivity index, total eccentricity index, Zagreb eccentricity indices, etc. One of most investigated eccentricity-based indexes is the eccentric connectivity index, which was proposed by Sharma et al. [15]. The eccentric connectivity index is defined as:

$$\xi^c(G) = \sum_{x \in V_G} \deg_G(x) ec_G(x). \quad (1)$$

The eccentric connectivity index has been shown to give high level of predictability of pharmaceutical properties and provides leads for the development of safe and useful anti-HIV compounds [7].

The eccentric adjacency index (also known as Ediz eccentric connectivity index [8]) is the modification of eccentric connectivity index and is defined as follows:

$$\xi^{ad}(G) = \sum_{x \in V_G} \frac{S_G(x)}{ec_G(x)}. \tag{2}$$

Relationship of eccentric connectivity index and eccentric adjacency index has been investigated by Gupta et al. [10]. The eccentric distance sum was introduced by Gupta et al. [11], which was defined as:

$$\xi^{d}(G) = \sum_{x \in V_G} ec_G(x) D_G(x),$$

where $D_G(x) = \sum_{y \in V_G} d_G(y, x)$ is the sum of all distances from the vertex x.

In recent years, finding the extremal bounds for some topological indices in terms of graph structure parameters, has turned out to be a useful direction in extremal graph theory and many results are obtained. In [1], the authors found the extremal conjugated trees with respect to eccentric connectivity index and eccentric adjacency index. In [2], the authors determined the largest unicyclic graphs with a given girth and largest tree with a fixed diameter with respect to eccentric adjacency index. Akhter [3] derived the extremal trees for eccentric connectivity and eccentric adjacency indices in terms of other graph invariants including matching number, bipartition size, independence number and domination number. Hua [12] determined the smallest value of Wiener index among all n-vertex-connected graphs with k cut edges. Hua et al. [13] characterized the graphs with the smaller eccentric distance sum among all n-vertex-connected graphs with k cut edges. For further studies on topological indices of graphs with given parameters, we refer [4–6,9,14,16–20] to the readers.

Motivated by the work referred above, we continue the research on the eccentric adjacency index of graphs with some given parameters. In this paper, we find the graphs with the largest eccentric adjacency index among the n-vertex graphs with a given number of cut edges and characterize the extremal graphs.

2 The Connected Graphs with a Given Number of Cut Edges

Let $\mathcal{G}(n, k)$ be the set of n-vertex-connected graphs with k cut edges, where $0 \le k \le n-1 (k \ne n-2)$. Denote by K_{n-k}^{k} the graph obtained by attaching k pendent vertices to a unique vertex of a complete graph K_{n-k}. In this section, we find an n-vertex-connected graph in $\mathcal{G}(n, k)$ with largest eccentric adjacency index. First, we prove some lemmas which will be crucial to the proof of our main result (Fig. 1).

Lemma 1 *Let H_1 and H_2 be two vertex-disjoint-connected graphs each of order at least 2 with $u \in V_{H_1}$ and $v \in V_{H_2}$. Let G_1 be the graph obtained by connecting u and*

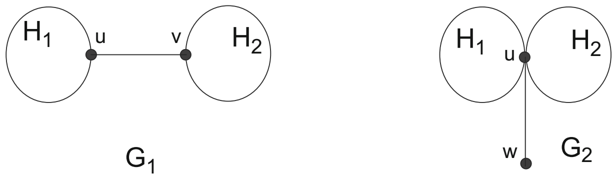

Fig. 1 The graphs G_1 and G_2 in Lemma 1

v by an edge uv, and G_2 be the graph obtained by identifying u with v and introducing a pendent edge uw with pendent vertex w, respectively. Then $\xi^{ad}(G_1) < \xi^{ad}(G_2)$.

Proof For each vertex $x \in V_{G_1}$, we have

$$\begin{aligned} ec_{G_1}(x) &= \max\{ec_{H_1}(x),\ d_{H_1}(x,u) + 1 + ec_{H_2}(v)\}, \text{ if } x \in V_{H_1}, \\ ec_{G_1}(x) &= \max\{ec_{H_2}(x),\ d_{H_2}(x,v) + 1 + ec_{H_1}(u)\}, \text{ if } x \in V_{H_2}. \end{aligned} \tag{3}$$

For each vertex $x \in V_{G_2} \backslash \{u, w\}$, we have

$$\begin{aligned} ec_{G_2}(x) &= \max\{ec_{H_1}(x),\ d_{H_1}(x,u) + ec_{H_2}(v),\ d_{H_1}(x,u) + 1\}, \text{ if } x \in V_{H_1}, \\ ec_{G_2}(x) &= \max\{ec_{H_2}(x),\ d_{H_2}(x,v) + ec_{H_1}(u),\ d_{H_2}(x,v) + 1\}, \text{ if } x \in V_{H_2}. \end{aligned} \tag{4}$$

Now, it is easily seen that the eccentricities of u and w in G_2 are as follows:

$$\begin{aligned} ec_{G_2}(u) &= \max\{ec_{H_1}(u),\ ec_{H_2}(v)\}, \\ ec_{G_2}(w) &= \max\{ec_{H_2}(u) + 1,\ ec_{H_2}(v) + 1\}. \end{aligned} \tag{5}$$

Note that from (3) and (4), we get $ec_{G_1}(x) \geq ec_{G_2}(x)$ for each $x \in V_{G_1} \backslash \{u, v\}$. By the construction of G_1 and G_2, for each $x \in V_{G_1} \backslash (\{u, v, w\} \cup \Gamma_{H_1}(u) \cup \Gamma_{H_2}(v))$, we have $S_{G_2}(x) = S_{G_1}(x)$. For each $x \in \Gamma_{H_1}(u)$, we have

$$\begin{aligned} S_{G_1}(x) &= \sum_{y \in \Gamma_{H_1}(x) \backslash \{u\}} \deg_{H_1}(y) + \deg_{H_1}(u) + 1, \\ S_{G_2}(x) &= \sum_{y \in \Gamma_{H_1}(x) \backslash \{u\}} \deg_{H_1}(y) + \deg_{H_1}(u) + \deg_{H_2}(v) + 1. \end{aligned} \tag{6}$$

For each $x \in \Gamma_{H_2}(v)$, we have

$$\begin{aligned} S_{G_1}(x) &= \sum_{y \in \Gamma_{H_2}(x) \backslash \{v\}} \deg_{H_2}(y) + \deg_{H_2}(v) + 1, \\ S_{G_2}(x) &= \sum_{y \in \Gamma_{H_2}(x) \backslash \{v\}} \deg_{H_2}(y) + \deg_{H_1}(u) + \deg_{H_2}(v) + 1. \end{aligned} \tag{7}$$

Furthermore, the sum of degrees of neighbors of the vertices u, v and w in G_1 and G_2 is given by

$$\begin{aligned}
S_{G_1}(u) &= \sum_{y\in\Gamma_{H_1}(u)} \deg_{H_1}(y) + \deg_{H_2}(v) + 1,\\
S_{G_1}(v) &= \sum_{y\in\Gamma_{H_2}(v)} \deg_{H_2}(y) + \deg_{H_1}(u) + 1,\\
S_{G_2}(u) &= \sum_{y\in\Gamma_{H_1}(u)} \deg_{H_1}(y) + \sum_{y\in\Gamma_{H_2}(v)} \deg_{H_2}(y) + 1,\\
S_{G_2}(w) &= \deg_{H_1}(u) + \deg_{H_2}(v) + 1.
\end{aligned} \tag{8}$$

Therefore, from (3) to (8), we obtain

$$\begin{aligned}
\xi^{ad}(G_1) - \xi^{ad}(G_2) \le & \sum_{x\in\Gamma_{H_1}(u)} \left(\frac{\sum_{y\in\Gamma_{H_1}(x)\setminus\{v\}} \deg_{H_1}(y) + \deg_{H_1}(u) + 1}{ec_{G_2}(x)} \right)\\
& - \sum_{x\in\Gamma_{H_1}(u)} \left(\frac{\sum_{y\in\Gamma_{H_1}(x)\setminus\{v\}} \deg_{H_1}(y) + \deg_{H_1}(u) + \deg_{H_2}(v) + 1}{ec_{G_2}(x)} \right)\\
& + \sum_{x\in\Gamma_{H_2}(v)} \left(\frac{\sum_{y\in\Gamma_{H_2}(x)\setminus\{v\}} \deg_{H_2}(y) + \deg_{H_2}(v) + 1}{ec_{G_2}(x)} \right)\\
& - \sum_{x\in\Gamma_{H_2}(v)} \left(\frac{\sum_{y\in\Gamma_{H_2}(x)\setminus\{v\}} \deg_{H_2}(y) + \deg_{H_1}(u) + \deg_{H_2}(v) + 1}{ec_{G_2}(x)} \right)\\
& + \frac{\sum_{y\in\Gamma_{H_1}(u)} \deg_{H_1}(y) + \deg_{H_2}(v) + 1}{\max\{ec_{H_1}(u),\ ec_{H_2}(v) + 1\}}\\
& - \frac{\sum_{y\in\Gamma_{H_1}(u)} \deg_{H_1}(y) + \sum_{y\in\Gamma_{H_2}(v)} \deg_{H_2}(y) + 1}{\max\{ec_{H_1}(u),\ ec_{H_2}(v)\}}\\
& + \frac{\sum_{y\in\Gamma_{H_2}(v)} \deg_{H_2}(y) + \deg_{H_1}(u) + 1}{\max\{ec_{H_2}(v),\ ec_{H_1}(u) + 1\}}\\
& - \frac{\deg_{H_1}(u) + \deg_{H_2}(v) + 1}{\max\{ec_{H_1}(u) + 1,\ ec_{H_2}(v) + 1\}}\\
= & - \left(\sum_{x\in\Gamma_{H_1}(u)} \frac{\deg_{H_2}(v)}{ec_{G_2}(x)} + \sum_{x\in\Gamma_{H_2}(v)} \frac{\deg_{H_2}(u)}{ec_{G_2}(x)} \right)
\end{aligned}$$

$$+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\deg_{H_2}(v)+1}{\max\{ec_{H_1}(u),\ ec_{H_2}(v)+1\}}$$
$$-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{\max\{ec_{H_1}(u),\ ec_{H_2}(v)\}}$$
$$+\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+\deg_{H_1}(u)+1}{\max\{ec_{H_2}(v),\ ec_{H_1}(u)+1\}}$$
$$-\frac{\deg_{H_1}(u)+\deg_{H_2}(v)+1}{\max\{ec_{H_1}(u)+1,\ ec_{H_2}(v)+1\}}$$

Case I If $ec_{H_1}(u)\geq ec_{H_2}(v)+1$, then

$$\begin{aligned}\xi^{ad}(G_1)-\xi^{ad}(G_2)=&-\left(\sum_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\&+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\deg_{H_2}(v)+1}{ec_{H_1}(u)}\\&-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{ec_{H_1}(u)}\\&+\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+\deg_{H_1}(u)+1}{ec_{H_1}(u)+1}\\&-\frac{\deg_{H_1}(u)+\deg_{H_2}(v)+1}{ec_{H_1}(u)+1}\\=&-\left(\sum_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\&+\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)-\deg_{H_2}(v)}{ec_{H_1}(u)+1}\\&-\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)-\deg_{H_2}(v)}{ec_{H_1}(u)}<0.\end{aligned}$$

Case II If $ec_{H_2}(v)\geq ec_{H_1}(u)+1$, then

$$\begin{aligned}\xi^{ad}(G_1)-\xi^{ad}(G_2)=&-\left(\sum_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\&+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\deg_{H_2}(v)+1}{ec_{H_2}(v)+1}\\&-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{ec_{H_2}(v)}\\&+\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+\deg_{H_1}(u)+1}{ec_{H_2}(v)}\end{aligned}$$

$$
\begin{aligned}
&-\frac{\deg_{H_1}(u)+\deg_{H_2}(v)+1}{ec_{H_2}(v)+1}\\
&=-\left(\sum\nolimits_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum\nolimits_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\
&\quad+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)-\deg_{H_1}(u)}{ec_{H_2}(v)+1}\\
&\quad-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)-\deg_{H_1}(u)}{ec_{H_2}(v)}<0.
\end{aligned}
$$

Case III If $ec_{H_1}(u)=ec_{H_2}(v)$, then

$$
\begin{aligned}
\xi^{ad}(G_1)-\xi^{ad}(G_2)&=-\left(\sum\nolimits_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum\nolimits_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\
&\quad+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\deg_{H_2}(v)+1}{ec_{H_2}(v)+1}\\
&\quad-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{ec_{H_2}(v)}\\
&\quad+\frac{\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+\deg_{H_1}(u)+1}{ec_{H_2}(v)+1}\\
&\quad-\frac{\deg_{H_1}(u)+\deg_{H_2}(v)+1}{ec_{H_2}(v)+1}\\
&=-\left(\sum\nolimits_{x\in\Gamma_{H_1}(u)}\frac{\deg_{H_2}(v)}{ec_{G_2}(x)}+\sum\nolimits_{x\in\Gamma_{H_2}(v)}\frac{\deg_{H_2}(u)}{ec_{G_2}(x)}\right)\\
&\quad+\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{ec_{H_2}(v)+1}\\
&\quad-\frac{\sum_{y\in\Gamma_{H_1}(u)}\deg_{H_1}(y)+\sum_{y\in\Gamma_{H_2}(v)}\deg_{H_2}(y)+1}{ec_{H_2}(v)}\qquad<0.
\end{aligned}
$$

This completes the proof. □

In the following lemma, we prove an elementary result.

Lemma 2 *Let $G\ncong K_n$ be an n-vertex-connected graph, and $u,v\in V_G$ be non-adjacent vertices of G. Then $\xi^{ad}(G)<\xi^{ad}(G+uv)$.*

Proof Observe that $d_G(u,v)\geq 2$ and $d_{G+uv}(u,v)=1$. Let $x\in V_G$ and A be the set of eccentric vertices of x in G, such that $ec_G(x)=d_G(x,u)+d_G(u,v)+d_G(v,y)$, for all $y\in A$. Then

$$
\begin{aligned}
ec_{G+uv}(x)&=d_{G+uv}(x,u)+d_{G+uv}(u,v)+d_{G+uv}(v,y)\\
&<d_G(x,u)+d_G(u,v)+d_G(v,y)=ec_G(x).
\end{aligned}
\tag{9}
$$

The eccentricities of other vertices of G are same in G and $G+uv$. The sum of degrees of neighbors of the vertices u and v in $G+uv$ is given by

$$\begin{aligned} S_{G+uv}(u) &= S_G(u) + \deg_G(v) + 1, \\ S_{G+uv}(v) &= S_G(v) + \deg_G(u) + 1. \end{aligned} \tag{10}$$

For each $x \in \Gamma_G(u)$ and $y \in \Gamma_G(v)$, we have

$$S_{G+uv}(x) = S_G(x) + 1, \quad S_{G+uv}(y) = S_G(y) + 1. \tag{11}$$

Therefore, from (9) to (11), we obtain

$$\begin{aligned} \xi^{ad}(G) - \xi^{ad}(G+uv) \leq & \frac{S_G(u)}{ec_{G+uv}(u)} - \frac{S_G(u) + \deg_G(v) + 1}{ec_{G+uv}(u)} + \frac{S_G(v)}{ec_{G+uv}(v)} \\ & - \frac{S_G(v) + \deg_G(u) + 1}{ec_{G+uv}(v)} + \sum_{x \in \Gamma_G(u)} \left(\frac{S_G(x)}{ec_{G+uv}(x)} \right. \\ & \left. - \frac{S_G(x) + 1}{ec_{G+uv}(x)} \right) + \sum_{y \in \Gamma_G(v)} \left(\frac{S_G(y)}{ec_{G+uv}(y)} - \frac{S_G(y) + 1}{ec_{G+uv}(y)} \right) \\ = & -\frac{\deg_G(v) + 1}{ec_{G+uv}(u)} - \frac{\deg_G(u) + 1}{ec_{G+uv}(v)} - \sum_{x \in \Gamma_G(u)} \frac{1}{ec_{G+uv}(x)} \\ & - \sum_{y \in \Gamma_G(v)} \frac{1}{ec_{G+uv}(y)} \\ < & \ 0. \end{aligned}$$

This completes the proof. □

Lemma 3 *Let H be a complete graph of order $n \geq 2$, and $v_1, \ldots, v_t \in V_H$ be some distinct vertices of H, where $2 \leq t \leq n$. Let $H_1, H_2, \ldots, H_t$ be the non-trivial connected graphs corresponding to a vertex $v_1, v_2, \ldots, v_t$, respectively, and $u_1 \in V_{H_1}, u_2 \in V_{H_2}, \ldots, u_t \in V_{H_t}$. Let G_3 be the graph obtained from H by identifying a vertex $u_j \in V_{H_j}$ to a vertex $v_j \in V_H$ for $j = 1, \ldots, t$, respectively. Let G_4 be the graph obtained from H by identifying the vertices $u_1, u_2, \ldots, u_t$ to a vertex, say $v_1 \in V_H$, of v_j's. Then $\xi^{ad}(G_3) < \xi^{ad}(G_4)$.*

Proof The order of both G_3 and G_4 is defined as $n = \sum_{j=1}^{t} |H_j| - t + |H|$. For each vertex $u \in V_{H_j}$, we have

$$\begin{aligned} ec_{G_3}(u) &= \max\{ec_{H_j}(u),\ d_{H_j}(u, u_j) + 1 + ec_{H_l}(u_l),\ l \neq j\}, \\ ec_{G_4}(u) &= \max\{ec_{H_j}(u),\ d_{H_j}(u, u_j) + ec_{H_l}(u_l),\ l \neq j\}. \end{aligned} \tag{12}$$

For each $w \in V_H \backslash \{v_1, v_2, \ldots, v_t\}$

$$ec_{G_3}(w) = ec_{G_4}(w) = \max\{1 + ec_{H_j}(v_j),\ j = 1, 2, \ldots, t\}. \tag{13}$$

From (12) and (13), it is obvious that $ec_{G_3}(u) \geq ec_{G_4}(u)$ for each $u \in V_{H_j}$. Let $A = V_{H_j} \backslash (V_H \cap V_{H_j}) \cup \Gamma_{H_j}(u_j)$. Note that $S_{G_3}(x) = S_{G_4}(x)$ for each $x \in A$, where $1 \leq j \leq t$. For each $v_1, v_2, \ldots, v_t \in V_H \cap V_{H_j}$, $j = 1, \ldots, t$,

$$S_{G_3}(v_l) = S_H(v_l) + \sum_{\substack{j=1,\\ j\neq l}}^{t} \deg_{H_j}(u_j) + S_{H_l}(u_l), \text{ for } l = 1, 2, \ldots, t. \tag{14}$$

Also

$$\begin{aligned} S_{G_3}(x) &= S_{G_4}(x) = S_H(x) + \sum_{j=1}^{t} \deg_{H_j}(u_j), \ \forall\, x \in V_H \backslash \{v_1, v_2, \ldots, v_t\}, \\ S_{G_3}(x) &= S_{H_j}(x) + \deg_H(v_j), \qquad \forall\, x \in \Gamma_{H_j}(u_j), \ \text{where } 1 \leq j \leq t. \end{aligned} \tag{15}$$

From (14) and (15), we obtain

$$\begin{aligned} \xi^{ad}(G_3) &= \sum_{l=1}^{t} \frac{1}{ec_{G_3}(v_l)} \left(S_H(v_l) + \sum_{\substack{j=1\\ j\neq l}}^{t} \deg_{H_j}(u_j) + S_{H_l}(u_l) \right) \\ &\quad + \sum_{x \in V_H \backslash \{v_1, v_2, \ldots, v_t\}} \frac{1}{ec_{G_3}(x)} \left(S_H(x) + \sum_{j=1}^{t} \deg_{H_j}(u_j) \right) \\ &\quad + \sum_{j=1}^{t} \sum_{x \in \Gamma_{H_j}(u_j)} \frac{1}{ec_{G_3}(x)} (S_{H_j}(x) + \deg_H(v_j)) + \sum_{j=1}^{t} \sum_{x \in A} \frac{S_{H_j}(x)}{ec_{G_3}(x)} \\ &= \sum_{x \in V_H} \frac{S_H(x)}{ec_{G_3}(x)} + \sum_{l=1}^{t} \sum_{\substack{j=1\\ j\neq l}}^{t} \frac{\deg_{H_j}(u_j)}{ec_{G_3}(v_l)} + \sum_{l=1}^{t} \frac{S_{H_l}(u_l)}{ec_{G_3}(v_l)} \\ &\quad + \sum_{j=1}^{t} \sum_{x \in V_H \backslash \{v_1, v_2, \ldots, v_t\}} \frac{\deg_{H_j}(u_j)}{ec_{G_3}(x)} + \sum_{j=1}^{t} \sum_{x \in \Gamma_{H_j}(u_j)} \frac{S_{H_j}(x)}{ec_{G_3}(x)} \\ &\quad + \sum_{j=1}^{t} \sum_{x \in \Gamma_{H_j}(u_j)} \frac{\deg_H(v_j)}{ec_{G_3}(x)} + \sum_{j=1}^{t} \sum_{x \in A} \frac{S_{H_j}(x)}{ec_{G_3}(x)}. \end{aligned} \tag{16}$$

Furthermore, the sum of the degrees of neighbors of v_1 in G_4 is as follows:

$$S_{G_4}(v_1) = S_H(v_1) + \sum_{j=1}^{t} S_{H_j}(u_j). \tag{17}$$

Also

$$\begin{aligned}
S_{G_4}(x) &= S_H(x) + \sum_{j=1}^{t} \deg_{H_j}(u_j),\ \forall\, x \in \Gamma_H(v_1),\\
S_{G_4}(x) &= S_{H_l}(x) + \deg_H(v_l)\\
&\quad + \sum_{\substack{j=1\\ j\neq l}}^{t} \deg_{H_j}(u_j), \qquad \forall\, x \in \Gamma_{H_l}(u_l),\ \text{where } 1 \le l \le t.
\end{aligned} \tag{18}$$

From (17) and (18), we obtain

$$\begin{aligned}
\xi^{ad}(G_4) &= \frac{1}{ec_{G_4}(v_1)}\left(S_H(v_1) + \sum_{j=1}^{t} S_{H_j}(u_j)\right) + \sum_{x\in\Gamma_H(v_1)} \frac{1}{ec_{G_4}(x)}\Bigg(S_H(x)\\
&\quad + \sum_{j=1}^{t} \deg_{H_j}(u_j)\Bigg) + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{1}{ec_{G_4}(x)}\Bigg(S_{H_l}(x)\\
&\quad + \deg_H(v_1) + \sum_{\substack{j=1\\ j\neq l}}^{t} \deg_{H_j}(u_j)\Bigg) + \sum_{j=1}^{t}\sum_{x\in A} \frac{S_{H_j}(x)}{ec_{G_4}(x)}\\
&= \sum_{x\in\Gamma_H(v_1)} \frac{S_H(x)}{ec_{G_4}(x)} + \sum_{j=1}^{t}\sum_{x\in\Gamma_H(v_1)} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)} + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{S_{H_l}(x)}{ec_{G_4}(x)}\\
&\quad + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_H(v_1)}{ec_{G_4}(x)} + \sum_{l=1}^{t}\sum_{\substack{j=1\\ j\neq l}}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)}\\
&\quad + \sum_{j=1}^{t}\sum_{x\in A} \frac{S_{H_j}(x)}{ec_{G_4}(x)}.
\end{aligned}$$

Since H is a complete graph, $\Gamma_H(v_1) = V_H \backslash \{v_1\}$ and the degree of every vertex is same.

$$\begin{aligned}
\xi^{ad}(G_4) &= \sum_{j=1}^{t} \frac{S_{H_j}(u_j)}{ec_{G_4}(v_1)} + \sum_{x\in V_H\backslash\{v_1\}} \frac{S_H(x)}{ec_{G_4}(x)} + \sum_{j=1}^{t}\sum_{x\in V_H\backslash\{v_1\}} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)}\\
&\quad + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{S_{H_l}(x)}{ec_{G_4}(x)} + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_H(v_1)}{ec_{G_4}(x)}\\
&\quad + \sum_{l=1}^{t}\sum_{\substack{j=1\\ j\neq l}}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)} + \sum_{j=1}^{t}\sum_{x\in A} \frac{S_{H_j}(x)}{ec_{G_4}(x)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in V_H} \frac{S_H(x)}{ec_{G_4}(x)} + \sum_{j=1}^{t} \frac{S_{H_j}(u_j)}{ec_{G_4}(v_1)} \qquad (19)\\
&\quad + \sum_{j=1}^{t}\left(\sum_{x\in V_H\setminus\{v_1,v_2,\ldots,v_t\}} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)} + \sum_{x\in\{v_2,\ldots,v_t\}} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)}\right)\\
&\quad + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{S_{H_l}(x)}{ec_{G_4}(x)} + \sum_{l=1}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_H(v_1)}{ec_{G_4}(x)}\\
&\quad + \sum_{l=1}^{t}\sum_{\substack{j=1\\ j\neq l}}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)} + \sum_{j=1}^{t}\sum_{x\in A} \frac{S_{H_j}(x)}{ec_{G_4}(x)}.
\end{aligned}
$$

Thus, from (16) and (19), we obtain

$$
\begin{aligned}
\xi^{ad}(G_3) - \xi^{ad}(G_4) \le &\sum_{j=2}^{t}\left(\frac{S_{H_j}(u_j)}{ec_{G_4}(v_j)} - \frac{S_{H_j}(u_j)}{ec_{G_4}(v_1)}\right)\\
&+ \sum_{j=2}^{t}\sum_{x\in\Gamma_{H_j}(u_j)}\left(\frac{\deg_H(v_j)}{ec_{G_4}(x)} - \frac{\deg_H(v_1)}{ec_{G_4}(x)}\right)\\
&+ \sum_{j=2}^{t}\left(\frac{\deg_{H_j}(u_j)}{ec_{G_4}(v_1)} - \frac{\deg_{H_j}(u_j)}{ec_{G_4}(v_j)}\right)\\
&- \sum_{l=1}^{t}\sum_{\substack{j=1\\ j\neq l}}^{t}\sum_{x\in\Gamma_{H_l}(u_l)} \frac{\deg_{H_j}(u_j)}{ec_{G_4}(x)}\\
&< 0.
\end{aligned}
$$

This completes the proof. □

By elementary calculations, one can easily derive the following lemma.

Lemma 4 *Let K^k_{n-k} be an n-vertex-connected graph as described above, where $0 \le k \le n-1 (k \neq n-2)$. Then*

$$
\xi^{ad}(K^k_{n-k}) = \begin{cases} \dfrac{n(n-1)^2}{2} & \text{for} k = 0,\\ \dfrac{1}{2}((n-k-1)^2(n-k) + (n-1)^2 + 2k) & \text{for} k \ge 1. \end{cases}
$$

Proof If $k = 0$ then $K^k_{n-k} \cong K_{n-k}$ and $\xi^{ad}(K^k_{n-k}) = \dfrac{n(n-1)^2}{2}$. Since there are $n-k-1$ vertices of eccentricity 2 and the sum of degrees of its neighbors $((n-k-2)(n-k-1)+(n-1))$, one vertex of eccentricity 1 and the sum of degrees of its neighbors $(n-k-1)^2+k$, and k pendent vertices of eccentricity 2 and the sum of degrees of its neighbor $n-1$ in K_{n-k}^k, $k \geq 1$. Therefore, by Eq. (2), we obtain the following:

$$\begin{aligned}\xi^{ad}(K_{n-k}^k) &= \frac{(n-k-1)[(n-k-2)(n-k-1)+(n-1)]}{2} \\ &\quad + \frac{(n-k-1)^2+k}{1} + \frac{k(n-1)}{2} \\ &= \frac{1}{2}(n-k-1)^2(n-k-2+2) + \frac{1}{2}(n-1)(n-k-1+k) + k \\ &= \frac{1}{2}((n-k-1)^2(n-k) + (n-1)^2 + 2k).\end{aligned}$$

This completes the proof. □

The following theorem gives the n-vertex-connected graph with larger eccentric adjacency index among all the graphs in $\mathcal{G}(n,k)$, where $0 \leq k \leq n-1 (k \neq n-2)$.

Theorem 1 *Let $G \in \mathcal{G}(n,k)$ be an n-vertex-connected graph with k cut edges. Then*

$$\xi^{ad}(G) \leq \begin{cases} \frac{n(n-1)^2}{2} & \text{for} \quad k=0, \\ \frac{1}{2}((n-k-1)^2(n-k)+(n-1)^2+2k) & \text{for} \quad k \geq 1, \end{cases}$$

equality hold if and only if $G \cong K_{n-k}^k$.

Proof Let $G_{max} \in \mathcal{G}(n,k)$ be a graph with the largest eccentric adjacency index among all n-vertex-connected graphs with k cut edges. Let $E' = \{e_1, e_2, \ldots, e_k\} \subseteq E_{G_{max}}$ be the set of all cut edges of G_{max}. Then, all edges in E' must be pendent edges and incident at a common vertex of G_{max}, say w. For $k=0$, the graph G_{max} has no cut edges and its each component is a clique or a single vertex. If G_{max} is not the graph as described above, then we can add an edge e between two non-adjacent vertices of G_{max} and obtain a new graph $G_{max}+uv$ having no cut edges. But by Lemma 2, we get $\xi^c(G_{max}) < \xi^c(G_{max}+uv)$ and it contradicts our assumption.

Therefore, now we have $1 \leq k \leq n-1$ and $k \neq n-2$. If G_{max} has an internal cut edge uv, then we can construct a new graph by identifying u with v and introducing a pendent edge uw with pendent vertex w and denote it by G_2. It is obvious that G_2 has k cut edges. Thus, by Lemma 3, we obtain $\xi^c(G_{max}) < \xi^c(G_2)$, which is a contradiction. When $k=n-1$ we have G_{max} is a tree, and thus, we have $G_{max} \cong S_n = K_{n-1}$, as our claim.

Next, we suppose that $1 \leq k \leq n-3$. Now let 2-edge-connected graph G_3 with order $n-k$ and k pendent edges is an induced subgraph of G_{max}. If $G_3 \ncong K_{n-k}$, then we can add edges into G_3. Similar to the above argument, we can deduce a new graph with a larger eccentric adjacency index than G_{max}, and therefore, $G_3 \cong K_{n-k}$ in G_{max}. Moreover, we can conform that all k pendent edges in G_{max} must be attached at the same vertex of K_{n-k}. Let $G_4 \ncong G_{max}$ be a graph with k pendent edges and these vertices attached at v_i vertices of G_4. Then, we can transform the k pendent edges to exactly one vertex of clique K_{n-k} of G_4. Therefore, by Lemma 1, we construct a new graph with a larger eccentric adjacency index than that of G_{max}, which is a contradiction.

Therefore, from all the above discussion, we must have $G_{max} \cong K_{n-k}^{k}$. By Lemma 4, we have $\xi^{ad}(K_{n-k}^{k}) = \frac{n(n-1)^2}{2}$ for $k = 0$, and $\xi^{ad}(K_{n-k}^{k}) = \frac{1}{2}((n-k-1)^2(n-k) + (n-1)^2 + 2k)$ for $k \geq 1$ and this completes the proof. □

The following result is the consequence of Theorem 1 for $k = n - 1$.

Theorem 2 (Akhter and Farooq [2]) *Let G be an n-vertex-connected graph, $n \geq 2$. Then, $\xi^{ad}(G) \leq \frac{n^2 - 1}{2}$ with equality if and only if $G \cong S_n$.*

References

1. Akhter, S., Farooq, R.: Computing the eccentric connectivity index and eccentric adjacency index of conjugated trees. Util. Math. **(accepted)**
2. Akhter, S., Farooq, R.: On the eccentric adjacency index of unicyclic graphs and trees. Asian Eur. J. Math. (2020). https://doi.org/10.1142/S179355712050028X
3. Akhter, S.: Two degree distance based topological indices of chemical trees. IEEE Access. **7**, 95653–95658 (2019)
4. Chen, S., Liu, W.: Extremal Zagreb indices of graphs with a given number of cut edges. Graphs Comb. **30**, 109–118 (2014)
5. Deng, H.: On the minimum Kirchhoff index of graphs with a given number of cut edges. MATCH Commun. Math. Comput. Chem. **63**, 171–180 (2010)
6. Du, Z., Zhou, B.: On the Estrada index of graphs with given number of cut edges. Electron. J. Linear Algebra **22**, 586–592 (2011)
7. Dureja, H., Gupta, S., Madan, A.K.: Predicting anti-HIV-1 activity of 6-arylbenzonitriles: computational approach using superaugmented eccentric connectivity topochemical indices. J. Mol. Graph. Model. **26**, 1020–1029 (2008)
8. Ediz, S.: On the Ediz eccentric connectivity index of a graph. Optoelectron. Adv. Mater. Rapid Commun. **5**(11), 1263–1264 (2011)
9. Farooq, R., Akhter, S., Rada, J.: Total eccentricity index of trees with fixed pendent vertices **(submitted)**
10. Gupta, S., Singh, M., Madan, A.K.: Predicting anti-HIV activity: computational approach using a novel topological descriptor. J. Comput. Aided Mol. Des. **15**(7), 671–678 (2001)
11. Gupta, S., Singh, M., Madan, A.K.: Eccentric distance sum: a novel graph invariant for predicting biological and physical properties. J. Math. Anal. Appl. **275**, 386–401 (2002)
12. Hua, H.: Wiener and Schultz molecular topological indices of graphs with specified cut edges. MATCH Commun. Math. Comput. Chem. **61**, 643–651 (2009)
13. Hua, H., Zhang, S., Xu, K.: Further results on the eccentric distance sum. Discrete Appl. Math. **160**, 170–180 (2012)
14. Liu, H., Lu, M., Tian, F.: On the spectral radius of graphs with cut edges. Linear Algebra Appl. **389**, 139–145 (2004)
15. Sharma, V., Goswami, R., Madan, A.K.: Eccentric connectivity index: a novel highly discriminating topological descriptor for structure–property and structure–activity studies. J. Chem. Inf. Comput. Sci. **37**, 273–282 (1997)
16. Wang, S., Wang, C., Chen, L.: On the maximum and minimum multiplicative Zagreb indices of graphs with given number of cut edges (2017). arXiv preprint arXiv:1705.02482
17. Wu, R., Chen, H., Deng, H.: On the monotonicity of topological indices and the connectivity of a graph. Appl. Math. Comput. **298**, 188–200 (2017)
18. Wu, Y.R., He, S., Shu, J.L.: Largest spectral radius among graphs with cut edges. J. East China Norm. Univ. Nat. Sci. Ed. **3**, 67–74 (2007)

19. Xu, K., Trinajstić, N.: Hyper Wiener and Harary indices of graphs with cut edges. Util. Math. **84**, 153–163 (2011)
20. Zhao, Q., Li, S.C.: On the maximum Zagreb indices of graphs with k cut edges. Acta Appl. Math. **111**, 93–106 (2010)