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Every Planar Graph Without 4-Cycles and 5-Cycles is (2, 6)-Colorable

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Abstract

A graph is $(d_1, \ldots, d_r)$-colorable if the vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the subgraph induced by V_i is at most d_i for each $i \in \{1, \ldots, r\}$. In this paper, we prove that every planar graph without 4-cycles and 5-cycles is (2, 6)-colorable, which improves the result of Sittitrai and Nakprasit, who proved that every planar graph without 4-cycles and 5-cycles is (2, 9)-colorable.

Keywords Improper coloring · Planar graph · Discharging method

Mathematics Subject Classification 05C15

1 Introduction

All graphs in this paper are finite and simple. A graph is *planar* if it has a drawing without crossings; such a drawing is a *planar embedding* of a planar graph. A *plane graph* is a particular planar embedding of a planar graph. Given a plane graph G, denote the vertex set, edge set and face set by $V(G)$, $E(G)$ and $F(G)$, respectively. The degree of a vertex v, denoted by $d(v)$, is the number of edges incident to v. The degree of a face f, denoted by $d(f)$, is the length of a shortest boundary walk of f. The *girth* of a graph G is the length of its shortest cycle.

A graph G is called *improper* $(d_1, \ldots, d_r)$-colorable, or simply $(d_1, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ such that the maximum degree of the induced subgraph $G[V_i]$ of G is at most d_i for $1 \leqslant i \leqslant r$.

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Using this terminology, the Four Color Theorem can be reformulated as that every planar graph is (0, 0, 0, 0)-colorable and the Grötzsch Theorem can be restated as every triangle-free planar graph is (0, 0, 0)-colorable. Improper coloring of plane graphs is a kind of relaxation of coloring of plane graphs, which is regarded as an important method to solve important plane graph coloring problems. One important version of improper colorings of planar graphs is that three colors are allowed. Cowen et al. [6] showed that every planar graph is (2, 2, 2)-colorable. Eaton and Hull [7] proved that (2, 2, 2)-colorability is optimal by exhibiting a non-$(1, d_1, d_2)$-colorable planar graph for any given nonnegative integers d_1 and d_2. Stronger results can be obtained by adding some restrictions. Liu et al. [10] showed that planar graphs without 5-cycles and intersecting triangles are (1, 1, 0)-colorable. For every planar G without 4-cycles and 5-cycles, Chen et al. [3] showed that G is (2,0,0)-colorable, Xu et al. [13] showed that G is (1,1,0)-colorable, but whether it is (1, 0, 0)-colorable is still open and more open problems; see [13]. Another version of improper colorings of planar graphs is that two colors are allowed. Montassier and Ochem [11] constructed planar graphs of girth 4 that are not (i, j)-colorable for every nonnegative integer i, j. For all $k \geqslant 2$, Borodin et al. [1] constructed non-$(0, k)$-colorable graphs with maximum average degree arbitrarily close to $\frac{3k+2}{k+1} = 3 - \frac{1}{k+1}$. On the other hand, Kim et al. [9] proved that planar graphs with girth at least 11 are (1, 0)-colorable. For every planar graph G of girth 8, Borodin and Kostochka [2] showed G is (2, 0)-colorable. For every planar graph G of girth 7, Borodin and Kostochka [2] showed G is (4, 0)-colorable. For every planar graph G of girth 6, Borodin and Kostochka [2] showed G is (4, 1)-colorable and Havet and Seren [8] showed that G is (2, 2)-colorable. For every planar graph G of girth 5, Choi et al. [5] showed that G is (1, 10)-colorable, Borodin and Kostochka [2] showed G is (2, 6)-colorable, Havet and Seren [8] showed that G is (4, 4)-colorable and Choi and Raspaud [4] showed that G is (3,5)-colorable. More interesting results can be found in Montassier and Ochem [11].

Recently, Sittitrai and Nakprasit [12] showed that every planar graph G without 4-cycles and 5-cycles is (2, 9)-, (3, 5)- and (4, 4)-colorable and constructed non-$(1, k)$-colorable planar graphs without 4-cycles and 5-cycles for every positive integer k.

Theorem 1.1 [12] *Every planar graph without 4-cycles and 5-cycles is* (d_1, d_2)-*colorable, where* $(d_1, d_2) \in \{(2, 9), (3, 5), (4, 4)\}$.

Motivated by above observations and Theorem 1.1, we present the following result in this paper.

Theorem 1.2 *Every planar graph without 4-cycles and 5-cycles is* (2, 6)-*colorable.*

Other notations that we use in this paper are as follows. A k-vertex (k^+-vertex, k^--vertex) is a vertex of degree k (at least k, at most k). A k-face (k^+-face, k^--face) is a face of degree k (at least k, at most k). A k-face $f = [v_1 v_2 \ldots v_k]$ is a $(d_1, d_2 \ldots, d_k)$-*face* if $d(v_i) = d_i$ for $1 \leqslant i \leqslant k$. Let $uv \in E(G)$. We call u a *pendant* neighbor of v if uv is not incident to any 3-faces. Moreover, we call u a pendant k-neighbor (k^+-neighbor, k^--neighbor) of v if u is a k-vertex (k^+-vertex, k^--vertex). A 3-face f is called a *pendant face* of u if f is incident to a pendant 3-neighbor of u. A 3-face f is called a *bad* 3-face if f is incident to a 2-vertex and *good* 3-face otherwise. A 2-vertex is called a *bad* 2-vertex if it is incident to a 3-face and *good* 2-vertex otherwise.

The proof of Theorem 1.2 is shown in Sects. 2 and 3. In Sect. 2, a counterexample G to Theorem 1.2 is constructed and the structural properties of G are investigated. In Sect. 3, discharging technique is used to derive a contradiction. In all figures in this paper, a black point means all its incident edges are drawn, a white point otherwise.

2 Structural Properties

Suppose Theorem 1.2 is false. Let G be a counterexample to Theorem 1.2 with $|V(G)|$ minimized, and subject to that choose one with minimum number of edges. Clearly, G is connected and with minimum degree at least 2. Moreover, G itself is not (2, 6)-colorable and any proper subgraph of G is (2, 6)-colorable. Let $S = \{2, 6\}$ denote the color set such that the subgraph of G induced by the vertices colored 2 has maximum degree at most 2 and the subgraph of G induced by the vertices colored 6 has maximum degree at most 6. For a (partial) coloring of G, a vertex v colored i is i*-saturated* if v is adjacent to i neighbors colored i, where $i \in S$. For simplicity, we say a vertex is *saturated* if it is i-saturated for some $i \in S$.

Lemma 2.1 [12] *If a 2-vertex v is on a bad 3-face f of G, then the other face g incident to v is a 7^+-face.*

Lemma 2.2 [12] *Let f be a k-face of G where $k \geqslant 7$. Then, f has at most $k - 6$ incident bad 2-vertices.*

Lemma 2.3 [4] *If v is a 3^--vertex of G, then v is adjacent to two 4^+-neighbors, one of which is a 8^+-vertex.*

Lemma 2.4 *A 7^--vertex of G is adjacent to at least one 8^+-vertex.*

Proof Suppose otherwise that v is a k-vertex of G such that v is not adjacent to any 8^+-vertices, where $k \leqslant 7$. Denote the neighbors of v by $v_1, \ldots, v_k$. Then, $v_1, \ldots, v_k$ are 7^--vertices. By the minimality, $G - v$ admits a (2, 6)-coloring. If each neighbor of v is colored 6, then obviously we can color v with 2 to obtain a (2, 6)-coloring of G, a contradiction. Thus, assume that at least one neighbor of v is colored 2. Without loss of generality, assume that v_1 is colored 2. If v_i is 6-saturated in $G - v$, then v_i must be a 7-vertex and each neighbor of v_i other than v is colored 6 since v_i is a 7^--vertex, where $i \in \{2, \ldots, k\}$. Thus, recolor v_i with 2 if v_i is 6-saturated for each $i \in \{2, \ldots, k\}$. Then, at most $k - 1 \leqslant 6$ neighbors of v are colored 6 but each of them is not 6-saturated. Thus, we can color v with 6 and obtain a (2, 6)-coloring of G, a contradiction. □

Lemma 2.5 *A 9^--vertex of G is adjacent to at least one 4^+-vertex.*

Proof Suppose otherwise that v is a k-vertex of G such that v is not adjacent to any 4^+-vertices, where $k \leqslant 9$. Denote the neighbors of v by $v_1, \ldots, v_k$. Then, $v_1, \ldots, v_k$ are 3^--vertices. By the minimality, $G - v$ admits a (2, 6)-coloring. Note that a 3^--vertex cannot be 6-saturated. If there are at most six neighbors of v colored 6, we can color v with 6 and obtain a (2, 6)-coloring of G, a contradiction. Thus, assume that there are at least seven neighbors of v colored 6 (which implies that $k \geqslant 7$). Without loss of generality, assume that $v_1, \ldots, v_7$ are colored 6 and $v_8, \ldots, v_k$ are colored 2. If v_i is 2-saturated in $G - v$, then v_i must be a 3-vertex and each neighbor of v_i other than v is colored 2 since v_i is a 3^--vertex, where $i \in \{8, \ldots, k\}$. Thus, recolor v_i with 6 if v_i is 2-saturated for each $i \in \{8, \ldots, k\}$. Then, at most $k - 7 \leqslant 2$ neighbors of v are colored 2 but each of them is not 2-saturated. Thus, we can color v with 2 and obtain a (2, 6)-coloring of G, a contradiction. □

Lemma 2.6 *There are no two adjacent 3-vertices in G.*

Proof Suppose otherwise that u and v are two adjacent 3-vertices of G. Since $G - uv$ is a graph with the same vertex set with G and fewer edges than G, $G - uv$ admits a (2, 6)-coloring c. If c is not a (2, 6)-coloring of G, then $c(u) = c(v)$ and at least one vertex in $\{u, v\}$ is saturated in $G - uv$. Since u and v are 3-vertices, neither u nor v can be 6-saturated. Thus, $c(u) = c(v) = 2$ and at least one vertex in $\{u, v\}$ is 2-saturated in $G - uv$. For each 2-saturated vertex in $\{u, v\}$, we recolor it with 6 since both of its two neighbors are colored 2 in $G - uv$ and obtain a new coloring c_1. Note that the new coloring c_1 is also a (2, 6)-coloring of G, a contradiction. □

The following lemma is straightforward.

Lemma 2.7 *Suppose that u is a 2-vertex of G, v and w are two neighbors of u. Then, for any (2, 6)-coloring of $G - u$, one of v and w is 2-saturated and the other is 6-saturated.*

Lemma 2.8 *Let $[uvw]$ be a bad 3-face of G where u is bad. Then, one of v and w is a 5^+-vertex and the other is a 8^+-vertex or one of v and w is a 9^+-vertex and the other is a 4^+-vertex.*

Proof By the minimality, $G - u$ admits a (2, 6)-coloring. By Lemma 2.7, one of v and w is 2-saturated and the other is 6-saturated. Without loss of generality, assume that v is 2-saturated and w is 6-saturated. Then, v is a 4^+-vertex of G and w is a 8^+-vertex of G. We only need to show that if v is a 4-vertex, then w cannot be a 8-vertex. Suppose otherwise that v is a 4-vertex and w is a 8-vertex. Since v is 2-saturated and w is 6-saturated, each neighbor of v other than u and w is colored 2 and each neighbor of w other than v and u is colored 6. Then, recolor v with 6, w with 2. Now, we can color u with 2, a contradiction. □

A bad 3-face f is called a *terrible 3-face* if f is incident to a 4-vertex. By Lemma 2.8, a terrible face is a $(2, 4, 9^+)$-face and a bad 3-face which is not terrible is a $(2, 5^+, 8^+)$-face (see Fig. 1).

Lemma 2.9 *A k-vertex u of G is incident to at most $(k - 8)$ terrible 3-faces, where $9 \leqslant k \leqslant 14$.*

Proof Suppose otherwise that a k-vertex u of G is incident to $(k - 7)$ terrible 3-faces, where $9 \leqslant k \leqslant 14$ (see Fig. 2). Denote the terrible 3-faces incident to u by $[uv_1w_1]$,

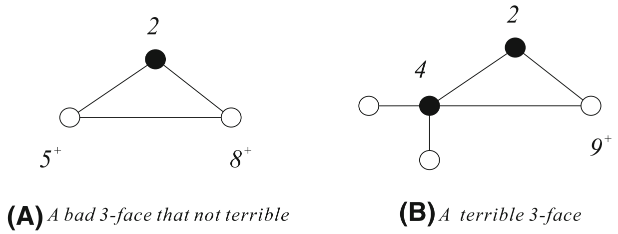

Fig. 1 A bad 3-face that is not terrible and a terrible 3-face

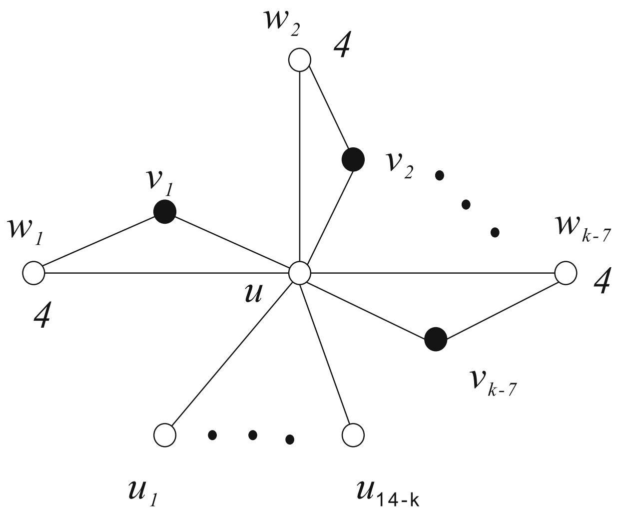

Fig. 2 u is a k-vertex, where $9 \leqslant k \leqslant 14$

…, $[uv_{k-7}w_{k-7}]$, where $v_1, \ldots, v_{k-7}$ are bad 2-vertices and $w_1, \ldots, w_{k-7}$ are 4-vertices. The other neighbors of u are denoted by $u_1, \ldots, u_{14-k}$. Since $k \geqslant 9$, u is incident to at least two terrible 3-faces.

By the minimality, $G - v_1$ admits a $(2, 6)$-coloring c. By Lemma 2.7, one of w_1 and u is 2-saturated and the other is 6-saturated. Since w_1 is a 4-vertex, it cannot be 6-saturated. This implies that w_1 is 2-saturated and u is 6-saturated, that is, each neighbor of w_1 other than v_1 and u is colored 2 and six neighbors of u other than v_1 and w_1 are colored 6. There are two cases to be considered.

Case 1 There exists some $i \in \{2, \ldots, k-7\}$ such that $c(v_i) = c(w_i) = 6$. Recolor v_i with 2 since no neighbors of v_i are colored 2. Then, u is not 6-saturated. Thus, we can color v_1 with 6, a contradiction.

Case 2 At most one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{2, \ldots, k-7\}$. We claim that exactly one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{2, \ldots, k-7\}$. Suppose otherwise that there exists some $i \in \{2, \ldots, k-7\}$ such that each vertex in $\{v_i, w_i\}$ is colored 2. Then, there are at most $k - 9 + 14 - k = 5$ neighbors of u colored 6. Recall that u is 6-saturated, which implies that u has six neighbors colored 6, a contradiction. We also claim that each vertex in $\{u_1, \ldots, u_{14-k}\}$ is colored 6. Suppose otherwise that there exists $i \in \{1, \ldots, 14-k\}$ such that u_i is colored 2. Then, there

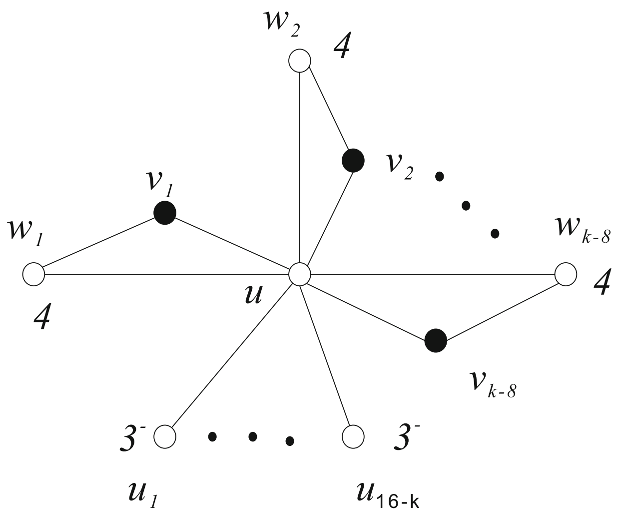

Fig. 3 u is a k-vertex of G, where $9 \leqslant k \leqslant 16$

are at most $14-k-1+k-8=5$ neighbors of u colored 6, a contradiction. Assume that there exists some $i \in \{2,\ldots,k-7\}$ such that w_i is colored 2 but not 2-saturated. Note that v_i is colored 6 now. Recolor v_i with 2; then, u is not 6-saturated and we can color v_1 with 6, a contradiction. Thus, assume that each w_i is either 2-saturated or colored 6, where $i \in \{2,\ldots,k-7\}$. For each $i \in \{2,\ldots,k-7\}$, recolor w_i with 6 if w_i is 2-saturated (note that now two neighbors of w_i other than v_i and u are both colored 2), and recolor v_i with 6 if w_i is colored 6 (note that now v_i is colored 2 and w_i is not 6-saturated since it is a 4-vertex). So, until now, each neighbor of u other than w_1 and v_1 is colored 6. Thus, recolor u with 2. Since w_1 is 2-saturated in $G-v$, each neighbor of w_1 other than u is colored 2. Recolor w_1 with 6 and obviously w_1 is not 6-saturated. Then, color v_1 with 6, a contradiction. □

Lemma 2.10 *No k-vertex of G is incident to $(k-8)$ terrible 3-faces and also has $(16-k)$ pendant 3^--neighbors, where $9 \leqslant k \leqslant 16$.*

Proof Suppose otherwise that a k-vertex u of G is incident to $(k-8)$ terrible 3-faces and also has $(16-k)$ pendant 3^--neighbors, where $9 \leqslant k \leqslant 16$ (see Fig. 3). Denote the terrible 3-faces incident to u by $[uv_1w_1]$,. . .,$[uv_{k-8}w_{k-8}]$, where $v_1,\ldots,v_{k-8}$ are bad 2-vertices and $w_1,\ldots,w_{k-8}$ are 4-vertices. Denote the pendant 3^--neighbors of u by $u_1,\ldots,u_{16-k}$. Since $k \geqslant 9$, u is incident to at least one terrible 3-face.

By the minimality, $G-v_1$ admits a (2, 6)-coloring c. By Lemma 2.7, one vertex in $\{u, w_1\}$ is 2-saturated and the other is 6-saturated. Since a 4-vertex cannot be 6-saturated, w_1 is 2-saturated and u is 6-saturated; that is, each neighbor of w_1 other than v_1 and u is colored 2 and six neighbors of u other than w_1 and v_1 are colored 6. Erase the colors of u and w_1. Since the two neighbors of w_1 other than v_1 and u are both colored 2, color w_1 with 6. Obviously, w_1 is not 6-saturated. Note that u has seven neighbors colored 6 now, there are two cases to be considered.

Case 1 u is a 9-vertex. Then, u is incident to only one terrible 3-face and has seven pendant 3^{-}-neighbors such that six of them are colored 6 and one colored 2. Without loss of generality, assume that $c(u_1) = 2$. Recolor u_1 with 6 if u_1 is 2-saturated, since now u_1 is a 3-vertex such that two neighbors of u_1 other than u are both colored 2. Then, recolor u with 2 since now at most one neighbor of u is colored 2 but not 2-saturated. Obviously, u is not 2-saturated. Since neither w_1 nor u is saturated now, we can color v_1 with 2 or 6, a contradiction.

Case 2 u is a k-vertex with $10 \leqslant k \leqslant 16$.

First, assume that at least two vertices in $\{u_1, \ldots, u_{16-k}\}$ are colored 2. We claim that there exists some $i \in \{2, \ldots, k-8\}$ such that $c(v_i) = c(w_i) = 6$. Suppose otherwise that at most one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{2, \ldots, k-8\}$. Then, at most $16 - k - 2 + k - 9 = 5$ neighbors of u are colored 6. Recall that u is 6-saturated, which implies that u has six neighbors colored 6, a contradiction. Without loss of generality, assume that $c(v_2) = c(w_2) = 6$. Recolor v_2 with 2 since no neighbors of v_2 are colored 2. Noting that u has six neighbors colored 6 and no one is 6-saturated now (a 4^{-}-vertex cannot be 6-saturated), recolor u with 6. So until now, two neighbors of v_1 are both colored 6. Color v_1 with 2, a contradiction.

Now assume that at most one vertex in $\{u_1, \ldots, u_{16-k}\}$ is colored 2. Without loss of generality, assume that u_1 is colored 2. Recolor u_1 with 6 if it is 2-saturated, since now u_1 is a 3-vertex such that two neighbors of u_1 other than u are both colored 2. Assume that there exists some $i \in \{2, \ldots, k-8\}$, such that $c(v_i) = c(w_i) = 6$. By the argument in Case 1, we can color v_1 with 6, a contradiction. Thus, assume that at most one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{2, \ldots, k-8\}$. Without loss of generality, assume that each vertex in $\{v_i, w_i\}$ is colored 2 for $i \in \{2, \ldots, j\}$ and exactly one vertex in $\{v_i, w_i\}$ is colored 2 for $i \in \{j+1, \ldots, k-8\}$, where $2 \leqslant j \leqslant k-8$. We claim that $j = 2$. Suppose otherwise that $j \geqslant 3$, then at most $k-8-(j+1)+1+16-k = 8-j \leqslant 5$ neighbors of u are colored 6. Recall that u is 6-saturated, which implies that u has six neighbors colored 6, a contradiction. Assume that there exists some $i \in \{3, \ldots, k-8\}$ such that w_i is colored 2 but not 2-saturated. Clearly, v_i is colored 6 and not 6-saturated. Recolor v_i with 2. Then, recolor u with 6 since only six neighbors of u are colored 6 and no one is 6-saturated (a 4^{-}-vertex cannot be 6-saturated). So until now, two neighbors of v_1 are both colored 6 and we can color v_1 with 2, a contradiction. Thus, assume that for each $i \in \{3, \ldots, k-8\}$, either w_i is 2-saturated (and v_i is colored 6) or w_i is colored 6 (and v_i is colored 2). For each $i \in \{3, \ldots, k-8\}$, recolor w_i with 6 if w_i is 2-saturated, since two neighbors of w_i other than u and w are both colored 2 and v_i is colored 6, recolor v_i with 6 if w_i is colored 6, since now v_i is colored 2 and only one neighbor of v_i is colored 6 but not 6-saturated. Then, recolor v_2 with 6 since both w_2 and v_2 are colored 2. Clearly, w_2 is not 2-saturated. So, until now, among all neighbors of u, only u_1, w_2 may be colored 2 and neither of them is 2-saturated. Thus, we recolor u with 2. Then, color v_1 with 6 since only one neighbor of v_i is colored 6 but not 6-saturated, a contradiction. □

Lemma 2.11 *No k-vertex of G is incident to $(k-8)$ terrible 3-faces and m bad 3-faces that are not terrible and also has $(16-2m-k)$ pendant 3^{-}-neighbors, where $9 \leqslant k \leqslant 12$ and $m \geqslant 2$.*

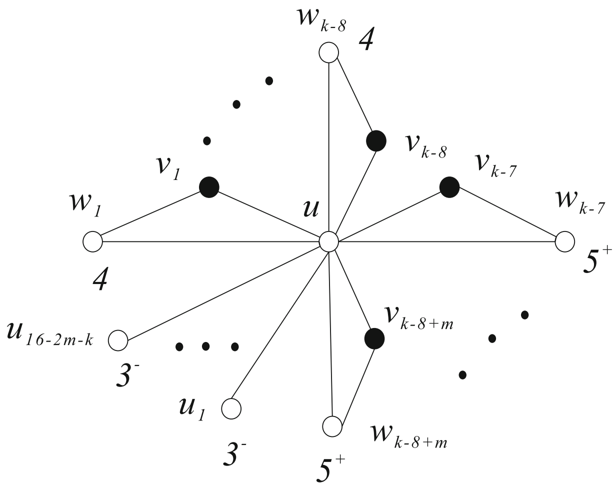

Fig. 4 u is a k-vertex of G, where $9 \leqslant k \leqslant 12$ and $m \geqslant 2$

Proof Suppose otherwise that a k-vertex u of G is incident to $(k-8)$ terrible 3-faces and m bad 3-faces that not terrible and also has $(16-2m-k)$ pendant 3^--neighbors, where $9 \leqslant k \leqslant 12$ and $m \geqslant 2$ (see Fig. 4). Denote the bad 3-faces incident to u by $[uv_1w_1]$, ..., $[uv_{k-8+m}w_{k-8+m}]$, where $v_1, \ldots, v_{k-8+m}$ are bad 2-vertices, $w_1, \ldots, w_{k-8}$ are 4-vertices, $w_{k-7}, \ldots, w_{k-8+m}$ are 5^+-vertices. Denote the pendant 3^--neighbors of u by $u_1, \ldots, u_{16-2m-k}$. Since $9 \leqslant k \leqslant 14$, u is incident to at least one terrible 3-face $[uv_1w_1]$.

By the minimality, $G - v_1$ admits a (2, 6)-coloring c. By Lemma 2.7, one of u and w_1 is 2-saturated and the other is 6-saturated. Since a 4-vertex cannot be 6-saturated, w_1 is 2-saturated and u is 6-saturated. Since u is 6-saturated, six neighbors of u other than v_1 and w_1 are colored 6. We claim that there exists some $i \in \{2, \ldots, k-8+m\}$ such that $c(v_i) = c(w_i) = 6$. Suppose otherwise that at most one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{2, \ldots, k-8+m\}$. Then, at most $k-8+m-1+16-2m-k = 7-m \leqslant 5$ neighbors of u are colored 6. Recall that u is 6-saturated, which implies that u has six neighbors colored 6, a contradiction. Without loss of generality, assume that $c(v_2) = c(w_2) = 6$. Recolor v_2 with 2 since no neighbors of v_2 are colored 2; then, u is not 6-saturated. Thus, we can color v_1 with 6, a contradiction. □

Let u be a k-vertex of G where $8 \leqslant k \leqslant 14$. We call u a *special vertex* if u satisfies one of the following conditions: (1) u is a 8-vertex and u is incident to at least one bad 3-face that not terrible; (2) u is a 9^+-vertex and u is incident to one bad 3-face that is not terrible and $(k-8)$ terrible 3-faces and u also has $(14-k)$ pendant 3^--neighbors (see Fig. 5). Let $N_0(u)$ be the vertex set containing u and bad 2-vertices adjacent to u, as well as 4-vertices adjacent to u.

Lemma 2.12 *Let $[uvw]$ be a bad 3-face of G which is not terrible, where u is a special vertex. Suppose that u is 6-saturated in a (2, 6)-coloring of $G - \{v, w\}$. Then, we can recolor the vertices in set $N_0(u)$ such that u is not saturated in $G - \{v, w\}$.*

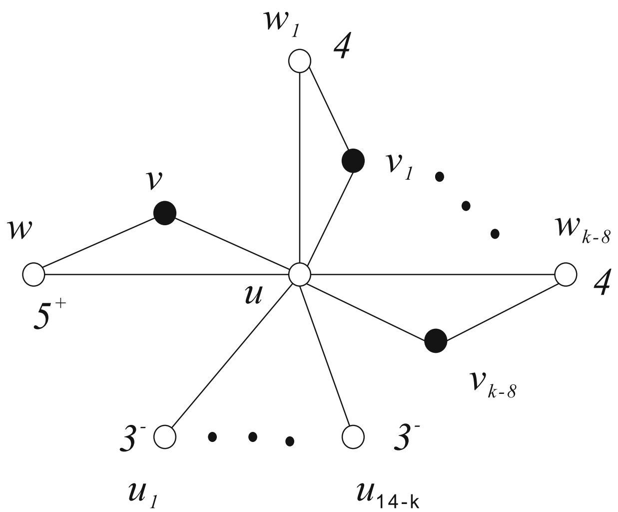

Fig. 5 A special k-vertex, where $9 \leqslant k \leqslant 14$

Proof Suppose that u is a 8-vertex. Since u is 6-saturated, each neighbor of u in $G - \{v, w\}$ is colored 6. Thus, recolor u with 2 and obviously u is not 6-saturated.

Suppose that u is a k-vertex where $9 \leqslant k \leqslant 14$ (see Fig. 5). Denote the terrible 3-faces incident to u by $[uv_1w_1]$, …, $[uv_{k-8}w_{k-8}]$, where $v_1, \ldots, v_{k-8}$ are bad 2-vertices and $w_1, \ldots, w_{k-8}$ are 4-vertices, the pendant 3^--neighbors of u by $u_1, \ldots, u_{14-k}$. There are two cases under consideration.

Case 1 There exists some $i \in \{1, \ldots, k-8\}$ such that each vertex in $\{v_i, w_i\}$ is colored 6. Recolor v_i with 2, then u is not 6-saturated.

Case 2 At most one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{1, \ldots, k-8\}$. We claim that exactly one vertex in $\{v_i, w_i\}$ is colored 6 for each $i \in \{1, \ldots, k-8\}$. Suppose otherwise that there exists $i \in \{1, \ldots, k-8\}$ such that v_i and w_i are both colored 2. Then, at most $k - 8 - 1 + 14 - k = 5$ neighbors of u are colored 6. Recall that u is 6-saturated, which implies that u has six neighbors colored 6, a contradiction. We also claim that for each $i \in \{1, \ldots, k-8\}$, u_i is colored 6. Suppose otherwise that there exists some $i \in \{1, \ldots, k-8\}$ such that u_i is colored 2. Then, at most $13 - k + k - 8 = 5$ neighbors of u are colored 6, a contradiction. First, assume that there exists one $i \in \{1, \ldots, k-8\}$ such that w_i is colored 2 but not 2-saturated. Without loss of generality, assume that w_1 is colored 2 but not 2-saturated and v_1 is colored 6. Recolor v_1 with 2. This implies that u is not 6-saturated. Now assume that for each $i \in \{1, \ldots, k-8\}$, w_i is either 2-saturated or colored 6. For each $i \in \{1, \ldots, k-8\}$, recolor w_i with 6 if w_i is 2-saturated, recolor v_i with 6 if w_i is colored 6 (note that now v_i is colored 2). So, until now, each neighbor of u other than v and w is colored 6. Recolor u with 2. Obviously, u is not 2-saturated. □

Lemma 2.13 *No bad 3-face of G is incident to two special vertices.*

Proof Suppose otherwise that there exists a bad 3-face f of G such that f is incident to two special vertices. Denote the bad 3-face by $f = [uvw]$, where v is a bad 2-vertex,

u is a special k_1-vertex and w is a special k_2-vertices, where $8 \leqslant k_1, k_2 \leqslant 14$. Since G has no 4-cycles, $N_0(u) \cap N_0(w) = \emptyset$.

By the minimality, $G - v$ admits a (2, 6)-coloring c. By Lemma 2.7, one of u and w is 2-saturated and the other is 6-saturated. Denote the color of w by c_1, where $c_1 \in S$. Erase the color of w. Clearly, u is still saturated. Recolor the vertices in set $N_0(u)$ such that u is not saturated by Lemma 2.12. Denote the new color of u by c_2, where $c_2 \in S$. Erase the color of u and color w with c_1. Since $N_0(u) \cap N_0(w) = \emptyset$, the recoloring of vertices in set $N_0(u)$ does not change the colors of w's neighbors other than u. Therefore, the color of w is well defined and w is still saturated. Recolor the vertices in set $N_0(w)$ such that w is not saturated by Lemma 2.12. Denote the new color of w by c_3. Color u with c_2. Since $N_0(u) \cap N_0(w) = \emptyset$, the recoloring of vertices in set $N_0(w)$ does not change the colors of of u's neighbors other than w. Therefore, the color of u is well defined. If $c_2 = c_3$, then we can color v with $S \backslash c_2$, a contradiction. If $c_2 \neq c_3$, then we can color v with any color in S since neither u nor w is saturated, a contradiction. □

Lemma 2.14 *No 5-vertex of G is incident to two bad 3-faces, each of which has a special vertex.*

Proof Suppose otherwise that there exists a 5-vertex u of G such that u is incident to two bad 3-faces, each of which has a special vertex. Denote the bad 3-faces incident to u by $[uvw]$ and $[uxy]$, where v, x are bad 2-vertices and w, y are special vertices, the remaining neighbor of u by z. By the minimality, $G - v$ admits a (2, 6)-coloring c. By Lemma 2.7, one of u and w is 2-saturated and the other is 6-saturated. Since a 5-vertex cannot be 6-saturated, u is 2-saturated and w is 6-saturated. Thus, two vertices in $\{x, y, z\}$ are colored 2. There are three cases to be considered.

Case 1 $c(x) = 2, c(y) = 2, c(z) = 6$. Recolor x with 6 since two neighbors of x are both colored 2. Then, u is not 2-saturated. Thus, we can color v with 2, a contradiction.

Case 2 $c(x) = 6, c(y) = 2, c(z) = 2$. Erase the color of u and recolor the vertices in set $N_0(w)$ such that w is not saturated by Lemma 2.12. Denote the new color of w by i, where $i \in S$. Color u with 6 since at most two neighbors of u are colored 6 and neither of them is 6-saturated. Obviously, u is not 6-saturated. Then, we can color v with 6 if w is colored 2 and with 2 if w is colored 6, a contradiction.

Case 3 $c(x) = 2, c(y) = 6, c(z) = 2$. Erase the color of u and recolor the vertices in set $N_0(w)$ such that w is not saturated by Lemma 2.12. Suppose that y is not 6-saturated. Color u with 6 since at most two neighbors of u are colored 6 and neither of them are 6-saturated. Obviously, u is not 6-saturated. Then, color v with 6 if w is colored 2 and with 2 if w is colored 6, a contradiction. Suppose that y is 6-saturated. Erase the color of x and recolor the vertices in set $N_0(y)$ such that y is not saturated by Lemma 2.12. Color x with 2 since at most one neighbor of x is colored 2 but not 2-saturated and color u with 6 since at most two neighbors of u are colored 6 and not 6-saturated. Obviously, u is not 6-saturated. Then, we can color v with 6 if w is colored 2 and with 2 if w is colored 6, a contradiction. □

3 Discharging

Let the initial charge of a vertex v of G be $\mu(v) = 2d(v) - 6$ and the initial charge of a face f of G be $\mu(f) = d(f) - 6$. By Euler's formula, we have

$$\sum_{v \in V(G)} \mu(v) + \sum_{f \in V(F)} \mu(f) = -12$$

Then, we design appropriate discharging rules and redistribute weights accordingly. After discharging, a new weight function μ^* is produced. The discharging procedure will preserve the total charge sum. One the other hand, we will show that $\mu^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$, which arrives at a contradiction:

$$-12 = \sum_{x \in V(G) \cup F(G)} \mu(x) = \sum_{x \in V(G) \cup F(G))} \mu^*(x) \geqslant 0$$

Here are the discharging rules:

(R1) Every k-vertex sends $\frac{2}{3}$ to each incident good 2-vertex or pendant 3-face, where $k = 4, 5, 6, 7$. Every 8^+-vertex sends $\frac{4}{3}$ to each incident good 2-vertex or pendant 3-face.

(R2) Every 4-vertex sends $\frac{2}{3}$ to each incident good 3-face that contains a 8^+-vertex and sends $\frac{4}{3}$ to each incident good 3-face that does not contain a 8^+-vertex. Every 5-vertex sends $\frac{4}{3}$ to each incident good 3-face. Every 6^+-vertex sends 2 to each incident good 3-face.

(R3) Suppose u is a nonspecial vertex.

- (R3.1) If u is a 4-vertex, then u sends $\frac{2}{3}$ to each incident bad 3-face.
- (R3.2) If u is a 5-vertex, then u sends 2 to each incident bad 3-face with a special vertex and sends $\frac{4}{3}$ to other incident bad 3-face.
- (R3.3) If u is a 6,7-vertex, then u sends 2 to each incident bad 3-face.
- (R3.4) If u is a 9^+-vertex, then u sends $\frac{8}{3}$ to each incident bad 3-face that is not terrible and sends $\frac{10}{3}$ to each incident terrible 3-face.

(R4) Suppose u is a special vertex. Then, u sends 2 to each incident bad 3-face that is not terrible and sends $\frac{10}{3}$ to each incident terrible 3-face.

(R5) Every 7^+-face sends 1 to each incident bad 2-vertex.

(R6) Every 3-face sends 1 to each incident 2-vertex.

Discharge rules are depicted in Fig. 6.

We will show that each vertex and each face have nonnegative final charge.

Claim 3.1 *Each k-vertex v of G has nonnegative final charge.*

Proof Recall that the minimum degree of G is 2, $k \geqslant 2$.

(1) Suppose that v is a 2-vertex. Then, $\mu(v) = 4 - 6 = -2$. Suppose that v is bad. By Lemma 2.1, a bad 2-vertex is incident to a 3-face and a 7^+-face. Then, $\mu^*(v) = -2+1+1 = 0$ by (R5) and (R6). Suppose that v is good. By Lemma 2.3, v has a 4^+-neighbor and a 8^+-neighbor. Then, $\mu^*(v) \geqslant -2+\frac{2}{3}+\frac{4}{3} = 0$ by (R1).

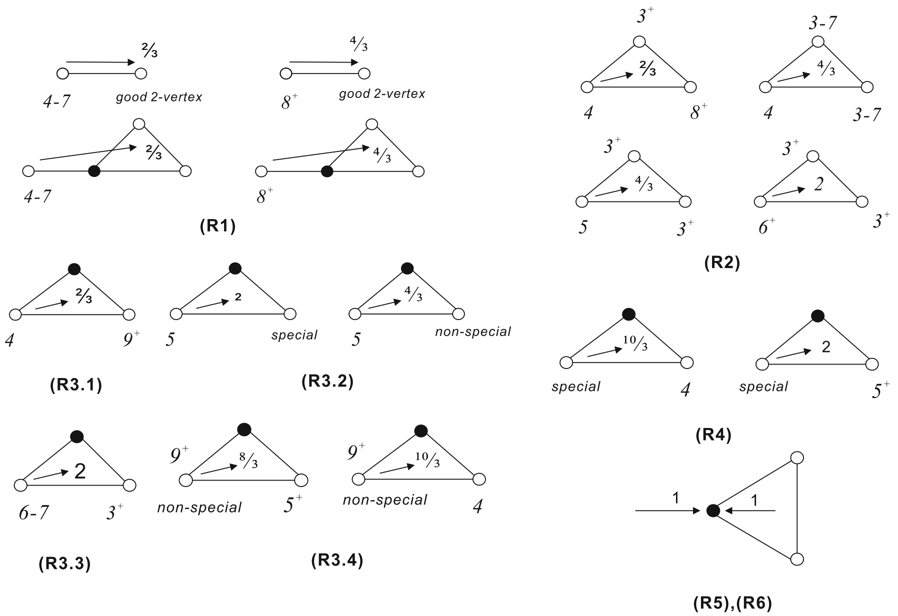

Fig. 6 Illustration of discharging rules

(2) Suppose that v is a 3-vertex. Then, $\mu(v) = 6 - 6 = 0$. Its charge holds to be nonnegative since no rules involved.

(3) Suppose that v is a 4-vertex. Then, $\mu(v) = 8 - 6 = 2$. Suppose that v is not incident to any 3-faces. By Lemma 2.4, v has at least one 8^+-neighbor. Then, $\mu^*(v) \geqslant 2 - \frac{2}{3} \times 3 = 0$ by (R1). Suppose that v is incident to one 3-face f. If f is a bad 3-face, then $\mu^*(v) \geqslant 2 - \frac{2}{3} - \frac{2}{3} \times 2 = 0$ by (R1) and (R3.1). If v is incident to a good 3-face with a 8^+-vertex, then $\mu^*(v) \geqslant 2 - \frac{2}{3} - \frac{2}{3} \times 2 = 0$ by (R1) and (R2). If v is incident to a good 3-face without a 8^+-vertex. By Lemma 2.4, v has a pendant 8^+-neighbor. Then, $\mu^*(v) \geqslant 2 - \frac{4}{3} - \frac{2}{3} = 0$ by (R1) and (R2). Suppose that v is incident to two 3-faces. If v is incident to at least one bad 3-face, then $\mu^*(v) \geqslant 2 - \frac{2}{3} - \frac{4}{3} = 0$ by (R2) and (R3.1). If v is not incident to any bad 3-face, then at least one of the two faces has a 8^+-vertex by Lemma 2.4. Thus, $\mu^*(v) \geqslant 2 - \frac{2}{3} - \frac{4}{3} = 0$ by (R2).

(4) Suppose that v is a 5-vertex. Then, $\mu(v) = 10 - 6 = 4$. Suppose that v is not incident to any 3-faces. Then, $\mu^*(v) \geqslant 4 - \frac{2}{3} \times 5 = \frac{2}{3} > 0$ by (R1). Suppose that v is incident to exactly one 3-face. Then, $\mu^*(v) \geqslant 4 - 2 - \frac{2}{3} \times 3 = 0$ by (R1) (R2) and (R3.2). Suppose that v is incident to two 3-faces. By Lemma 2.14, at most one of them has a special vertex. Then, $\mu^*(v) \geqslant 4 - 2 - \frac{4}{3} - \frac{2}{3} = 0$ by (R1) (R2) and (R3.2).

(5) Suppose that v is a 6-vertex. Then, $\mu(v) = 12 - 6 = 6$, $\mu^*(v) \geqslant 6 - 2 \times 3 = 0$ by (R1) (R2) and (R3.3).

(6) Suppose that v is a 7-vertex. Then, $\mu(v) = 14 - 6 = 8$, $\mu^*(v) \geqslant 8 - 2 \times 3 - \frac{2}{3} = \frac{4}{3} > 0$ by (R1) (R2) and (R3.3).

(7) Suppose that v is a 8-vertex. Then, $\mu(v) = 16 - 6 = 10$. Suppose that v is not incident to any 3-faces. By Lemma 2.5, v has at most seven pendant 3^--neighbors. Then, $\mu^*(v) \geqslant 10 - \frac{4}{3} \times 7 = \frac{2}{3} > 0$ by (R1). Suppose that v is incident to at least one 3-face f. Then, v is special. By Lemma 2.8, each 3-face incident to v is not terrible. Thus, $\mu^*(v) \geqslant 10 - 2 - \frac{4}{3} \times 6 = 0$ by (R1) (R2) and (R4).

(8) Suppose that v is a k-vertex, where $9 \leqslant k \leqslant 12$. Then, $\mu(v) = 2k - 6$. By Lemma 2.9, v is incident to at most $(k - 8)$ terrible 3-faces. Assume that v is incident to at most $(k-9)$ terrible 3-faces. Then, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k-9) - \frac{4}{3}(18 - k) = 0$ by (R1) (R2) and (R3.4). Assume that v is incident to $(k - 8)$ terrible 3-faces. Then, there exist five possible cases.

Case 1 v is not incident to any other 3-faces. By Lemma 2.10, v has at most $(15-k)$ pendant 3^--neighbors. Thus, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k-8) - \frac{4}{3}(15-k) = \frac{2}{3} > 0$ by (R1) (R2) and (R3.4).

Case 2 v is incident to at least one good 3-face. Then, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - 2 - \frac{4}{3}(14 - k) = 0$ by (R1) (R2) and (R3.4).

Case 3 v is not incident to any good 3-face but incident to one bad 3-face that is not terrible and has at most $(13 - k)$ pendant 3^--neighbors. Then, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - \frac{8}{3} - \frac{4}{3}(13 - k) = \frac{2}{3} > 0$ by (R1) (R2) and (R3.4).

Case 4 v is not incident to any good 3-face but incident to one bad 3-face that is not terrible and has $(14 - k)$ pendant 3^--neighbors. Then, v is special. Thus, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - 2 - \frac{4}{3}(14 - k) = 0$ by (R1) (R2) and (R4).

Case 5 v is not incident to any good 3-face but incident to at least $m \geqslant 2$ bad 3-faces that not terrible. By Lemma 2.11, v has at most $16-2m-k-1 = 15-2m-k$ pendant 3^--neighbors; thus, $\mu^*(v) \geqslant 2k-6-\frac{10}{3}(k-8)-\frac{8}{3}m-\frac{4}{3}(15-2m-k) = \frac{2}{3} > 0$ by (R1) (R2) and (R3.4).

(9) Suppose that v is a k-vertex, where $13 \leqslant k \leqslant 14$. Then, $\mu(v) = 2k - 6$. By Lemma 2.9, v is incident to at most $(k - 8)$ terrible 3-faces. Since $13 \leqslant k \leqslant 14$, $\lfloor \frac{k}{2} \rfloor = k - 7$, that is, v is incident to at most $(k - 7)$ 3-faces. Assume that v is incident to at most $(k-9)$ terrible 3-faces. Then, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k-9) - \frac{4}{3}(18 - k) = 0$ by (R1) (R2) and (R3.4). Assume that v is incident to $(k - 8)$ terrible 3-faces. Then, there exist four possible cases.

Case 1 v is not incident to any other 3-faces. By Lemma 2.10, v has at most $(15-k)$ pendant 3^--neighbors. Thus, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k-8) - \frac{4}{3}(15-k) = \frac{2}{3} > 0$ by (R1) (R2) and (R3.4).

Case 2 v is also incident to one good 3-faces. By (R1) (R2) and (R3.4), $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - 2 - \frac{4}{3}(14 - k) = 0$.

Case 3 v is also incident to one bad 3-face that is not terrible and has at most $(13 - k)$ pendant 3^--neighbors. By (R1) (R2) and (R3.4), $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - \frac{8}{3} - \frac{4}{3}(13 - k) = \frac{2}{3} > 0$.

Case 4 v is also incident to one bad 3-face that is not terrible and has $(14 - k)$ pendant 3^--neighbors. Then, v is special. Thus, $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3}(k - 8) - 2 - \frac{4}{3}(14 - k) = 0$ by (R1) (R2) and (R4).

(10) Suppose that v is a 15-vertex. Then, $\mu(v) = 30 - 6 = 24$. Note that v is incident to at most seven 3-faces. Assume that v is incident to at most six 3-faces. Then, $\mu^*(v) \geqslant 24 - \frac{10}{3} \times 6 - \frac{4}{3} \times 3 = 0$ by (R1) (R2) and (R3.4). Assume that v

is incident to seven 3-faces. If at least one of seven 3-faces is not terrible, then $\mu^*(v) \geqslant 24 - \frac{10}{3} \times 6 - \frac{8}{3} - \frac{4}{3} = 0$ by (R1) (R2) and (R3.4). If each of the seven 3-faces is terrible, then v has no pendant 3^--neighbors by Lemma 2.10. Thus, $\mu^*(v) \geqslant 24 - \frac{10}{3} \times 7 = \frac{2}{3} > 0$ by (R1) (R2) and (R3.4).

(11) Suppose that v is a 16-vertex. Then, $\mu(v) = 32 - 6 = 26$. Note that v is incident to at most eight 3-faces. Assume that v is incident to at most seven 3-faces. Then, $\mu^*(v) \geqslant 26 - \frac{10}{3} \times 7 - \frac{4}{3} \times 2 = 0$ by (R1) (R2) and (R3.4). Assume that v is incident to eight 3-faces. By Lemma 2.10, at least one of the eight 3-faces is not terrible. Thus, $\mu^*(v) \geqslant 26 - \frac{10}{3} \times 7 - \frac{8}{3} = 0$ by (R1) (R2) and (R3.4).

(12) Suppose that v is a k-vertex, $k \geqslant 17$. If k is odd, then $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3} \times \frac{k-1}{2} - \frac{4}{3} = \frac{1}{3}(k-17) \geqslant 0$ by (R4). If k is even, then $\mu^*(v) \geqslant 2k - 6 - \frac{10}{3} \times \frac{k}{2} = \frac{1}{3}(k - 18) \geqslant 0$ by (R4). □

Claim 3.2 *Each bad 3-face f of G has nonnegative final charge.*

Proof Since f is a 3-face, $\mu(f) = 3 - 6 = -3$. By Lemma 2.8, f is a $(2, 5^+, 8^+)$-face or a $(2, 4, 9^+)$-face.

(1) Suppose that $f = [uvw]$ is a $(2, 5^+, 8^+)$-face, which implies that f is not terrible. By Lemma 2.13, at most one vertex in $\{v, w\}$ is special. Assume that exactly one vertex in $\{v, w\}$ is special. Then, $\mu^*(f) \geqslant -3 + 2 + 2 - 1 = 0$ by (R3.2)–(R3.4) (R4) and (R6). Assume that neither v nor w is special. Then, $\mu^*(f) \geqslant -3 + \frac{4}{3} + \frac{8}{3} - 1 = 0$ by (R3.2)–(R3.4) and (R6).

(2) Suppose that $f = [uvw]$ is a $(2, 4, 9^+)$-face, which implies that f is terrible. Then, $\mu^*(f) \geqslant -3 + \frac{2}{3} + \frac{10}{3} - 1 = 0$ by (R3.1) (R3.4) (R4) and (R6). □

Claim 3.3 *Each good 3-face f of G has nonnegative final charge.*

Proof Since f is a 3-face, $\mu(f) = 3 - 6 = -3$. By Lemma 2.6, there are no adjacent two 3-vertices in G. Thus, f contains at most one 3-vertex.

(1) Suppose that f is not incident to any 3-vertices. Then, $f = [uvw]$ is a $(4^+, 4^+, 4^+)$-face. Suppose that at least one vertex in $\{u, v, w\}$ is a 8^+-vertex. Then, $\mu^*(f) \geqslant -3 + \frac{2}{3} + \frac{2}{3} + 2 = \frac{1}{3} > 0$ by (R2). Suppose that each vertex in $\{u, v, w\}$ is a vertex with degree 4–7. Then, $\mu^*(f) \geqslant -3 + \frac{4}{3} \times 3 = 1 > 0$ by (R2).

(2) Suppose that f is incident to exactly one 3-vertex u, which implies that $f = [uvw]$ is a $(3, 4^+, 4^+)$-face. Denote the pendant neighbor of u by x. By Lemma 2.6, x is a 4^+-vertex. Note that f is a pendant face of x. Suppose that x is a 8^+-vertex. If one vertex in $\{v, w\}$ is a 5^+-vertex, then $\mu^*(f) \geqslant -3 + \frac{4}{3} + \frac{4}{3} + \frac{2}{3} = \frac{1}{3} > 0$ by (R1) and (R2). If each vertex in $\{v, w\}$ is a 4-vertex, then $\mu^*(f) \geqslant -3 + \frac{4}{3} \times 3 = 1 > 0$ by (R1) and (R2). Suppose that x is a k-vertex, where $4 \leqslant k \leqslant 7$. By Lemma 2.5, one of v, w is a 8^+-vertex. Thus, $\mu^*(f) \geqslant -3 + 2 + \frac{2}{3} + \frac{2}{3} = \frac{1}{3} > 0$ by (R1) and (R2). □

Claim 3.4 *Each 6^+-face f has nonnegative final charge.*

Proof Suppose that f is a 6-face. Then, $\mu(f) = 6 - 6 = 0$. Its charge holds to be nonnative since no discharging rules involved. Suppose that f is a k-face, where $k \geqslant 7$. Then, $\mu(f) = k - 6$. By Lemma 2.2, f has at most $(k - 6)$ bad 2-vertices. Thus, $\mu^*(f) \geqslant k - 6 - (k - 6) \times 1 = 0$ by (R5). □

So until now, we have shown that $\mu^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$. Therefore, we complete the proof of Theorem 1.2.

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