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Berezin Number, Grüss-Type Inequalities and Their Applications

Ulaş Yamancı[1] · Remziye Tunç[2] · Mehmet Gürdal[2]

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Abstract
In this paper, we study the Berezin number inequalities by using the transform $C_{\alpha,\beta}(A)$ on reproducing kernel Hilbert spaces (RKHS). Moreover, we give Grüss-type inequalities for selfadjoint operators in RKHS.

Keywords Berezin number · Berezin symbol · Selfadjoint operators · Grüss inequality

Mathematics Subject Classification Primary 47A63

1 Introduction

Grüss [17] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}(x)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}(x).\frac{1}{b-a}\int_a^b g(x)\mathrm{d}(x)\right|$$
$$\leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

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✉ Ulaş Yamancı
ulasyamanci@sdu.edu.tr

Remziye Tunç
remziye_tunc_rt@hotmail.com

Mehmet Gürdal
gurdalmehmet@sdu.edu.tr

1 Department of Statistics, Suleyman Demirel University, 32260 Isparta, Turkey

2 Department of Mathematics, Suleyman Demirel University, 32260 Isparta, Turkey

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a, b]$, where ϕ, Φ, γ, Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

The discrete version of the Grüss' inequality can be found in [22] as following:

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be two n-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, \ldots, n$. Then, one has

$$\begin{aligned} &\left| \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i . \frac{1}{n}\sum_{i=1}^{n} b_i \right| \\ &\quad \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(R - r)(S - s), \end{aligned}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. In fact, the presented version of the discrete Grüss' inequality is due to Biernacki et al. [7]. For Grüss-type inequalities, we refer to [3,8,9,11,12] and references therein.

Let A be a selfadjoint linear operator on a complex Hilbert space $\mathcal{H}$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$ as follows (see for instance [14]).

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi\left(\overline{f}\right) = \Phi(f)^*$;

(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_{\mathcal{H}}$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation, we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and it is called the continuous functional calculus for the selfadjoint operator A.

If A is a selfadjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$ on $\mathcal{H}$. Therefore, if f and g are real-valued functions on $Sp(A)$, then the following basic property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{1}$$

in the operator order of $B(\mathcal{H})$.

Let Ω be an arbitrary set. Denote by $\mathcal{F}(\Omega)$ the set of all complex-valued functions on Ω. A reproducing kernel Hilbert space (RKHS for short) on the set Ω is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{F}(\Omega)$ with a function $k_\lambda : \Omega \times \Omega \to \mathcal{H}$, which is called the reproducing kernel enjoying the reproducing property $k_\lambda := k(., \lambda) \in \mathcal{H}$ for all $\lambda \in \Omega$ and $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$ holds for all $\lambda \in \Omega$ and all $f \in \mathcal{H}$ (see [24]). As it is known (see [2,24]),

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathcal{H}(\Omega)$.

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For any bounded linear operator A on $\mathcal{H}$, the Berezin transform of A is the function $\widetilde{A}$ defined by (see [23])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

The Berezin set and the Berezin number for operator A are defined by (see [19,20])

$$\mathrm{Ber}(A) := \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \text{ and } \mathrm{ber}(A) := \sup \left\{ \left| \widetilde{A}(\lambda) \right| : \lambda \in \Omega \right\},$$

respectively. Recently, some Berezin number inequalities have been obtained by authors [5,15,16,25–27].

The numerical range and numerical radius of A in $\mathcal{B}(\mathcal{H})$ are, respectively, defined by

$$W(A) := \{\langle Af, f \rangle : f \in \mathcal{H}, \ \|f\| = 1\} \text{ and } w(A) := \sup\{|z| : z \in W(A)\}.$$

The Berezin set and the Berezin number have a relationship with the numerical range and the numerical radius as follows:

$$\mathrm{Ber}(A) \subset W(A) \text{ and } \mathrm{ber}(A) \leq w(A) \leq \|A\|.$$

For the numerical radius and its applications, we refer to [1,4,6,10,13,21], and references therein. The numerical radius inequality for the product of two operators is following:

$$w(AB) \leq 4w(A)\, w(B)$$

for the bounded linear operators A, B on the Hilbert space $\mathcal{H}$. In that case that $AB = BA$, then

$$w(AB) \leq 2w(A)\, w(B)$$

(see [18] for detailed information). So, the following questions are natural:

Is it true that the above inequality is also provided for Berezin number of operators? For which operator classes, there exists a number $C > 0$ such that

$$\mathrm{ber}\,(AB) \le C\mathrm{ber}\,(A)\,\mathrm{ber}\,(B)\,? \tag{2}$$

In this paper, we study inequality (2) by using the transform $C_{\alpha,\beta}(A)$ on reproducing kernel Hilbert spaces (RKHS). Moreover, we give Grüss-type inequalities for selfadjoint operators in RKHS.

2 Berezin Number Inequalities for Two Operators

Let $\alpha, \beta \in \mathbb{C}$ and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator. We define the following transform [11]

$$C_{\alpha,\beta}(A) := \left(A^* - \overline{\alpha} I\right)(\beta I - A),$$

where A^* denotes the adjoint of A. The transform $C_{\alpha,\beta}(.)$ has some interesting properties for $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$ as following:

(i) $C_{\alpha,\beta}(I) := (1-\overline{\alpha})(\beta - 1)\,I$ and $C_{\alpha,\alpha}(A) := -(\alpha I - T)^*(\alpha I - A)$.
(ii) $\left[C_{\alpha,\beta}(A)\right]^* = C_{\beta,\alpha}(A)$ and $C_{\overline{\beta},\overline{\alpha}}(A^*) - C_{\alpha,\beta}(A) = A^*A - AA^*$.

A bounded linear operator A on the RKHS $\mathcal{H}$ is said to be accretive if $\mathrm{Re}\,\widetilde{A}(\lambda) \ge 0$ for any $\lambda \in \Omega$. Using this property, we have

$$\mathrm{Re}\,\widetilde{C_{\alpha,\beta}(A)}(\lambda) = \mathrm{Re}\,\widetilde{C_{\beta,\alpha}(A)}(\lambda) = \frac{1}{4}|\beta - \alpha|^2 - \left\|\left(A - \frac{\beta+\alpha}{2} I\right)\widehat{k}_\lambda\right\|^2$$

for any scalars $\alpha, \beta \in \mathbb{C}$ and $\lambda \in \Omega$. So we can give a simple result.

Lemma 1 *For $A \in \mathcal{B}(\mathcal{H}(\Omega))$ and complex numbers α, β, the following statements are equivalent:*

(i) *The transforms $C_{\alpha,\beta}(A)$ and $C_{\overline{\alpha},\overline{\beta}}(A^*)$ are accretive;*
(ii) $\left\|A\widehat{k}_\lambda - \frac{\beta+\alpha}{2}\widehat{k}_\lambda\right\| \le \frac{1}{2}|\beta-\alpha|$ *and* $\left\|A^*\widehat{k}_\lambda - \frac{\overline{\beta}+\overline{\alpha}}{2}\widehat{k}_\lambda\right\| \le \frac{1}{2}|\beta-\alpha|$ *for any $\lambda \in \Omega$.*

Theorem 1 *Let $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ be accretive transform for $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then,*

$$\mathrm{ber}\,(BA) \le 3\mathrm{ber}\,(A)\,\mathrm{ber}\,(B) + \frac{1}{4}|\beta-\alpha|\,|\gamma-\delta|\,.$$

Proof By hypothesis, $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, and then, from Lemma 1 we get $\left\|A\widehat{k}_\lambda - \frac{\beta+\alpha}{2}\widehat{k}_\lambda\right\| \le \frac{1}{2}|\beta-\alpha|$ and $\left\|B^*\widehat{k}_\lambda - \frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_\lambda\right\| \le \frac{1}{2}\left|\overline{\gamma}-\overline{\delta}\right|$ for any $\lambda \in \Omega$.

Using the Schwarz inequality, we get that

$$\begin{aligned} &\left|\left\langle A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda, B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\rangle\right| \\ &\quad \le \left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| \left\|B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\| \end{aligned} \tag{3}$$

for all $\lambda, \eta \in \Omega$.

Since $\left\|f - \left\langle f, \widehat{k}_\lambda\right\rangle \widehat{k}_\lambda\right\| = \inf_{\phi\in\mathbb{C}} \left\|f - \phi\widehat{k}_\lambda\right\|$ for any $f \in \mathcal{H}$ and $\lambda \in \Omega$, we have

$$\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| \le \left\|A\widehat{k}_\lambda - \frac{\beta+\alpha}{2}\widehat{k}_\lambda\right\| \le \frac{1}{2}|\beta-\alpha|$$

and

$$\left\|B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\| \le \left\|B^*\widehat{k}_\eta - \frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_\eta\right\| \le \frac{1}{2}|\gamma-\delta|$$

for all $\lambda, \eta \in \Omega$. Hence, we have

$$\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| \left\|B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\| \le \frac{1}{4}|\beta-\alpha|\,|\gamma-\delta| \tag{4}$$

for all $\lambda, \eta \in \Omega$. On the other hand,

$$\begin{aligned} &\left\langle A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda, B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\rangle \\ &\quad = \left\langle BA\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle + \widetilde{A}(\lambda)\widetilde{B}(\eta)\left\langle \widehat{k}_\lambda, \widehat{k}_\eta\right\rangle \\ &\qquad - \widetilde{A}(\lambda)\left\langle B\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle - \left\langle A\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle \widetilde{B}(\eta) \end{aligned}$$

for all $\lambda, \eta \in \Omega$. Taking the modulus in the above equality, we have

$$\begin{aligned} &\left|\left\langle A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda, B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\rangle\right| \\ &\quad \ge \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| - \left|\widetilde{A}(\lambda)\left\langle B\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle + \left\langle A\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\widetilde{B}(\eta) - \widetilde{A}(\lambda)\widetilde{B}(\eta)\left\langle \widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| \\ &\quad \ge \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| - \left|\widetilde{A}(\lambda)\left\langle B\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| - \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\widetilde{B}(\eta)\right| - \left|\widetilde{A}(\lambda)\widetilde{B}(\eta)\left\langle \widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left|\left\langle A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda, B^*\widehat{k}_\eta - \widetilde{B^*}(\eta)\widehat{k}_\eta\right\rangle\right| \\ &\qquad + \left|\widetilde{A}(\lambda)\left\langle B\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| + \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\widetilde{B}(\eta)\right| + \left|\widetilde{A}(\lambda)\widetilde{B}(\eta)\left\langle \widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| \\ &\quad \ge \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\right| \end{aligned} \tag{5}$$

for all $\lambda, \eta \in \Omega$. So we have for $\lambda = \eta$ from (3)-(5)

$$\begin{aligned}&\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|+\left|\widetilde{A}\left(\lambda\right)\widetilde{B}\left(\lambda\right)\right|+\left|\widetilde{A}\left(\lambda\right)\widetilde{B}\left(\lambda\right)\right|\\&\quad+\left|\widetilde{A}\left(\lambda\right)\widetilde{B}\left(\lambda\right)\right|\geq\left|\widetilde{BA}\left(\lambda\right)\right|.\end{aligned}\tag{6}$$

Taking the supremum in (6) over $\lambda \in \Omega$, we get that

$$\operatorname{ber}\left(BA\right)\leq 3\operatorname{ber}\left(A\right)\operatorname{ber}\left(B\right)+\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|.$$

This gives the desired result. □

Now, we consider a different approach in the following result.

Theorem 2 *Let $C_{\alpha,\beta}\left(A\right)$ and $C_{\gamma,\delta}\left(B\right)$ be accretive transform for $A, B \in \mathcal{B}\left(\mathcal{H}\right)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then,*

$$\operatorname{ber}\left(BA\right)\leq\operatorname{ber}\left(A\right)\operatorname{ber}\left(B\right)+\frac{1}{4}\left|\beta-\alpha\right|\left(\left|\gamma-\delta\right|+\left|\gamma+\delta\right|\right).$$

Proof We can state the following inequality from the Schwarz inequality and the assumptions

$$\begin{aligned}\left|\left\langle A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda}, B^{*}\widehat{k}_{\eta}-\frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_{\eta}\right\rangle\right|&\leq\left\|A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda}\right\|\left\|B^{*}\widehat{k}_{\eta}-\frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_{\eta}\right\|\\&\leq\left\|A\widehat{k}_{\lambda}-\frac{\beta+\alpha}{2}\widehat{k}_{\lambda}\right\|\left\|B^{*}\widehat{k}_{\eta}-\frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_{\eta}\right\|\\&\leq\frac{1}{4}\left|\beta-\alpha\right|\left|\gamma-\delta\right|\end{aligned}\tag{7}$$

for all $\lambda, \eta \in \Omega$.

Since

$$\begin{aligned}&\left\langle A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda}, B^{*}\widehat{k}_{\eta}-\frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_{\eta}\right\rangle\\&\quad=\left\langle BA\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle-\widetilde{A}\left(\lambda\right)\left\langle B\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle-\frac{\gamma+\delta}{2}\left\langle A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle\end{aligned}$$

on taking the modulus in this inequality, we obtain

$$\begin{aligned}&\left|\left\langle A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda}, B^{*}\widehat{k}_{\eta}-\frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_{\eta}\right\rangle\right|\\&\quad\geq\left|\left\langle BA\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle\right|-\left|\widetilde{A}\left(\lambda\right)\left\langle B\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle\right|-\left|\frac{\gamma+\delta}{2}\right|\left|\left\langle A\widehat{k}_{\lambda}-\widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda},\widehat{k}_{\eta}\right\rangle\right|\end{aligned}\tag{8}$$

for all $\lambda, \eta \in \Omega$. Then, we have for $\lambda = \eta \in \Omega$ from (7) and (8)

$$\begin{aligned}\left|\widetilde{BA}(\lambda)\right| &\le \left|\widetilde{A}(\lambda)\widetilde{B}(\lambda)\right| + \left|\frac{\gamma+\delta}{2}\right|\left|\widetilde{A\widehat{k}_\lambda - \widetilde{A}(\lambda)}\right| + \frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ &\le \left|\widetilde{A}(\lambda)\widetilde{B}(\lambda)\right| + \left|\frac{\gamma+\delta}{2}\right|\left\|A\widehat{k}_\lambda - \frac{\beta+\alpha}{2}\widehat{k}_\lambda\right\| + \frac{1}{4}|\beta-\alpha||\gamma-\delta| \\ &\le \left|\widetilde{A}(\lambda)\widetilde{B}(\lambda)\right| + \frac{1}{4}|\beta-\alpha|\left(|\gamma-\delta| + |\gamma+\delta|\right).\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$ in the above inequality, we have

$$\text{ber}(BA) \le \text{ber}(A)\,\text{ber}(B) + \frac{1}{4}|\beta-\alpha|\left(|\gamma-\delta| + |\gamma+\delta|\right).$$

This proves the theorem. □

By using arguments in above theorem, we can get the following result.

Corollary 1 *Let $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ be accretive transform for $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then,*

$$\text{ber}(BA) \le \text{ber}(A)\,\text{ber}(B) + |\gamma+\delta|\,\text{ber}(A) + \frac{1}{4}|\beta-\alpha||\gamma-\delta|.$$

Proof Indeed, we have that

$$\begin{aligned}\left\langle A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda, B^*\widehat{k}_\eta - \frac{\overline{\gamma}+\overline{\delta}}{2}\widehat{k}_\eta\right\rangle &= \left\langle BA\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle - \widetilde{A}(\lambda)\left\langle B\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle \\ &\quad - \frac{\gamma+\delta}{2}\left\langle A\widehat{k}_\lambda, \widehat{k}_\eta\right\rangle + \frac{\gamma+\delta}{2}\widetilde{A}(\lambda)\left\langle \widehat{k}_\lambda, \widehat{k}_\eta\right\rangle\end{aligned}$$

for all $\lambda, \eta \in \Omega$. Taking the supremum on $\lambda = \eta \in \Omega$ and using the same arguments in the proof of the above theorem, we get

$$\text{ber}(BA) \le \text{ber}(A)\,\text{ber}(B) + |\gamma+\delta|\,\text{ber}(A) + \frac{1}{4}|\beta-\alpha||\gamma-\delta|$$

for the operators $A, B \in \mathcal{B}(\mathcal{H})$. □

3 Grüss-Type Inequality

Now, we give a Grüss-type inequality for selfadjoint operators on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 3 *Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f and g are continuous on $[m, M]$, then*

$$\widetilde{f(A)g(A)}(\mu)-\widetilde{f(A)}(\mu)\widetilde{g(A)}(\lambda)-\frac{\gamma+\Gamma}{2}\left[\widetilde{g(A)}(\mu)-\widetilde{g(A)}(\lambda)\right]$$
$$\leq\frac{1}{2}.(\Gamma-\gamma)\left[||g(A)||^{2}+\left(\widetilde{g(A)}(\lambda)\right)^{2}-2\widetilde{g(A)}(\lambda)\widetilde{g(A)}(\mu)\right]^{1/2} \tag{9}$$

for any $\lambda,\mu\in\Omega$*, where* $\gamma=\min_{t\in[m,M]}f(t)$*,* $\Gamma=\max_{t\in[m,M]}f(t)$

Proof Indeed, we have the identity

$$\begin{aligned}&\langle(f(A)-\xi.1_{\mathcal{H}})\left(g(A)-\langle g(A)\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle.1_{\mathcal{H}}\right)\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle\\&\quad=\langle f(A)g(A)\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle-\xi.\left[\langle g(A)\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle-\langle g(A)\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle\right]\\&\qquad-\langle g(A)\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle\langle f(A)\widehat{k}_{\mu},\widehat{k}_{\mu}\rangle\end{aligned} \tag{10}$$

for each $\xi\in\mathbb{R}$ and $\lambda,\mu\in\Omega$.

Taking the modulus in (10), we obtain

$$\begin{aligned}&\left|\widetilde{f(A)g(A)}(\mu)-\xi.\left[\widetilde{g(A)}(\mu)-\widetilde{g(A)}(\lambda)\right]-\widetilde{g(A)}(\lambda)\widetilde{f(A)}(\mu)\right|\\&\quad=\left|\left\langle\left(g(A)-\widetilde{g(A)}(\lambda).1_{\mathcal{H}}\right)\widehat{k}_{\mu},(f(A)-\xi.1_{H})\widehat{k}_{\mu}\right\rangle\right|\\&\quad\leq\left\|g(A)\widehat{k}_{\mu}-\widetilde{g(A)}(\lambda)\widehat{k}_{\mu}\rangle\right\|\left\|f(A)\widehat{k}_{\mu}-\xi\widehat{k}_{\mu}\right\|\\&\quad=\left[\left\|g(A)\widehat{k}_{\mu}\right\|^{2}+\left(\widetilde{g(A)}(\lambda)\right)^{2}-2\widetilde{g(A)}(\lambda)\widetilde{g(A)}(\mu)\right]^{\frac{1}{2}}\\&\qquad\times\left\|\left(f(A)\widehat{k}_{\mu}-\lambda\widehat{k}_{\mu}\right)\right\|\\&\quad\leq\left[\|g(A)\|^{2}+\left(\widetilde{g(A)}(\lambda)\right)^{2}-2\widetilde{g(A)}(\lambda)\widetilde{g(A)}(\mu)\right]^{\frac{1}{2}}\\&\qquad\times\|f(A)-\lambda.1_{\mathcal{H}}\|\end{aligned} \tag{11}$$

for any $\lambda,\mu\in\Omega$.

Since $\gamma=\min_{t\in[m,M]}f(t)$ and $\Gamma=\max_{t\in[m,M]}f(t)$, by the property (1) we have that $\gamma\leq\widetilde{f(A)}(\mu)\leq\Gamma$ for each $\mu\in\Omega$ which is obviously equivalent to

$$\left|\widetilde{f(A)}(\mu)-\frac{\gamma+\Gamma}{2}\left\|\widehat{k}_{\mu}\right\|^{2}\right|\leq\frac{1}{2}(\Gamma-\gamma)$$

or with

$$\left|\left(f(A)-\widetilde{\frac{\gamma+\Gamma}{2}}1_{\mathcal{H}}\right)(\mu)\right|\leq\frac{1}{2}(\Gamma-\gamma)$$

for each $\mu\in\Omega$.

Taking the supremum in this inequality, we get

$$\left\| f(A) - \frac{\gamma + \Gamma}{2} .1_{\mathcal{H}} \right\| \leq \frac{1}{2} (\Gamma - \gamma),$$

which together with the inequality (11) applied for $\xi = \frac{\gamma+\Gamma}{2}$ produces the desired results. □

As a special case of the above theorem, we can give the following result.

Corollary 2 *Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. Then,*

$$\text{ber}\left(f^2(A) - (f(A))^2\right) \leq \frac{1}{2}(\Gamma - \gamma)\left[\|f(A)\|^2 - \text{ber}\left((f(A))^2\right)\right]^{\frac{1}{2}}$$

for each $\lambda \in \Omega$, where $\gamma = \min_{t \in [m,M]} f(t)$, $\Gamma = \max_{t \in [m,M]} f(t)$.

Proof Taking $f = g$ and $\lambda = \mu$ in (9), then

$$\left| \widetilde{f^2(A)}(\lambda) - \left(\widetilde{f(A)}(\lambda)\right)^2 \right| \leq \frac{1}{2}(\Gamma - \gamma)\left[\|f(A)\|^2 - \left(\widetilde{f(A)}(\lambda)\right)^2\right]^{\frac{1}{2}}$$

for all $\lambda \in \Omega$. Taking the supremum on $\lambda \in \Omega$ in above inequality, we have

$$\text{ber}\left(f^2(A) - (f(A))^2\right) \leq \frac{1}{2}(\Gamma - \gamma)\left[\|f(A)\|^2 - \text{ber}\left((f(A))^2\right)\right]^{\frac{1}{2}}$$

for any selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$. This proves the theorem. □

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