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Every Planar Graph Without Pairwise Adjacent 3-, 4-, and 5-Cycle is DP-4-Colorable

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Abstract

DP-coloring is a generalization of list coloring in simple graphs. Many results in list coloring can be generalized in those of DP-coloring. Kim and Ozeki showed that every planar graph without k-cycles where $k = 3, 4, 5$, or 6 is DP-4-colorable. Recently, Kim and Yu extended the result on 3- and 4-cycles by showing that every planar graph without triangles adjacent to 4-cycles are DP-4-colorable. Xu and Wu showed that every planar graph without 5-cycles adjacent simultaneously to 3-cycles and 4-cycles is 4-choosable. In this paper, we extend the results on 3-, 4-, and 5-cycles as follows. Let G be a planar graph without pairwise adjacent 3-, 4-, and 5-cycle. We prove that each precoloring of a 3-cycle of G can be extended to a DP-4-coloring of G. As a consequence, each planar graph without pairwise adjacent 3-, 4-, and 5-cycle is DP-4-colorable.

Keywords DP-coloring · List coloring · Planar graph · Cycle

MSC code 05C15

1 Introduction

Every graph in this paper is finite, simple, and undirected. Embedding a graph G in the plane, we let $V(G)$, $E(G)$, and $F(G)$ denote the vertex set, edge set, and face set of G. For $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of G induced by U. For $X, Y \subseteq V(G)$ where X and Y are disjoint, we let $E_G(X, Y)$ be the set of all edges in G with one endpoint in X and the other in Y.

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The concept of choosability was introduced by Vizing [9] and by Erdős et al. [5], independently. A *k-assignment* L of a graph G assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex v. A graph G is L-colorable if there is a proper coloring f where $f(v) \in L(v)$. If G is L-colorable for every k-assignment L, then we say G is *k-choosable*.

Dvořák and Postle [4] introduced a generalization of list coloring in which they called *correspondence coloring*. But following Bernshteyn et al. [3], we call it *DP-coloring*.

Definition 1 Let L be an assignment of a graph G. We call H a *cover* of G if it satisfies all the followings:

(i) The vertex set of H is $\bigcup_{u \in V(G)}(\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$;
(ii) $H[u \times L(u)]$ is a complete graph for every $u \in V(G)$;
(iii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (may be empty).
(iv) If $uv \notin E(G)$, then no edges of H connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

Definition 2 An (H, L)-coloring of G is an independent set in a cover H of G with size $|V(G)|$. We say that a graph is *DP-k-colorable* if G has an (H, L)-coloring for every k-assignment L and every cover H of G. The *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$, is the minimum number k such that G is DP-k-colorable.

If we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$, then G has an (H, L)-coloring if and only if G is L-colorable. Thus DP-coloring is a generalization of list coloring. This also implies that $\chi_{DP}(G) \geq \chi_l(G)$. In fact, the difference of these two chromatic numbers can be arbitrarily large. For graphs with average degree d, Bernshteyn [2] showed that $\chi_{DP}(G) = \Omega(d/\log d)$, while Alon [1] showed that $\chi_l(G) = \Omega(\log d)$.

Dvořák and Postle [4] showed that $\chi_{DP}(G) \leq 5$ for every planar graph G. This extends a major result on list coloring by Thomassen [8]. On the other hand, Voigt [10] gave an example of a planar graph which is not 4-choosable (thus not DP-4-colorable). It is of interest to obtain sufficient conditions for planar graphs to be DP-4-colorable. Kim and Ozeki [6] showed that every planar graph without k-cycles is DP-4-colorable for each $k = 3, 4, 5, 6$. Kim and Yu [7] extended the result on 3- and 4-cycles by showing that every planar graph without triangles adjacent to 4-cycles is DP-4-colorable.

Let $\mathcal{A}$ denote the family of planar graphs G without pairwise adjacent 3-, 4-, and 5-cycle. In this paper, we extend the results on 3-, 4-, and 5-cycles as follows.

Theorem 1.1 *Let $G \in \mathcal{A}$. Then each precoloring of a* 3*-cycle can be extended to a DP-4-coloring of G.*

The following corollary is immediate.

Corollary 1.2 *Every planar graph without pairwise adjacent* 3-, 4-, *and* 5-*cycle is DP*-4-*colorable.*

Corollary 1.2 generalizes the aforementioned result by Kim and Yu [7] and the following result by Xu and Wu [11].

Theorem 1.3 [11] *Every planar graph without* 5*-cycles adjacent simultaneously to* 3*-cycles and* 4*-cycles is* 4*-choosable.*

2 Preliminaries

First, we introduce some notations and definitions. A k-vertex (k^+-vertex, k^--vertex, respectively) is a vertex of degree k (at least k, at most k, respectively). The same notations are applied to faces.

A $(d_1, d_2, \ldots, d_k)$-*face* f is a face of degree k where vertices on f have degree $d_1, d_2, \ldots, d_k$ in a cyclic order. A $(d_1, d_2, \ldots, d_k)$-*vertex* v is a vertex of degree k where faces incident to v have degree $d_1, d_2, \ldots, d_k$ in a cyclic order. Note that some face may appear more than one time in the order. We say xy is a *chord* in a cycle C if $x, y \in V(C)$ but $xy \in E(G) - E(C)$. An *internal chord* of C is a chord inside C while *external chord* of C is a chord outside C

A graph $C(m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{m+n-2}$ with an internal chord x_1x_m. A graph $C(l, m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{l+m+n-4}$ with internal chords x_1x_l and x_1x_{l+m-2}. A graph $C(m, n, p, q)$ can be defined similarly. We use $int(C)$ and $ext(C)$ to denote the sets of vertices inside and outside a cycle C, respectively. The cycle C is a *separating cycle* if $int(C)$ and $ext(C)$ are not empty. We use $B(f)$ to denote a boundary of a face f. It is straightforward to see that if f is a 5^--face, then $B(f)$ is a cycle.

3 Structures

Proof of Theorem 1.1 Let G be a minimal counterexample to Theorem 1.1 with $|V(G)|$ minimized and a precolored 3-cycle C_0. □

Lemma 3.1 *G has no separating* 3*-cycles.*

Proof Suppose to the contrary that there exists G contains a separating 3-cycle C. Note that C is not necessary C_0. By symmetry, we assume $V(C_0) \subseteq V(C) \cup int(C)$. By the minimality of G, a precoloring of C_0 can be extended to $V(C) \cup int(C)$. After C is colored, then again the coloring on C can be extended to $ext(C)$. Thus we have a DP-4-coloring of G, a contradiction. □

Since C_0 is not a separating 3-cycle by Lemma 3.1, we may assume that C_0 is the boundary of the outer face D of G in the remaining of the paper.

Definition 3 Let H be a cover of G with a list assignment L. Let $G' = G - F$ where F is an induced subgraph of G. A list assignment L' is a *restriction of* L on G' if $L'(u) = L(u)$ for each vertex in G'. A graph H' is a *restriction of* H on G' if $H' = H[\{\{v\} \times L(v) : v \in V(G')\}]$. Assume G' has an (H', L')-coloring with an independent set I' in H' such that $|I'| = |V(G)| - |V(F)|$.

A *residual list assignment* L^* of F is defined by

$$L^*(x) = L(x) - \bigcup_{ux \in E(G)} \{c' \in L(x) : (u, c)(x, c') \in E(H) \text{ and } (u, c) \in I'\}$$

for each $x \in V(F)$.

A *residual cover* H^* is defined by $H^* = H[\{\{x\} \times L^*(x) : x \in V(F)\}]$.

From above definitions, we have the following fact.

Lemma 3.2 *Assume I' is a (H', L')-coloring of G'. A residual cover H^* is a cover of F with an assignment L^*. Furthermore, if F is (H^*, L^*)-colorable, then G is (H, L)-colorable.*

Proof One can check from the definitions of a cover and a residual cover that H^* is a cover of F with an assignment L^*.

Suppose that F is (H^*, L^*)-colorable. Then H^* has an independent set I^* with $|I^*| = |F|$. It follows from Definition 3 that no edges connect H^* and I'. Additionally, I' and I^* are disjoint. Altogether, we have that $I = I' \cup I^*$ is an independent set in H with $|I| = (|V(G)| - |V(F)|) + |V(F)| = |V(G)|$. Thus G is (H, L)-colorable. □

Lemma 3.3 *Every vertex not on C_0 has degree at least* 4.

Proof Suppose to the contrary that G has a vertex x not on C_0 with degree at most 3. Let L be a 4-assignment and let H be a cover of G such that G has no (H, L)-coloring. By the minimality of G, the subgraph $G' = G - x$ an (H', L')-coloring where L' (and H') is a restriction of L (and H, respectively) on G'. Thus there is an independent set I' with $|I'| = |G'|$ in H'. Consider a residual list assignment L^* on x. Since $|L(x)| = 4$ and $d(x) \leq 3$, we obtain $|L^*(x)| \geq 1$. Clearly, $\{(x, c)\}$ where $c \in L^*(x)$ is an independent set in $G[\{x\}]$. Thus $G[\{x\}]$ is (H^*, L^*)-colorable. It follows from Lemma 3.2 that G is (H, L)-colorable, a contradiction. □

Lemma 3.4 *(a) A* 5*-cycle has no chords.*
(b) A bounded 3*-face f is not adjacent to a* 4*-face g.*
(c) If bounded 3*-faces f and g are adjacent, then $B(f) \cup B(g) = C(3, 3)$.*
(d) If a bounded 3*-face f is adjacent to a* 5*-face g, then $B(f) \cup B(g) = C(3, 5)$.*
(e) If $C(3, 5)$ is obtained from a 6*-cycle C with a chord, then C has exactly one chord.*
(f) If bounded 3*-faces f and g are adjacent, then f is not adjacent to a bounded* 5^-*-face.*

Proof (a) Let $C = rstuv$ be a 5-cycle. Suppose that rt is a chord. Then we have three pairwise adjacent cycles rst, $rtuv$, and $rstuv$, contrary to $G \in \mathcal{A}$.
(b) Let $B(f) = uvw$ and $B(g) = vwxy$. Suppose that $u = x$ or y. We have that $d(v) = 2$ or $d(w) = 2$, contrary to Lemma 3.3. Thus $x \neq u \neq y$. We obtain a 5-cycle $uwxyv$ with a chord vw, contrary to (a).
(c) Let $B(f) = uvw$ and $B(g) = uvx$. Since both f and g are bounded, we have that $w \neq x$. Thus $B(f) \cup B(g) = C(3, 3)$.

(d) Let $B(f) = uvz$ and $B(g) = uvwxy$. If $z \in \{w, x, y\}$, then a 5-cycle $B(g)$ has a chord, contrary to (a).
(e) Let C be a cycle $uzvwxy$ with a chord uv. Suppose to the contrary that C contains another chord st. By (a) and (b), st is not a chord in a 5-cycle $uvwxy$. By symmetry, we may assume $s = z$ and $t = x$ or $t = y$. Then we have three pairwise adjacent cycles uvz, $uvwxy$, and $xyuz$ or $uvzy$, contrary to $G \in \mathcal{A}$.
(f) Let $B(f) = uvw$ and $B(g) = vwx$. By (c), $u \neq x$. Suppose f is adjacent to a k-face h where $k \leq 5$. By (b), $h = 3$ or 5.

- $B(h) = uvz$.
  By (c), $z \neq w$. Suppose to the contrary that $z = x$. Then $d(v) = 3$, contrary to Lemma 3.3. Thus $z \neq x$. Altogether, we have three pairwise adjacent 3-cycles uvw, $uvxw$, $zuwxv$, contrary to $G \in \mathcal{A}$.
- $B(h) = rstuv$.
  By (a), $w \notin \{r, s, t\}$. By (e), $x \notin \{r, s, t\}$. Altogether, we have three pairwise adjacent cycles uvw, $uvxw$, and $rstuv$, contrary to $G \in \mathcal{A}$.

Thus h is not a 5^--face. □

Lemma 3.4 (f) yields this immediate consequence.

Corollary 3.5 *For $k \geq 4$, a k-vertex v in G is incident to at most $k - 2$ 3-faces.*

Lemma 3.6 *Let $C(l_1, \ldots, l_k)$ be obtained from a cycle $C = x_1 \ldots x_m$ with k internal chords sharing a common endpoint x_1 such that $V(C) \cap V(C_0) = \emptyset$. Suppose x_2 or x_m is not the endpoint of any chord in C. If $d(x_1) \leq k + 2$, then there exists $i \in \{2, 3, \ldots, m\}$ such that $d(x_i) \geq 5$.*

Proof By symmetry, let x_m be not an endpoint of any chord in C. Suppose to the contrary that $d(x_i) \leq 4$ for each $i = 2, 3, \ldots, m$. Let L be a 4-assignment and let H be a cover of G such that G has no (H, L)-coloring. By the minimality of G, the subgraph $G' = G - \{x_1, \ldots, x_m\}$ admits an (H', L')-coloring where L' (and H', respectively) is a restriction of L (and H, respectively) in G'. Thus there is an independent set I' with $|I'| = |G'|$ in H'.

Consider a residual list assignment L^* on F. Since $|L(v)| = 4$ for every $v \in V(G)$, we have $|L^*(x_1)| \geq 3$ and $|L^*(v)| \geq 3$ for each $v \in V(C)$ with an edge x_1v and $|L^*(x_i)| \geq 2$ for each of the remaining vertices x_i in $V(C)$. Let H^* be an residual cover of F. Since x_m is not an endpoint of a chord in C, we can choose a color c from $L^*(x_1)$ such that $|L^*(x_m) - \{c' : (x_1, c)(x_m, c') \in E(H^*)\}| \geq 2$. By choosing colors of $x_2, x_3, \ldots, x_m$ in this order, we obtain an independent set I^* with $|I^*| = m = |F|$. Thus F is (H^*, L^*)-colorable. It follows from Lemma 3.2 that G is (H, L)-colorable, a contradiction. □

Corollary 3.7 *For each $C(3, 5)$ such that $V(C(3, 5)) \cap V(C_0) = \emptyset$, there exists a vertex with degree at least 5.*

Proof Let $C(3, 5)$ be obtained from a 6-cycle $C = x_1 \ldots x_6$ with a chord x_1x_3 such that $V(C) \cap V(C_0) = \emptyset$. By Lemma 3.4 (e), C has no other chords. The proof is complete by Lemma 3.6. □

Lemma 3.8 *Let $F = C(3,5,3)$, $C(3,5,5)$ or $C(5,3,5)$ be obtained from a cycle $C = x_1 \dots x_m$ with two internal chords sharing an endpoint x_1 such that $V(C) \cap V(C_0) = \emptyset$. If $d(x_1) = 5$, then there exists $i \in \{2, \dots, m\}$ with $d(x_i) \geq 5$.*

Proof By Lemma 3.6, it suffices to show that x_2 or x_m is not an endpoint to a chord in C.

Let $F = C(3,5,3)$. It follows from Lemma 3.4 (e) that x_2 is not adjacent to x_4, x_5, or x_6. If x_2 is adjacent to x_7, then we have separating 3-cycle $x_1x_2x_7$, contrary to Lemma 3.1. Thus x_2 is not an endpoint of any chord of C.

Let $F = C(3,5,5)$. Suppose there exists a chord e of C where $e = x_2x_i$, for otherwise we have the desired condition. If $x_i = x_9$, then we have separating 3-cycle $x_1x_2x_9$, contrary to Lemma 3.1. It follows from Lemma 3.4 (e) that $i \notin \{4,5,6\}$. Then $x_i = x_7$ or x_8. By Lemma 3.4 (a), x_9 is not adjacent to x_6 or x_7. Thus x_9 is not an endpoint of any chord of C.

Let $F_1 = C(5,3,5)$. Suppose there exists a chord e of C where $e = x_2x_i$, for otherwise we have the desired condition. If $x_i = x_9$, then we have separating 3-cycle $x_1x_2x_9$, contrary to Lemma 3.1. It follows from Lemma 3.4 (e) that $i \notin \{4,5,6\}$. Then $x_i = x_7$ or x_8. By Lemma 3.4 (a), x_9 is not adjacent to x_6 or x_7. Thus x_9 is not an endpoint of any chord of C'. □

Corollary 3.9 *Let v be a 5-vertex with incident bounded faces $f_1, \dots, f_5$ in a cyclic order. Let $F = B_1 \cup B_2 \cup B_3$ where B_i denote $B(f_i)$ and $V(F) \cap V(C_0) = \emptyset$. If $(d(f_1), d(f_2), d(f_3)) = (3,5,3)$ or $(3,5,5)$, or $(d(f_1), d(f_2), d(f_3), d(f_4)) = (5,3,5,3)$, then there exists $w \in V(F)$ such that $d(w) \geq 5$ and $w \neq v$.*

Proof By Lemma 3.8, it suffices to show that $F = C(3,5,3)$, $C(3,5,5)$, or $C(5,3,5)$.

- $(d(f_1), d(f_2), d(f_3)) = (3,5,3)$.
  Let $B_1 = rsv$, $B_2 = vstuw$, and $B_3 = vwx$. It follows from Lemma 3.4 (d) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{v, w\}$. If $r = x$, then $d(v) = 3$, contrary to Lemma 3.3. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(3,5,3)$.
- $(d(f_1), d(f_2), d(f_3)) = (3,5,5)$.
  Let $B_1 = rsv$, $B_2 = vstuw$, and $B_3 = vwxyz$. It follows from Lemma 3.4 (d) that $V(B_1) \cap V(B_2) = \{s, v\}$. If $r = z$, then $d(v) = 3$, contrary to $d(v) = 5$. It follows from Lemma 3.4 (a) that neither r nor s is in $V(B_3)$. Thus $V(B_1) \cap V(B_3) = \{v\}$. Now consider $V(B_2) \cap V(B_3)$. It follows from Lemma 3.4 (a) that neither s nor u is in $V(B_3)$. Similarly, neither x nor z is in $V(B_2)$. Then $V(B_2) \cap V(B_3) = \{t = y, v, w\}$ or $\{v, w\}$. Note that $r \neq z$, for otherwise rsv is a separating cycle, contrary to Lemma 3.1. Moreover, $r \notin \{x, y\}$, otherwise the cycle $vwxyz$ has a chord, contrary to Lemma 3.4 (a). Thus $V(B_1) \cap V(B_3) = \{v\}$. If $V(B_2) \cap V(B_3) = \{t = y, v, w\}$, then we have three adjacent pairwise cycles rsv, $stzv$, $stuwv$, contrary to $G \in \mathcal{A}$. Thus $V(B_2) \cap V(B_3) = \{v, w\}$. Altogether we have $F = C(3,5,5)$.
- $(d(f_1), d(f_2), d(f_3), d(f_4)) = (5,3,5,3)$.
  Let $B_1 = rstuv$, $B_2 = uvw$, $B_3 = vwxyz$, and $B_4 = vpz$. It follows from Lemma 3.4 (e) that $V(B_1) \cap V(B_2) = \{u, v\}$ and $V(B_2) \cap V(B_3) = \{v, w\}$.

Consider $V(B_1) \cap V(B_3)$. It follows from Lemma 3.4 (a) that neither r nor u is in $V(B_3)$. Similarly, neither w nor z is in $V(B_2)$. By Lemma 3.4 (a) that neither r nor u is in $V(B_3)$.

- Suppose $\{s, t\} \subseteq V(B_1) \cap V(B_3)$.
  Then $\{x, y\} \subseteq V(B_1) \cap V(B_3)$ and $s = y$ and $t = x$. Consequently, we have three adjacent pairwise cycles uvw, $uvwt$, $rstuv$, contrary to $G \in \mathcal{A}$.
- Suppose $s \in V(B_1) \cap V(B_3)$ but $t \notin V(B_1) \cap V(B_3)$.
  Then $\{v, s\} = V(B_1) \cap V(B_3)$. Consequently $s = x$ or $s = y$. In the former case, we have three adjacent pairwise cycles uvw, $stuw$, $vwxyz$, contrary to $G \in \mathcal{A}$. In the later case, we have three adjacent pairwise cycles pvz, $srvz$, $vwxyz$, contrary to $G \in \mathcal{A}$.
- Suppose $t \in V(B_1) \cap V(B_3)$ but $s \notin V(B_1) \cap V(B_3)$.
  Then $\{v, t\} = V(B_1) \cap V(B_3)$. Consequently $t = x$ or $t = y$. In the former case, we have three adjacent pairwise cycles uvw, $uvwt$, $vwxyz$, contrary to $G \in \mathcal{A}$. In the later case, we have three adjacent pairwise cycles uvw, $uwxy$, $vwxyz$, contrary to $G \in \mathcal{A}$.

Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $B_1 \cup B_2 \cup B_3 = C(5, 3, 5)$.

□

Corollary 3.10 *Let v be a 6-vertex with consecutive incident faces $f_1, \ldots, f_6$. Let $F = B_1 \cup B_2 \cup B_3 \cup B_4$ where B_i denote $B(f_i)$ and $V(F) \cap V(C_0) = \emptyset$. If $(d(f_1), d(f_2), d(f_3), d(f_4)) = (3, 5, 3, 5)$, then there exists $w \in V(F)$ such that $w \neq v$ and $d(w) \geq 5$.*

Proof By Lemma 3.6, it suffices to show that $F = C(3, 5, 3, 5)$. Similar to the proof of corollary 3.9, one can show that $B_1 \cup B_2 \cup B_3 = C(3, 5, 3)$ and $B_2 \cup B_3 \cup B_4 = C(5, 3, 5)$. Let $V(B_1) = \{t, u, v\}$ and $V(B_4) = \{v, w, x, y, z\}$ where $t \in V(B_2)$ and $w \in V(B_3)$. It only remains to show that $u \notin \{x, y, z\}$. If $u = x$ or y, then the cycle $vwxyz$ has a chord vx or vy, contrary to Lemma 3.4 (a). If $u = z$, then vtz is a separating cycle, contrary to Lemma 3.1. Thus $F = C(3, 5, 3, 5)$. □

4 Discharging Process

We are now ready to present a discharging procedure that will complete the proof of Theorem 1.1. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = 2d(v) - 6$, each face $f \neq D$ has an initial charge of $\mu(f) = d(f) - 6$ and $\mu(D) = d(D) + 6 = 9$. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = 0$. Let $\mu^*(x)$ be the charge of $x \in V \cup F$ after the discharge procedure. We prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(D) > 0$ to get a contradiction.

Let $w(v \to f)$ be the charge transferred from a vertex v to an incident face f. We say that v is a *flaw* vertex if v is a $(3, 5, 3, 5^+)$-vertex. The discharging rules are as follows.

(R1) Let f be a 3-face.

(R1.1) For a 4-vertex v not in C_0,

$$w(v \to f) = \begin{cases} \frac{3}{5}, & \text{if } v \text{ is flaw and } f \text{ is a } (4, 5^+, 5^+)\text{-face}, \\ \frac{4}{5}, & \text{if } v \text{ is flaw and } f \text{ is a } (4, 4, 5^+)\text{-face}, \\ 1, & \text{otherwise.} \end{cases}$$

(R1.2) For a 5^+-vertex v not in C_0,

$$w(v \to f) = \begin{cases} \frac{7}{5}, & \text{if } f \text{ is a } (4, 4, 5^+)\text{-face with two incident flaw vertices}, \\ \frac{6}{5}, & \text{if } f \text{ is a } (4, 4^+, 5^+)\text{-face with exactly one flaw vertex}, \\ 1, & \text{otherwise.} \end{cases}$$

(R2) Let f be a 4-face.

For a 4^+-vertex v not in C_0, $w(v \to f) = \frac{1}{2}$.

(R3) Let f be a 5-face.

(R3.1) For a 4-vertex v not in C_0,

$$w(v \to f) = \begin{cases} 0, & \text{if } v \text{ is a flaw vertex with four 4-neighbors}, \\ \frac{1}{10}, & \text{if } v \text{ is a flaw vertex with exactly one } 5^+\text{-neighbor}, \\ \frac{1}{5}, & \text{if } v \text{ is a flaw vertex with at least two } 5^+\text{-neighbors}, \\ \frac{1}{5}, & \text{if } v \text{ is a } (3, 5, 4, 5)\text{-vertex}, \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

(R3.2) For a 5-vertex v not in C_0,

$$w(v \to f) = \begin{cases} \frac{7}{10}, & \text{if } f \text{ is a } (4, 4, 4, 4, 5)\text{-face with five adjacent } 4^-\text{-faces}, \\ \frac{3}{5}, & \text{if } f \text{ is a } (4, 4, 4, 4, 5)\text{-face with at least one adjacent } 5^+\text{-face}, \\ \frac{2}{5}, & \text{if } f \text{ is a } (4, 4, 4, 5, 5^+)\text{-face} \\ \frac{3}{10}, & \text{otherwise.} \end{cases}$$

(R3.3) For a 6^+-vertex v not in C_0,

$$w(v \to f) = \begin{cases} \frac{4}{5}, & \text{if } f \text{ is a } (4, 4, 4, 4, 6^+)\text{-face}, \\ \frac{2}{5}, & \text{if } f \text{ is incident to a } 5^+\text{-vertex other than } v. \end{cases}$$

(R4) The outerface D gets $\mu(v)$ from each incident vertex v and gives 2 to each 4- or 5-face or 3-face sharing exactly one vertex with D, $\frac{12}{5}$ to each 3-face sharing one edge with D.

It suffices to check that each $x \in V(G) \cup F(G)$ has nonnegative final charge and D has positive final charge. By (R4), we have $\mu^*(v) = 0$ for each $v \in V(C_0)$. Thus we only consider a vertex v not on C_0.

Let v be a vertex with neighbors $v_1, v_2, \ldots, v_{d(v)}$ in a cyclic order. Let $f_1, f_2, \ldots, f_{d(v)}$ be incident faces of v in a cyclic order with v_i and v_{i+1} incident to f_i where $i+1$ is taken in modulo $d(v)$. Thus v is a $(d(f_1), d(f_2), \ldots, d(f_{d(v)})$-vertex.

Case 1 v is a 4-vertex but v is not a flaw vertex.

It follows from Lemma 3.4 (b) that a 3-face is not adjacent to a 4-face, and from Corollary 3.5 that v has at most two incident 3-faces. Thus it suffices to consider that v is a $(4^+, 4^+, 4^+, 4^+)$-, $(3, 5, 4, 5)$-, $(3, 5^+, 4, 6^+)$-, $(3, 5^+, 5^+, 5^+)$-, $(3, 3, 5^+, 5^+)$-, or $(3, 5^+, 3, 5^+)$-vertex.

- v is $(4^+, 4^+, 4^+, 4^+)$-vertex.
  Then v sends charge at most $\frac{1}{2}$ to each incident face by (R2) and (R3.1). Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{2} = 0$.
- v is a $(3, 5, 4, 5)$-vertex.
  Then $w(v \to f_1) \leq 1$ by (R1.1), $w(v \to f_2) = w(v \to f_4) = \frac{1}{5}$ by (R3.1), and $w(v \to f_3) = \frac{1}{2}$ by (R2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{1}{5} - 1 - \frac{1}{2} > 0$.
- v is a $(3, 5^+, 4, 6^+)$-vertex.
  Then $w(v \to f_1) \leq 1$ by (R1.1), $w(v \to f_2) \leq \frac{1}{3}$ by (R3.1), and $w(v \to f_3) = \frac{1}{2}$ by (R2). Thus $\mu^*(v) \geq \mu(v) - 1 - \frac{1}{2} - \frac{1}{3} > 0$.
- v is a $(3, 5^+, 5^+, 5^+)$-vertex.
  Then $w(v \to f_1) \leq 1$ by (R1.1) and $w(v \to f_i) \leq \frac{1}{3}$ for $2 \leq i \leq 4$ by (R3.1). Thus $\mu^*(v) \geq \mu(v) - 1 - 3 \times \frac{1}{3} = 0$.
- v is a $(3, 3, 5^+, 5^+)$-vertex.
  It follows from Lemma 3.4 (b) that f_3 and f_4 are 6^+-faces. Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$ by (R1.1).
- v is a $(3, 5^+, 3, 5^+)$-vertex.
  Since v is not a flaw vertex, f_2 and f_4 are 6^+-faces. Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$ by (R1.1).

Case 2 v is a flaw vertex, that is v is a $(3, 5, 3, 5^+)$-vertex.

- Each adjacent vertex of v is a 4-vertex.
  Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq 1$ by (R1.1) and $w(v \to f_2) = w(v \to f_4) = 0$ by (R3.1). Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$.
- v is adjacent to exactly one 5^+-vertex.
  Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq 1$, $\min\{w(v \to f_1), w(v \to f_3)\} \leq \frac{4}{5}$ by (R1.1), and $\max\{w(v \to f_2), w(v \to f_4)\} = \frac{1}{10}$ by (R3.1). Thus $\mu^*(v) \geq \mu(v) - 1 - \frac{4}{5} - 2 \times \frac{1}{10} = 0$.
- v is adjacent to at least two 5^+-vertices and incident to a $(4, 5^+, 5^+)$-face.
  Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq 1$, $\min\{w(v \to f_1), w(v \to f_3)\} = \frac{3}{5}$ by (R1.1), and $\max\{w(v \to f_2), w(v \to f_4)\} = \frac{1}{5}$ by (R3.1). Thus $\mu^*(v) \geq \mu(v) - 1 - \frac{3}{5} - 2 \times \frac{1}{5} = 0$.
- v is adjacent to at least two 5^+-vertices but neither f_1 nor f_3 is a $(4, 5^+, 5^+)$-face.
  Then f_1 and f_3 are $(4, 4, 5^+)$-faces. It follows that $w(v \to f_1) = w(v \to f_3)\} = \frac{4}{5}$ by (R1.1) and $\max\{w(v \to f_2), w(v \to f_4)\} \leq \frac{1}{5}$ by (R3.1). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{4}{5} - 2 \times \frac{1}{5} = 0$.

Case 3 v is a 5-vertex but is not a $(3, 5, 3, 5, 5)$-vertex. It follows from Corollary 3.5 that v is incident to at most three 3-faces. Since a 3-face is not adjacent to a 4-face [Lemma 3.4 (b)], we may assume v is a $(3^+, 4^+, 4^+, 4^+, 4^+)$-, $(3, 3, 4^+, 3^+, 4^+)$-, $(3, 6^+, 3, 6^+, 5^+)$-, $(3, 5, 3, 6^+, 6^+)$-, $(3, 6^+, 3, 5, 5)$-, or $(3, 5, 3, 5, 6^+)$-vertex.

- v is a $(3^+, 4^+, 4^+, 4^+, 4^+)$-vertex.
  It follows that $w(v \to f_1) \leq \frac{7}{5}$ by (R1.2), (R2), (R3.2).
  - Suppose v is a $(3^+, 5^+, 5^+, 5^+, 5^+)$-vertex.
    Then each incident 5-face is adjacent to a 5^+-face. Consequently, $\max\{w(v \to f_2), \ldots, w(v \to f_5)\} \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - 4 \times \frac{3}{5} > 0$.
  - Suppose there exists $i \in \{2, \ldots, 5\}$ such that $d(f_i) = 4$.
    Then $w(v \to f_i) \leq \frac{1}{2}$ by (R2) and $\max\{w(v \to f_2), \ldots, w(v \to f_5)\} \leq \frac{7}{10}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{1}{2} - 3 \times \frac{7}{10} = 0$.
- v is a $(3, 3, 4^+, 3^+, 4^+)$-vertex.
  It follows from Lemma 3.4 (f) that f_3 and f_5 are 6^+-faces. Since both f_1 and f_2 are 3-faces, each of common incident vertices is not a flaw vertex. Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{6}{5}$ and $w(v \to f_4) \leq \frac{7}{5}$ by (R1.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - \frac{7}{5} > 0$.
- v is a $(3, 6^+, 3, 6^+, 5^+)$- or a $(3, 5, 3, 6^+, 6^+)$-vertex.
  If v is a $(3, 6^+, 3, 6^+, 5^+)$-vertex, then $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{7}{5}$ by (R1.2) and $w(v \to f_5) \leq \frac{7}{10}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - \frac{7}{10} > 0$. The proof is similar for v is a $(3, 5, 3, 6^+, 6^+)$-vertex.
- v is a $(3, 6^+, 3, 5, 5)$-vertex.
  Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{7}{5}$ by (R1.2) and $\max\{w(v \to f_4), w(v \to f_5)\} \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - 2 \times \frac{3}{5} = 0$.
- v is a $(3, 5, 3, 5, 6^+)$-vertex.
  If f_3 is a $(4^+, 5, 5^+)$-face, then f_2 or f_4 is incident to at least two 5^+-vertex. If f_3 is a $(4, 4, 5)$-vertex, then applying Lemma 3.7 to f_3 and f_4 yields that f_4 is incident to at least two 5^+-vertex. Consequently, $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{7}{5}$ by (R1.2) and $\max\{w(v \to f_2), w(v \to f_4)\} \leq \frac{7}{10}$ and $\min\{w(v \to f_2), w(v \to f_4)\} \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - \frac{7}{10} - \frac{2}{5} > 0$.

Case 4 v is a $(3, 5, 3, 5, 5)$-vertex.

- f_1 and f_3 are $(4, 4, 5)$-faces.
  Applying Corollary 3.9 to f_1, f_2, f_3, we have that f_2 is incident to at least two non-adjacent 5^+-vertices (including v). Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{7}{5}$ by (R1.2) and $w(v \to f_2) \leq \frac{3}{10}$ by (R3.2).
  - Suppose f_4 is incident to exactly one 5^+-vertex.
    Applying Corollary 3.9 to f_3, f_4, f_5, we have that f_5 is incident to at least two non-adjacent 5^+-vertices. Then $w(v \to f_4) \leq \frac{3}{5}$ and $w(v \to f_5) \leq \frac{3}{10}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - 2 \times \frac{3}{10} - \frac{3}{5} = 0$.
  - Suppose f_4 and f_5 are incident to at least two 5^+-vertices. Then $\max\{w(v \to f_4), w(v \to f_5)\} \leq \frac{2}{5}$ by (R3.2) Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - \frac{3}{10} - 2 \times \frac{2}{5} > 0$.

- f_1 or f_3 is a $(5, 5^+, 5^+)$-face.
  By symmetry, we assume f_1 is a $(5, 5^+, 5^+)$-face. It follows that f_2 and f_5 are incident to at least two 5^+-vertices. Then $w(v \to f_1) \leq 1$, $w(v \to f_3) \leq \frac{7}{5}$ by (R1.2) and $w(v \to f_4) \leq \frac{3}{5}$, $\max\{w(v \to f_2), w(v \to f_5)\} \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - 1 - \frac{3}{5} - 2 \times \frac{2}{5} > 0$.
- f_1 and f_3 are $(4, 5, 5^+)$-faces with incident 5^+-vertex x and y, respectively, where $x \neq v \neq y$.
  Then $\max\{w(v \to f_1), w(v \to f_3)\} \leq \frac{6}{5}$ by (R1.2).

  - Suppose x and y are not incident to f_2.
    It follows that f_4 and f_5 are incident to at least two 5^+-vertices. Consequently, $w(v \to f_2) \leq \frac{7}{10}$, $\max\{w(v \to f_4), w(v \to f_5)\} \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - \frac{7}{10} - 2 \times \frac{2}{5} > 0$.
  - Suppose x and y are incident to f_2.
    Then $w(v \to f_2) = \frac{3}{10}$, $\max\{w(v \to f_4), w(v \to f_5)\} \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - \frac{3}{10} - 2 \times \frac{3}{5} > 0$.
  - Suppose x is incident to f_2 but y is not.
    Then f_4 is incident to at least two 5^+-vertices. Consequently, $\max\{w(v \to f_2), w(v \to f_4)\} \leq \frac{2}{5}$, $w(v \to f_3) \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - 2 \times \frac{2}{5} - \frac{3}{5} > 0$.

- f_1 is a $(4, 4, 5)$-face but f_3 is a $(4, 5, 5^+)$-face with its two incident 5^+-vertices are also incident to f_2.
  It follows that $w(v \to f_1) \leq \frac{7}{5}$, $w(v \to f_3) \leq \frac{6}{5}$ by (R1.2) and $w(v \to f_2) \leq \frac{2}{5}$ by (R1.2). Applying Corollary 3.9 to f_1, f_5, f_4, we have that f_4 or f_5 is incident to at least two 5^+-vertices.

  - Suppose f_4 is incident to at least two non-adjacent 5^+-vertices.
    Then $w(v \to f_4) \leq \frac{3}{10}$, $w(v \to f_5) \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{6}{5} - \frac{2}{5} - \frac{3}{10} - \frac{3}{5} > 0$. The proof is similar for f_5 is incident to at least two non-adjacent 5^+-vertices.
  - Suppose a 5^+-vertex u incident to f_4 or f_5 is adjacent to v.
    From assumption on f_1 and f_3, we have that u is not incident to f_1 and is not incident to f_3. It follows that u is incident to f_4 and f_5. Then $\max\{w(v \to f_4, w(v \to f_5)\} \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{6}{5} - 3 \times \frac{2}{5} > 0$.

- f_1 is a $(4, 4, 5)$-face but f_3 is a $(4, 5, 5^+)$-face with its two 5^+-vertices are also incident to f_4.
  It follows that $w(v \to f_1) \leq \frac{7}{5}$, $w(v \to f_3) \leq \frac{6}{5}$ by (R1.2). Applying Corollary 3.9 to f_5, f_1, f_2, we have that f_2 or f_5 is incident to at least two 5^+-vertices.

  - Suppose f_2 is incident to a 5^+-vertex u where $u \neq v$.
    We have that u is not incident to f_1 and f_3 by assumption on f_1 and f_3. This implies u is not adjacent to v. Then $w(v \to f_2) \leq \frac{3}{10}$, $w(v \to f_4) \leq \frac{2}{5}$, and $w(v \to f_5) \leq \frac{3}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{6}{5} - \frac{3}{10} - \frac{2}{5} - \frac{3}{5} > 0$.
  - Suppose f_5 is incident to a 5^+-vertex u where $u \neq v$.
    If u is not incident to f_4, then u is not adjacent to v. It follows that $w(v \to$

$f_2) \leq \frac{7}{10}$, $w(v \to f_4) \leq \frac{2}{5}$, and $w(v \to f_5) \leq \frac{3}{10}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{6}{5} - \frac{7}{10} - \frac{2}{5} - \frac{3}{10} = 0$.
If u is also incident to f_4, then f_4 is a $(4^+, 4^+, 5, 5^+, 5^+)$-vertex. It follows that $w(v \to f_2) \leq \frac{7}{10}$, $w(v \to f_4) \leq \frac{3}{10}$ and $w(v \to f_5) \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - \frac{7}{5} - \frac{6}{5} - \frac{7}{10} - \frac{3}{10} - \frac{2}{5} = 0$.

Case 5 v is a 6-vertex.

From Lemma 3.4 (a) that a 3-face is not adjacent to a 4-face, it suffices to consider v is a $(3, 3, k_3, k_4, k_5, k_6)$-, $(3^+, 4^+, 3^+, 4^+, 4^+, 4^+)$-, $(3^+, 4^+, 4^+, 3^+, 4^+, 4^+)$-, $(3, 5, 3, 5, 3, 5)$-, or a $(3, 5^+, 3, 5^+, 3, 6^+)$-vertex.

- v is a $(3, 3, k_3, k_4, k_5, k_6)$-vertex.
  It follows from Lemma 3.4 (b) that f_3 and f_6 are 6^+-faces. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{7}{5} > 0$ by (R1.2), (R2), and (R3.3).
- v is a $(3^+, 4^+, 3^+, 4^+, 4^+, 4^+)$-vertex (or $(3^+, 4^+, 4^+, 3^+, 4^+, 4^+)$-vertex, respectively).
  Then v sends charge at most $\frac{7}{5}$ to f_1 and f_3 (or f_4, respectively) by (R1.2), (R2), (R3.3), and v sends charge at most $\frac{4}{5}$ to each of the remaining incident faces by (R2), (R3.3). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{7}{5} - 4 \times \frac{4}{5} = 0$.
- v is a $(3, 5, 3, 5, 3, 5)$-vertex.
  - Suppose at least two incident 5-faces of v, say f_2 and f_4, are incident to at least two 5^+-vertices.
    Then $\max\{w(v \to f_1), w(v \to f_3), w(v \to f_5)\} \leq \frac{7}{5}$ by (R1.2), $\max\{w(v \to f_2), w(v \to f_4)\} = \frac{2}{5}$ and $w(v \to f_6) \leq \frac{4}{5}$ by (R3.3). Thus $\mu^*(v) \geq \mu(v) - 3 \times \frac{7}{5} - 2 \times \frac{2}{5} - \frac{4}{5} > 0$.
  - Suppose two incident 5-faces of v, say f_2 and f_4, are incident to exactly one 5^+-vertex.
    Applying Lemma 3.10 to f_1, f_2, f_3, f_4, we obtain that f_1 is incident to two 5^+ vertices, say v_1 and v, that are also incident to f_6. Applying Lemma 3.10 to f_2, f_3, f_4, f_5, we obtain that f_5 is incident to two 5^+ vertices, say v_5 and v, that are also incident to f_6. Then $\max\{w(v \to f_1), w(v \to f_5)\} \leq \frac{6}{5}$ and $w(v \to f_3) \leq \frac{7}{5}$ by (R1.2). Moreover, $\max\{w(v \to f_2), w(v \to f_4)\} \leq \frac{4}{5}$ and $w(v \to f_6) \leq \frac{2}{5}$ by (R3.2). Thus $\mu^*(v) \geq \mu(v) - 2\times\frac{6}{5} - \frac{7}{5} - 2\times\frac{4}{5} - \frac{2}{5} > 0$.

Case 6 v is a d-vertex with $d \geq 7$.

- v is a $(3, 3, k_3, \ldots, k_d)$-vertex.
  It follows from Lemma 3.4 that f_3 and f_d are two 6^+-faces. Thus $\mu^*(v) \geq \mu(v) - (d-2) \times \frac{7}{5} = 2d - 6 - (d-2) \times \frac{7}{5} > 0$ by (R1.2), (R2), and (R3.3).
- v has no adjacent incident 3-faces.
  It follows that v is incident to at most $\frac{d}{2}$ 3-faces. Since v sends charge at most $\frac{7}{5}$ to each of its incident 3-faces by (R1.2) and v sends charge at most $\frac{4}{5}$ to each of the remaining incident faces by (R2) and (R3.3), we have $\mu^*(v) \geq \mu(v) - \frac{d}{2} \times \frac{7}{5} - \frac{d}{2} \times \frac{4}{5} = (2d - 6) - d \times \frac{11}{10} > 0$.

Let f be a face in G. Let $V(f) \cap V(D) \neq \emptyset$. If $d(f) = 3$, then f gets $\frac{12}{5}$ from D when f shares an edge with D, 2 from D when f shares exactly one vertex with D. Note that each vertex of f in $int(C_0)$ sends at least $\frac{1}{2}$ to f. It follows that $\mu^*(f) \geq -3 + \min\{\frac{12}{5} + \frac{1}{2}, 2 + \frac{1}{2} \times 2\} = 0$. If $d(f) \in \{4, 5\}$, then it gains 2 from D. Thus $\mu^*(f) \geq d(f) - 6 + 2 \geq 0$. If $d(f) = 6$, then $\mu^*(f) = \mu(f) = 0$. If $d(f) \geq 7$, then $\mu^*(f) \geq (k-6) - k \times \frac{k-6}{k} = 0$. Thus we may assume that $V(f) \cap V(D) = \emptyset$ for the remaining of the paper. Let f be a 5^--face with vertices $v_1, v_2, \ldots, v_{d(v)}$ in a cyclic order.

Case 7 f is a 3-face.

- f is a (4, 4, 4)-face or each vertex of f is not a flaw vertex.
  Then $\mu^*(f) = \mu(f) + 3 \times 1 = 0$ by (R1.1).
- f is a $(4, 4, 5^+)$-face with exactly one incident flaw vertex, say v_1.
  Then $w(v_1 \to f) = \frac{4}{5}$, $w(v_2 \to f) = 1$ by (R1.1) and $w(v_3 \to f) = \frac{6}{5}$ (R1.2). Thus $\mu^*(f) = \mu(f) + \frac{4}{5} + 1 + \frac{6}{5} = 0$.
- f is a $(4, 4, 5^+)$-face with v_1 and v_2 are flaw vertices.
  Then $w(v_1 \to f) = w(v_2 \to f) = \frac{4}{5}$ by (R1.1) and $w(v_3 \to f) = \frac{7}{5}$ (R1.2). Thus $\mu^*(f) = \mu(f) + 2 \times \frac{4}{5} + \frac{7}{5} = 0$.
- f is a $(4, 5^+, 5^+)$-face and v_1 is a flaw vertex.
  Then $w(v_1 \to f) = \frac{3}{5}$ by (R1.1) and $w(v_2 \to f) = w(v_3 \to f) = \frac{6}{5}$ by (R1.2) Thus $\mu^*(f) = \mu(f) + \frac{3}{5} + 2 \times \frac{6}{5} = 0$.

Case 8 f is a 4-face.
We obtain $\mu^*(f) \geq \mu(f) + 4 \times \frac{1}{2} = 0$ by (R2).

Case 9 f is a 5-face.

- f is incident to at least three 5^+-vertices.
  It follows that each of its incident 4-vertex is adjacent to at least one 5^+-vertex. Then each of these 4-vertices sends charge at least $\frac{1}{10}$ to f by (R3.1) and each 5^+-vertex sends charge at least $\frac{3}{10}$ to f by (R3.2) and (R3.3). Thus $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{10} + 3 \times \frac{3}{10} > 0$.
- f is a $(4, 5^+, 4, 5^+, 4)$-face.
  Since v_1 and v_5 are adjacent to at least one 5^+-vertex and v_3 is adjacent to at least two 5^+-vertices, we have $\min\{w(v_1 \to f), w(v_5 \to f)\} \geq \frac{1}{10}$ and $w(v_3 \to f) \geq \frac{1}{5}$ by (R3.1). We have $\min\{w(v_2 \to f), w(v_4 \to f)\} \geq \frac{1}{10}$ by (R3.2) and (R3.3). Thus $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{10} + \frac{1}{5} + 2 \times \frac{3}{10} = 0$.
- f is a $(4, 4, 4, 5^+, 5^+)$-face
  Since v_1 and v_3 are adjacent to at least one 5^+-vertex, we have $\min\{w(v_1 \to f), w(v_3 \to f)\} \geq \frac{1}{10}$ by (R3.1). We have $\min\{w(v_2 \to f), w(v_4 \to f)\} \geq \frac{2}{5}$ by (R3.2) and (R3.3). Thus $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{10} + 2 \times \frac{2}{5} = 0$.
- f is a $(4, 4, 4, 4, 6^+)$-face.
  Since v_1 and v_4 are adjacent to at least one 5^+-vertex, we have $\min\{w(v_1 \to f), w(v_4 \to f)\} \geq \frac{1}{10}$ by (R3.1). We have $w(v_5 \to f) = \frac{4}{5}$ by (R3.3). Thus $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{10} + \frac{4}{5} = 0$.
- f is a (4, 4, 4, 4, 5)-face with at least one adjacent 5^+-face f_i.
  Let a 4-vertex v_i be in f_i. It follows that a 4-vertex v_i is not a flaw vertex. Then $w(v_i \to f) = \frac{1}{3}$ by (R3.1), $\min\{w(v_1 \to f), w(v_4 \to f)\} \geq \frac{1}{10}$ by (R3.1), and $w(v_5 \to f) = \frac{3}{5}$ by (R3.2), Thus $\mu^*(f) \geq \mu(f) + \frac{1}{3} + \frac{1}{10} + \frac{3}{5} > 0$.

- f is a (4, 4, 4, 4, 5)-face with five adjacent 4^--faces and a 4-vertex v_i which is not a flaw vertex.
  Then $w(v_i \to f) = \frac{1}{3}$ by (R3.1) and $w(v_5 \to f) = \frac{7}{10}$ by (R3.2). Thus $\mu^*(f) \geq \mu(f) + \frac{1}{3} + \frac{7}{10} > 0$.
- f is a (4, 4, 4, 4, 5)-face with five adjacent 4^--faces and four flaw vertices.
  It follows that each adjacent face of f is a 3-face. Let f_4 be a 3-face incident to v_4 and v_5 and let f_5 be a 3-face incident to v_1 and v_5. Applying Corollary 3.9 to f, f_4, and f_5, we have that f_4 or f_5 is a $(4, 5, 5^+)$-face. By symmetry, let f_5 be a $(4, 5, 5^+)$-face. Consequently, v_1 is adjacent to at least two 5^+-vertices and v_4 is adjacent to at least one 5^+-vertex. It follows that $w(v_1 \to f) = \frac{1}{5}$ and $w(v_4 \to f) \geq \frac{1}{10}$ by (R3.1). We have $w(v_5 \to f) = \frac{7}{10}$ by (R3.2). Thus $\mu^*(f) \geq \mu(f) + \frac{1}{5} + \frac{1}{10} + \frac{7}{10} = 0$.
- f is a (4, 4, 4, 4, 4)-face.
  Applying Corollary 3.7 to f and its adjacent 3-face, we have that each adjacent 3-face of f is a $(4, 4, 5^+)$-face. This implies that each incident flaw vertex of f is adjacent to at least two 5^+-vertex. If v_i is a flaw vertex, then $w(v_i \to f) \geq \frac{1}{5}$, otherwise $w(v_i \to f) \geq \frac{1}{3}$ by (R3.1). Thus $\mu^*(f) \geq \mu(f) + 5 \times \frac{1}{5} = 0$.

Case 10 Consider the outerface D.

Let f_3', f' be the number of 3-faces sharing exactly one edge with D, 3-faces sharing exactly one vertex with D or 4-or 5-faces sharing vertices with D, respectively. Let $E(C_0, V(G) - C_0)$ be the set of edges between C_0 and $V(G) - C_0$ and let $e(C_0, V(G) - C_0)$ be its size. Then by (R4),

$$\mu^*(D) = 3 + 6 + \sum_{v \in C_0} (2d(v) - 6) - \frac{12}{5} f_3' - 2f' \tag{1}$$

$$= 9 + 2 \sum_{v \in C_0} (d(v) - 2) - 2 \times 3 - \frac{12}{5} f_3' - 2f' \tag{2}$$

$$= 3 - \frac{2}{5} f_3' + 2(e(C_0, V(G) - C_0) - f_3' - f') \tag{3}$$

So we may consider that each edge $e \in E(C_0, V(G) - C_0)$ gives a charge of 2 to D. Since each 5^--face is a cycle, it contains two edges in $E(C_0, V(G) - C_0)$. It follows that $e(C_0, V(G) - C_0) - f_3' - f' \geq 0$. Note that $f_3' \leq 3$. Thus $\mu^*(D) > 0$.

This completes the proof.

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