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On the Stabilization of a Memory-Type Porous Thermoelastic System

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Abstract
In this work, we consider a one-dimensional porous thermoelastic system with memory effects. We prove a general decay result, for which exponential and polynomial decay results are special cases, depending only on the kernel of the memory effects. Our result is established irrespective of the wave speeds of the system. The result obtained is new and improves previous results in the literature.

Keywords Stability · Asymptotic behavior of solutions · Porous system · Thermoelasticity · Relaxation function

Mathematics Subject Classification 35B35 · 35B40 · 93D20

1 Introduction

Elastic materials with voids have stimulated a lot on interest in recent years and many results have been published, most notably in the area of petroleum industry, material science, soil mechanics, foundation engineering, powder technology, and biology. It is widely known that an extension of the classical elasticity theory to porous media was established by Goodman and Cowin [1]. They introduced the concept of a continuum theory of granular materials with interstitial voids. In addition, Nunziato and Cowin [2] introduced the concept that the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. Later, Ieşan [3,5], and Ieşan and Quintanilla [6] added the temperature as well as the microtemperature elements to the theory. For

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extensive discussion on these materials, we refer interested reader to [7]–[10] and the references therein.

In this work, we are concerned with the asymptotic behavior of the solution of porous thermoelastic system with memory effects

$$
\begin{aligned}
&\rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x = 0,\\
&J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi - m\theta + \int_0^t g(t-s)\phi_{xx}(x,s)\mathrm{d}s = 0,\\
&c\theta_t - \kappa\theta_{xx} + \beta u_{tx} + m\phi_t = 0,\\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\\
&\phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \theta(x,0) = \theta_0(x),\\
&u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \theta(0,t) = \theta(1,t) = 0,
\end{aligned}
\tag{1.1}
$$

where $(x,t) \in (0,1) \times [0,+\infty)$, u is the longitudinal displacement, ϕ is the volume fraction of the solid elastic material, θ is the temperature difference, $u_0, u_1, \phi_0, \phi_1, \theta_0$ are given initial data, and $\rho, \mu, J, \delta, \xi, m, c, \kappa, \beta$ are constitutive constants which are positive. Furthermore, the constants μ and ξ satisfy $\mu\xi > b^2$, where $b \neq 0$ is a real number. The integral represents the memory effect and g is the relaxation function satisfying the following:

(H1) $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 decreasing function satisfying

$$
g(0) > 0, \qquad \delta - \int_0^\infty g(s)\mathrm{d}s = l > 0.
$$

(H2) There exists a nonincreasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$
g'(t) \leq -\zeta(t)g(t), \qquad t \geq 0.
$$

The basic evolution together with the constitutive equations, for one-dimensional theories of porous materials, with memory effect is

$$
\rho u_{tt} = T_x, \quad J\phi_{tt} = H_x + G, \quad \rho T_0\eta_t = q_x, \tag{1.2}
$$

and

$$
\begin{aligned}
&T = \mu u_x + b\phi - \beta\theta, \quad H = \delta\phi_x - \int_0^t g(t-s)\phi_x \mathrm{d}s,\\
&\eta = c\theta + \beta u_x + m\phi, \quad q = k\theta_x, \quad G = -bu_x - \xi\phi + m\theta,
\end{aligned}
\tag{1.3}
$$

respectively. Here, η is the entropy, T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, q is the heat flux vector, and T_0 is the absolute temperature in the reference configuration. By substituting (1.3) into (1.2), we obtain the first three equations in (1.1).

The asymptotic behavior of system (1.1) has been considered in the literature with various types of dissipative mechanisms. We first mention the case where the memory term in (1.1) is replaced with a porous dissipation. That is

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\phi_x - \beta\theta_x, && x \in (0, L),\ t > 0,\\ J\phi_{tt} &= \delta\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, && x \in (0, L),\ t > 0,\\ c\theta_t &= \kappa\theta_{xx} - \beta u_{xt} - m\phi_t, && x \in (0, L),\ t > 0. \end{aligned} \tag{1.4}$$

Casas and Quintanilla [11] considered (1.4) and used the semigroup theory together with the method developed by Liu and Zheng [12] to establish the exponential decay of the solutions. Whereas in absence of porous dissipation ($\tau = 0$), the same authors showed in [13] that the heat effect alone is not strong enough to exponentially stabilize the system. However, the heat effect together with microtemperature produced an exponential decay result. Similarly, when $\tau = 0$ and γu_{xxt} is added to the first equation in (1.4), Pamplona et al. [14] proved that the system lacks exponential stability, however, by taking some regular initial data, a polynomial stability is obtained. Also, for $\tau = 0$, Soufyane et al. [15] considered (1.4) with some boundary controls and obtained a general decay result, from which the usual exponential and polynomial decay rates are just special cases.

In the absence of the heat effect, (1.4) becomes

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\phi_x, && x \in (0, L),\ t > 0,\\ J\phi_{tt} &= \delta\phi_{xx} - bu_x - \xi\phi - \tau\phi_t, && x \in (0, L),\ t > 0. \end{aligned} \tag{1.5}$$

Quintanilla [16] considered (1.5) and proved that the porous dissipation is not strong enough to bring about an exponential decay. However, Apalara [17] considered the same system and proved that the system is exponentially stable provided the wave speeds of the two systems are equal. Equivalently, Apalara [18] replaced the porous dissipation in (1.5) with a memory term of the form $\int_0^t g(t-s)\phi_{xx}(x,s)\mathrm{d}s$ and obtained a general decay result depending on the kernel of the memory term and the wave speeds of the system. We refer reader to [19]–[23] and the references therein for more results.

Obviously, when $\mu = b = \xi$ and $m = \beta$ then (1.1) is equivalent to the following Timoshenko system

$$\begin{aligned} &\rho u_{tt} - \mu(u + \phi_x)_x + \beta\theta_x = 0,\\ &J\phi_{tt} - \delta\phi_{xx} + \mu\,(u_x + \phi) - \beta\theta + \int_0^t g(t-s)\phi_{xx}(x,s)\mathrm{d}s = 0,\\ &\rho_3\theta_t - \kappa\theta_{xx} + \beta(u_x + \phi)_t = 0. \end{aligned} \tag{1.6}$$

In the absence of memory term ($g = 0$), Almeida Júnior et al [24] considered (1.6) and proved that the system is exponentially stable if and only if

$$\frac{\mu}{\rho} = \frac{\delta}{J} \tag{1.7}$$

holds. Prior to the results in [24], Messaoudi and Fareh [25,26] considered (1.6) with initial data and fully Dirichlet boundary conditions and established some general decay results depending on (1.7) and the kernel g of the memory term. In other words, the viscoelastic dissipation given by the memory term is not strong enough to neutralize the condition of equal wave speeds required to obtain an exponential decay result as established in [24]. However, Apalara [27] recently proved that the memory term together with the heat effect is strong enough to uniformly stabilize system (1.6) without imposing condition (1.7).

The main question which can be asked here is the following: Is the memory term together with the heat effect strong enough to exponentially stabilize system (1.1) irrespective of the wave speeds as in [27] for Timoshenko system? The aim of the present work is to give a positive answer to the question by considering (1.1) and establish a general stability result without imposing (1.7). Our result depends only on the kernel g of the memory term. Meanwhile, from $(1.1)_1$ and the boundary conditions, it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^1 u(x,t)\mathrm{d}x = 0. \tag{1.8}$$

So, by solving (1.8) and using the initial data of u, we obtain

$$\int_0^1 u(x,t)\mathrm{d}x = t\int_0^1 u_1(x)\mathrm{d}x + \int_0^1 u_0(x)\mathrm{d}x.$$

Consequently, if we let

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\mathrm{d}x - \int_0^1 u_0(x)\mathrm{d}x,$$

we obtain

$$\int_0^1 \overline{u}(x,t)\mathrm{d}x = 0, \quad \forall t \geq 0.$$

Consequently, the use of Poincaré's inequality for $\overline{u}$ is justified. Furthermore, simple substitution shows that $(\overline{u}, \phi, \theta)$ satisfies system (1.1) with initial data for $\overline{u}$ given as

$$\overline{u}_0(x) = u_0(x) - \int_0^1 u_0(x)\mathrm{d}x \text{ and } \overline{u}_1(x) = u_1(x) - \int_0^1 u_1(x)\mathrm{d}x.$$

Henceforth, we work with $\overline{u}$ instead of u but write u for simplicity of notation.

For the well-posedness result, we consider the following space

$$H_*^1(0,1)=H^1(0,1)\cap L_*^2(0,1),\ \text{ where } L_*^2(0,1)=\left\{u\in L^2(0,1)\mid \int_0^1 u(x)\mathrm{d}x=0\right\}$$

and state without proof the following result.

Proposition 1.1 *Let* $(u_0,\phi_0,\theta_0)\in H_*^1(0,1)\times\left(H_0^1(0,1)\right)^2$ *and* $(u_1,\phi_1)\in\left(L^2(0,1)\right)^2$ *be given. Assume that (H1) and (H2) are satisfied, then problem (1.1) has a unique global solution* (u,ϕ,θ) *which satisfies*

$$\begin{aligned}&u\in C(\mathbb{R}_+,H_*^1(0,1))\cap C^1(\mathbb{R}_+,L^2(0,1)),\ \phi\in C(\mathbb{R}_+,H_0^1(0,1))\\&\cap C^1(\mathbb{R}_+,L^2(0,1)),\theta\in C(\mathbb{R}_+,H_0^1(0,1)).\end{aligned}$$

Moreover, if $(u_0,\phi_0,\theta_0)\in H^2(0,1)\cap H_*^1(0,1)\times\left(H^2(0,1)\cap H_0^1(0,1)\right)^2$ *and* $(u_1,\phi_1)\in H_*^1(0,1)\times H_0^1(0,1)$ *then the solution satisfies*

$$\begin{aligned}&u\in C(\mathbb{R}_+,H^2(0,1)\cap H_*^1(0,1))\cap C^1(\mathbb{R}_+,H_*^1(0,1))\cap C^2(\mathbb{R}_+,L^2(0,1)),\\&\phi\in C(\mathbb{R}_+,H^2(0,1)\cap H_0^1(0,1))\cap C^1(\mathbb{R}_+,H_0^1(0,1))\cap C^2(\mathbb{R}_+,L^2(0,1)),\\&\theta\in C(\mathbb{R}_+,H^2(0,1)\cap H_0^1(0,1))\cap C^1(\mathbb{R}_+,H_0^1(0,1)).\end{aligned}$$

Remark 1.2 The proof can be established using the Galerkin method.

The rest of our paper is organized as follows. In Sect. 2, we state and prove some technical lemmas. In Sect. 3, we state and prove our stability result. We use c_1 throughout this paper to denote a generic positive constant.

2 Technical Lemmas

In this section, we state and prove some technical lemmas needed in the proof of our stability result.

Lemma 2.1 *Under assumptions (H1) and (H2), the energy functional* E*, defined by*

$$\begin{aligned}E(t)=&\ \frac{1}{2}\int_0^1\left[\rho u_t^2+\mu u_x^2+J\phi_t^2+c\theta^2+\left(\delta-\int_0^t g(s)\mathrm{d}s\right)\phi_x^2+\xi\phi^2+2bu_x\phi\right]\mathrm{d}x\\&+\frac{1}{2}g\circ\phi_x,\end{aligned}\tag{2.1}$$

satisfies

$$E'(t)=-\kappa\int_0^1\theta_x^2\mathrm{d}x+\frac{1}{2}g'\circ\phi_x-\frac{1}{2}g(t)\int_0^1\phi_x^2\mathrm{d}x\le 0,\tag{2.2}$$

where

$$(g \circ \phi_x)(t) = \int_0^1 \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))^2 \mathrm{d}s\mathrm{d}x.$$

Proof Multiplying the first three equations of (1.1) by u_t, ϕ_t, and θ, respectively, integrating by parts over (0, 1), and using the boundary conditions, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + c\theta^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right]\mathrm{d}x \\
&\quad - \int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)\mathrm{d}s\mathrm{d}x \\
&= -\kappa\int_0^1 \theta_x^2\mathrm{d}x. \qquad (2.3)
\end{aligned}$$

The last term in the left hand side of (2.3) is estimated as follows.

$$\begin{aligned}
-\int_0^1 \phi_{xt}\int_0^t g(t-s)\phi_x(s)\mathrm{d}s\mathrm{d}x &= \int_0^1 \phi_{xt}\int_0^t g(t-s)(\phi_x(t) - \phi_x(s))\mathrm{d}s\mathrm{d}x \\
&\quad - \int_0^t g(s)\mathrm{d}s\int_0^1 \phi_x\phi_{tx}\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}g\circ\phi_x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(s)\mathrm{d}s\int_0^1 \phi_x^2\mathrm{d}x - \frac{1}{2}g'\circ\phi_x + \frac{1}{2}g(t)\int_0^1 \phi_x^2\mathrm{d}x. \qquad (2.4)
\end{aligned}$$

The substitution of (2.4) into (2.3), bearing in mind (2.1), yields (2.2). □

Remark 2.2 The energy functional $E(t)$ defined by (2.1) is nonnegative. In fact, it can easily be verified that

$$\begin{aligned}
\mu u_x^2 + 2bu_x\phi + \xi\phi^2 = \frac{1}{2}\Bigg[&\mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \xi\left(\phi + \frac{b}{\xi}u_x\right)^2 \\
&+ \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\Bigg].
\end{aligned}$$

So, using the fact that $\mu\xi > b^2$, we obtain

$$\mu u_x^2 + 2bu_x\phi + \xi\phi^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\right] > 0.$$

Consequently,

$$E(t) > \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu_1 u_x^2 + J\phi_t^2 + c\theta^2 + \left(\delta - \int_0^t g(s)\mathrm{d}s\right)\phi_x^2 + \xi_1\phi^2\right]\mathrm{d}x + \frac{1}{2}g\circ\phi_x, \qquad (2.5)$$

where $2\mu_1 = \mu - \frac{b^2}{\xi} > 0$ and $2\xi_1 = \xi - \frac{b^2}{\mu} > 0$.

Lemma 2.3 *The functional*

$$F_1(t) := -c \int_0^1 \theta \left(\int_0^x u_t(y,t)\mathrm{d}y \right) \mathrm{d}x$$

satisfies, along the solution of (1.1)*, for any* $\varepsilon_1 > 0$*, the estimate*

$$F_1'(t) \leq -\frac{\beta}{2} \int_0^1 u_t^2 \mathrm{d}x + \varepsilon_1 \int_0^1 (u_x^2 + \phi^2)\mathrm{d}x + c_1 \int_0^1 \phi_t^2 \mathrm{d}x + c_1 \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \theta_x^2 \mathrm{d}x. \tag{2.6}$$

Proof Direct computations using (1.1) yield

$$\begin{aligned} F_1'(t) &= -\beta \int_0^1 u_t^2 \mathrm{d}x + \kappa \int_0^1 \theta_x u_t \mathrm{d}x + \frac{c\beta}{\rho} \int_0^1 \theta^2 \mathrm{d}x - \frac{c\mu}{\rho} \int_0^1 \theta u_x \mathrm{d}x \\ &\quad - \frac{cb}{\rho} \int_0^1 \theta\phi \mathrm{d}x + m \int_0^1 \phi_t \left(\int_0^x u_t(y,t)\mathrm{d}y \right) \mathrm{d}x. \end{aligned} \tag{2.7}$$

By Young's inequality, for any $\varepsilon_1 > 0$, we obtain

$$\kappa \int_0^1 \theta_x u_t \mathrm{d}x \leq \frac{\beta}{4} \int_0^1 u_t^2 \mathrm{d}x + \frac{\kappa^2}{\beta} \int_0^1 \theta_x^2 \mathrm{d}x \tag{2.8}$$

$$-\frac{c\mu}{\rho} \int_0^1 \theta u_x \mathrm{d}x \leq \varepsilon_1 \int_0^1 u_x^2 \mathrm{d}x + \frac{c^2\mu^2}{4\rho^2\varepsilon_1} \int_0^1 \theta^2 \mathrm{d}x \tag{2.9}$$

$$-\frac{cb}{\rho} \int_0^1 \theta\phi \mathrm{d}x \leq \varepsilon_1 \int_0^1 \phi^2 \mathrm{d}x + \frac{c^2 b^2}{4\rho^2\varepsilon_1} \int_0^1 \theta^2 \mathrm{d}x \tag{2.10}$$

$$m \int_0^1 \phi_t \left(\int_0^x u_t(y,t)\mathrm{d}y \right) \mathrm{d}x \leq \frac{\beta}{4} \int_0^1 \left(\int_0^x u_t(y,t)\mathrm{d}y \right)^2 \mathrm{d}x + \frac{m^2}{\beta} \int_0^1 \phi_t^2 \mathrm{d}x. \tag{2.11}$$

The combination of (2.7)–(2.11) yields

$$\begin{aligned} F_1'(t) &\leq -\frac{3\beta}{4} \int_0^1 u_t^2 \mathrm{d}x + \frac{\kappa^2}{\beta} \int_0^1 \theta_x^2 \mathrm{d}x + \varepsilon_1 \int_0^1 u_x^2 \mathrm{d}x + c_1 \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \theta^2 \mathrm{d}x \\ &\quad + \varepsilon_1 \int_0^1 \phi^2 \mathrm{d}x + \frac{m^2}{\beta} \int_0^1 \phi_t^2 \mathrm{d}x + \frac{\beta}{4} \int_0^1 \left(\int_0^x u_t(y,t)\mathrm{d}y \right)^2 \mathrm{d}x. \end{aligned} \tag{2.12}$$

By Cauchy-Schwarz inequality, it is clear that

$$\left(\int_0^x u_t(y,t)\mathrm{d}y\right)^2 \le \left(\int_0^1 u_t \mathrm{d}x\right)^2 \le \int_0^1 u_t^2 \mathrm{d}x. \tag{2.13}$$

By substituting (2.13) into (2.12), and using Poincaré's inequality, we obtain (2.6). □

Lemma 2.4 *The functional*

$$F_2(t) := \rho \int_0^1 u u_t \mathrm{d}x$$

satisfies, along the solution of (1.1),

$$\begin{aligned} F_2'(t) \ &\le -\frac{\mu}{2}\int_0^1 u_x^2 \mathrm{d}x + \rho \int_0^1 u_t^2 \mathrm{d}x \\ &\quad + c_1 \int_0^1 \phi^2 \mathrm{d}x + c_1 \int_0^1 \theta_x^2 \mathrm{d}x. \end{aligned} \tag{2.14}$$

Proof By taking a derivative of F_2, using (1.1), and then integrating by parts, we obtain

$$F_2'(t) = -\mu \int_0^1 u_x^2 \mathrm{d}x + \rho \int_0^1 u_t^2 \mathrm{d}x - b \int_0^1 u_x \phi \mathrm{d}x + \beta \int_0^1 u_x \theta \mathrm{d}x.$$

Using Young's and Poincaré's inequalities as in the proof of Lemma 2.3, we obtain (2.14). □

Lemma 2.5 *The functional*

$$F_3(t) := -J \int_0^1 \phi_t \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x$$

satisfies, for some fixed $t_0 > 0$ *and for any* $\varepsilon_2 > 0$, *the estimate*

$$\begin{aligned} F_3'(t) \ &\le -\frac{J g_0}{2}\int_0^1 \phi_t^2 \mathrm{d}x + \varepsilon_2 \int_0^1 (u_x^2 + \phi^2 + \phi_x^2)\mathrm{d}x + c_1 \int_0^1 \theta_x^2 \mathrm{d}x \\ &\quad + c_1\left(1 + \frac{1}{\varepsilon_2}\right) g \circ \phi_x - c_1 g' \circ \phi_x, \end{aligned} \tag{2.15}$$

where $g_0 = \displaystyle\int_0^{t_0} g(s)\mathrm{d}s$.

Proof Differentiating F_3, taking into account (1.1), and using integrating by parts together with the boundary conditions, we obtain

$$
\begin{aligned}
F_3'(t) = & -J \int_0^t g(s)\mathrm{d}s \int_0^1 \phi_t^2 \mathrm{d}x - J \int_0^1 \phi_t \int_0^t g'(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \\
& + \delta \int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\mathrm{d}x \\
& - m \int_0^1 \theta \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \\
& + b \int_0^1 u_x \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \\
& + \xi \int_0^1 \phi \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \\
& - \int_0^1 \int_0^t g(t-s)\phi_x(s)\mathrm{d}s \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\mathrm{d}x. \qquad (2.16)
\end{aligned}
$$

Now, we estimate the terms in the right-hand side of (2.16), using Young's, Cauchy-Schwarz, and Poincaré's inequalities, and the fact that

$$
\int_0^t g(s)\mathrm{d}s \le \delta - l > 0. \qquad (2.17)
$$

So, for any $\delta_1, \varepsilon_2 > 0$, we obtain

$$
\begin{aligned}
I_1 &= -J \int_0^1 \phi_t \int_0^t g'(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \\
&\le J\delta_1 \int_0^1 \phi_t^2 \mathrm{d}x + \frac{J}{4\delta_1} \int_0^1 \left(\int_0^t g'(t-s)(\phi(t)-\phi(s))\mathrm{d}s \right)^2 \mathrm{d}x \\
&\le J\delta_1 \int_0^1 \phi_t^2 \mathrm{d}x + \frac{J}{4\delta_1} \int_0^1 \left(\int_0^t -g'(s)\mathrm{d}s \right) \int_0^t -g'(t-s)(\phi(t)-\phi(s))^2 \mathrm{d}s\mathrm{d}x \\
&\le J\delta_1 \int_0^1 \phi_t^2 \mathrm{d}x - \frac{c_1}{\delta_1} g' \circ \phi_x, \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \delta \int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\mathrm{d}x \\
&\le \frac{\varepsilon_2}{2} \int_0^1 \phi_x^2 \mathrm{d}x + \frac{\delta^2}{2\varepsilon_2} \int_0^1 \left(\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s \right)^2 \mathrm{d}x \\
&\le \frac{\varepsilon_2}{2} \int_0^1 \phi_x^2 \mathrm{d}x + \frac{\delta^2}{2\varepsilon_2} \int_0^t g(s)\mathrm{d}s \int_0^1 \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))^2 \mathrm{d}s\mathrm{d}x \\
&\le \frac{\varepsilon_2}{2} \int_0^1 \phi_x^2 \mathrm{d}x + \frac{c_1}{\varepsilon_2} g \circ \phi_x, \qquad (2.19)
\end{aligned}
$$

Similar to I_2, we have

$$I_3 = b\int_0^1 u_x \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \le \varepsilon_2 \int_0^1 u_x^2 \mathrm{d}x + \frac{c_1}{\varepsilon_2} g \circ \phi_x, \quad (2.20)$$

$$I_4 = \xi\int_0^1 \phi \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x \le \varepsilon_2 \int_0^1 \phi^2 \mathrm{d}x + \frac{c_1}{\varepsilon_2} g \circ \phi_x, \quad (2.21)$$

$$\begin{aligned} I_5 = -m\int_0^1 \theta \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\mathrm{d}x &\le \frac{m}{2}\int_0^1 \theta^2 \mathrm{d}x + \frac{m}{2} g \circ \phi, \\ &\le c_1 \int_0^1 \theta_x^2 \mathrm{d}x + c_1 g \circ \phi_x, \end{aligned} \quad (2.22)$$

$$\begin{aligned} I_6 = &-\int_0^1 \int_0^t g(t-s)\phi_x(s)\mathrm{d}s \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\mathrm{d}x \\ = &-\int_0^t g(s)\mathrm{d}s \int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\mathrm{d}x \\ &+\int_0^1 \left(\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\right)^2 \mathrm{d}x \\ &\le \frac{\varepsilon_2}{2}\int_0^1 \phi_x^2 \mathrm{d}x + \frac{1}{2\varepsilon_2}\left(\int_0^t g(s)\mathrm{d}s\right)^2 \int_0^1 \left(\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\right)^2 \mathrm{d}x \\ &+\int_0^1 \left(\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\mathrm{d}s\right)^2 \mathrm{d}x \\ &\le \frac{\varepsilon_2}{2}\int_0^1 \phi_x^2 \mathrm{d}x + \frac{c_1}{\varepsilon_2} g \circ \phi_x. \end{aligned} \quad (2.23)$$

Since the function g is positive, continuous and $g(0) > 0$, then, for any $t \ge t_0 > 0$, we have

$$\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s = g_0. \quad (2.24)$$

By substituting (2.18)–(2.23) into (2.16), and bearing in mind (2.24), we obtain

$$\begin{aligned} F_3'(t) \; &\le -J\,[g_0-\delta_1]\int_0^1 \phi_t^2 \mathrm{d}x + \varepsilon_2 \int_0^1 (u_x^2+\phi^2+\phi_x^2)\mathrm{d}x + c_1 \int_0^1 \theta_x^2 \mathrm{d}x \\ &+c\left(1+\frac{1}{\varepsilon_2}\right) g \circ \phi_x - \frac{c}{\delta_1} g' \circ \phi_x, \end{aligned}$$

for all $t \ge t_0$. By letting $\delta_1 = \dfrac{g_0}{2}$, we obtain (2.15). □

Lemma 2.6 *Let* (u, ϕ, θ) *be the solution of* (1.1). *Then the functional*

$$F_4(t) := J \int_0^1 \phi_t \phi \mathrm{d}x + \frac{b\rho}{\mu} \int_0^1 \phi \int_0^x u_t(y) \mathrm{d}y \mathrm{d}x$$

satisfies, for any $\varepsilon_3 > 0$, *the estimate*

$$\begin{aligned} F_4'(t) \leq & -\frac{l}{2} \int_0^1 \phi_x^2 \mathrm{d}x - \xi_1 \int_0^1 \phi^2 \mathrm{d}x + \varepsilon_3 \int_0^1 u_t^2 \mathrm{d}x + c_1 \left(1 + \frac{1}{\varepsilon_3}\right) \int_0^1 \phi_t^2 \mathrm{d}x \\ & + c_1 \int_0^1 \theta_x^2 \mathrm{d}x + c_1 g \circ \phi_x. \end{aligned} \tag{2.25}$$

Proof Direct differentiation of F_4, using (1.1) and integration by parts, yields

$$\begin{aligned} F_4'(t) = & -\delta \int_0^1 \phi_x^2 \mathrm{d}x + J \int_0^1 \phi_t^2 \mathrm{d}x - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 \mathrm{d}x + \frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) \mathrm{d}y \mathrm{d}x \\ & + \left(m - \frac{\beta b}{\mu}\right) \int_0^1 \phi\theta \mathrm{d}x + \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s) \mathrm{d}s \mathrm{d}x. \end{aligned} \tag{2.26}$$

In what follows, we use Cauchy-Schwarz and Young's inequalities, for $\delta_3 > 0$.

$$\begin{aligned} & \left(m - \frac{\beta b}{\mu}\right) \int_0^1 \phi\theta \mathrm{d}x \leq \delta_2 \int_0^1 \phi^2 \mathrm{d}x + \left(m - \frac{\beta b}{\mu}\right)^2 \frac{1}{4\delta_2} \int_0^1 \theta^2 \mathrm{d}x, \\ & \frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) \mathrm{d}y \mathrm{d}x \leq \varepsilon_3 \int_0^1 \left(\int_0^x u_t(y,t) \mathrm{d}y\right)^2 \mathrm{d}x + \left(\frac{b\rho}{\mu}\right)^2 \frac{1}{4\varepsilon_3} \int_0^1 \phi_t^2 \mathrm{d}x, \end{aligned} \tag{2.27}$$

by using (2.13), we obtain

$$\frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) \mathrm{d}y \mathrm{d}x \leq \varepsilon_3 \int_0^1 u_t^2 \mathrm{d}x + \frac{c_1}{\varepsilon_3} \int_0^1 \phi_t^2 \mathrm{d}x, \tag{2.28}$$

$$\begin{aligned} & \int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s) \mathrm{d}s \mathrm{d}x \\ & = \int_0^t g(s) \mathrm{d}s \int_0^1 \phi_x^2 \mathrm{d}x \\ & \quad - \int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t) - \phi_x(s)) \mathrm{d}s \mathrm{d}x \\ & \leq \left(\delta_3 + \int_0^t g(s) \mathrm{d}s\right) \int_0^1 \phi_x^2 \mathrm{d}x + \frac{1}{4\delta_3} \left(\int_0^t g(s) \mathrm{d}s\right) g \circ \phi_x. \end{aligned} \tag{2.29}$$

By substituting (2.27)–(2.29) into (2.26), and using Young's and Poincaré's inequalities together with (2.17) and the fact that $2\xi_1 = \xi - \frac{b^2}{\mu} > 0$ since $\mu\xi > b^2$ and $\mu > 0$, we arrive at

$$\begin{aligned} F_4'(t) \leq & -(l-\delta_3)\int_0^1 \phi_x^2 \mathrm{d}x - (2\xi_1 - \delta_2)\int_0^1 \phi^2 \mathrm{d}x + \varepsilon_3 \int_0^1 u_t^2 \mathrm{d}x \\ & + c\left(1 + \frac{1}{\varepsilon_3}\right)\int_0^1 \phi_t^2 \mathrm{d}x + \frac{c_1}{\delta_2}\int_0^1 \theta_x^2 \mathrm{d}x + \frac{c_1}{\delta_3} g \circ \phi_x. \end{aligned} \tag{2.30}$$

By taking $\delta_3 = \dfrac{l}{2}$ and $\delta_2 = \xi_1$, we obtain estimate (2.25). □

3 General Stability Result

In this section, we state and prove our result. To achieve this, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 3.1 *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + \frac{4\rho}{\beta}F_1(t) + F_2(t) + N_1 F_3(t) + N_2 F_4(t), \tag{3.1}$$

where N_1 and N_2 are positive real numbers to be chosen appropriately later, satisfies

$$c_3 E(t) \leq \mathcal{L}(t) \leq c_4 E(t), \qquad \forall t \geq 0, \tag{3.2}$$

for two positive constants c_3 and c_4.

Proof It follows that

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| \leq & \frac{4c\rho}{\beta}\int_0^1 \left|\theta \int_0^x u_t(y,t)\mathrm{d}y\right| \mathrm{d}x + \rho \int_0^1 \left| u_x \int_0^x u_t(y,t)\mathrm{d}y\right| \mathrm{d}x \\ & + JN_1 \int_0^1 \left|\phi_t \int_0^t g(t-s)(\phi(t)-\phi(s))\mathrm{d}s\right| \mathrm{d}x \\ & + JN_2 \int_0^1 |\phi_t \phi| \mathrm{d}x + \frac{b\rho}{\mu} N_2 \int_0^1 \left|\phi \int_0^x u_t(y)\mathrm{d}y\right| \mathrm{d}x. \end{aligned}$$

Exploiting Young's, Poincaré, and Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t) - NE(t)| \leq c_1 \int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2 + \theta^2\right) \mathrm{d}x + c_1 g \circ \phi_x.$$

Using (2.5), we obtain

$$|\mathcal{L}(t) - NE(t)| \leq c_1 E(t),$$

that is

$$(N - c_1)E(t) \le \mathcal{L}(t) \le (N + c_1)E(t).$$

By choosing N large enough, (3.2) follows. □

Next, we state and prove the main result of this section.

Theorem 3.2 *Let* $(u_0, \phi_0, \theta_0) \in H^1_*(0,1) \times \left(H^1_0(0,1)\right)^2$ *and* $(u_1, \phi_1) \in \left(L^2(0,1)\right)^2$ *be given. Assume* (H1) *and* (H2) *hold. Then, there exist positive constants* λ_1 *and* λ_2 *such that the energy functional given by* (2.1) *satisfies*

$$E(t) \le \lambda_1 e^{-\lambda_2 \int_0^t \zeta(s)\mathrm{d}s}, \quad \forall t \ge 0. \tag{3.3}$$

Proof By differentiating (3.1), recalling (2.2), (2.6), (2.14), (2.15), (2.25), and letting

$$\varepsilon_1 = \frac{\beta\mu}{16\rho}, \quad \varepsilon_3 = \frac{\rho}{2N_2}$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le\ & -[\kappa N - c_1 N_1 - c_1 N_2 - c_1] \int_0^1 \theta_x^2 \mathrm{d}x - \frac{\rho}{2}\int_0^1 u_t^2 \mathrm{d}x \\
& - \left[\frac{\mu}{4} - \varepsilon_2 N_1\right] \int_0^1 u_x^2 \mathrm{d}x - \left[\frac{J g_0}{2} N_1 - c_1 N_2 (1 + N_2) - c_1\right] \int_0^1 \phi_t^2 \mathrm{d}x \\
& - \left[\frac{l}{2} N_2 - \varepsilon_2 N_1\right] \int_0^1 \phi_x^2 \mathrm{d}x - [\xi_1 N_2 - \varepsilon_2 N_1 - c_1] \int_0^1 \phi^2 \mathrm{d}x \\
& + \left[\frac{N}{2} - c_1 N_1\right] g' \circ \phi_x + c_1 \left[N_2 + N_1\left(1 + \frac{1}{\varepsilon_2}\right)\right] g \circ \phi_x.
\end{aligned}$$

We choose N_2 large enough such that

$$\alpha_1 = \xi_1 N_2 - c_1 > 0,$$

then we choose N_1 large enough such that

$$\alpha_2 = \frac{J g_0}{2} N_1 - c_1 N_2 (1 + N_2) - c_1 > 0.$$

Next, we pick ε_2 small enough such that

$$\varepsilon_2 < \min\left(\frac{\alpha_1}{N_1}, \frac{\mu}{4N_1}, \frac{l N_2}{2N_1}\right),$$

consequently, we obtain

$$\alpha_3 = \frac{l}{2} N_2 - \varepsilon_2 N_1 > 0, \quad \alpha_4 = \frac{\mu}{4} - \varepsilon_2 N_1 > 0, \quad \alpha_5 = \alpha_1 - \varepsilon_2 N_1 > 0.$$

Finally, we choose N large enough such that (3.2) remains valid and

$$\alpha_6 = \kappa N - c_1 N_1 - c_1 N_2 - c_1 > 0, \quad \alpha_7 = \frac{N}{2} - c_1 N_2 > 0.$$

So, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\alpha_6 \int_0^1 \theta_x^2 \mathrm{d}x - \frac{\rho}{2} \int_0^1 u_t^2 \mathrm{d}x - \alpha_4 \int_0^1 u_x^2 \mathrm{d}x - \alpha_2 \int_0^1 \phi_t^2 \mathrm{d}x - \alpha_3 \int_0^1 \phi_x^2 \mathrm{d}x \\ & -\alpha_5 \int_0^1 \phi^2 \mathrm{d}x + \alpha_7 g' \circ \phi_x + c_1 g \circ \phi_x. \end{aligned} \tag{3.4}$$

On the other hand, from (2.1), using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} E(t) \leq & \ \frac{1}{2} \int_0^1 \left[c_1 \theta_x^2 + \rho u_t^2 + (\mu + b) u_x^2 + \left(\delta - \int_0^t g(s) \mathrm{d}s \right) \phi_x^2 + (\xi + b) \phi^2 + J \phi_t^2 \right] \mathrm{d}x + \frac{1}{2} g \circ \phi_x \\ \leq & \ c_1 \int_0^1 \left[\theta_x^2 + u_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \phi_t^2 \right] \mathrm{d}x + \frac{1}{2} g \circ \phi_x \end{aligned}$$

which implies that

$$-\int_0^1 \left[\theta_x^2 + u_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \phi_t^2 \right] \mathrm{d}x \leq -c' E(t) + c'' g \circ \phi_x. \tag{3.5}$$

The combination of (3.4) and (3.5) gives

$$\mathcal{L}'(t) \leq -k_1 E(t) + k_2 (g \circ \phi_x)(t), \quad \forall t \geq t_0, \tag{3.6}$$

for some positive constants k_1 and k_2. By multiplying (3.6) by $\zeta(t)$ and using (H2) and (2.2), we arrive at

$$\zeta(t) \mathcal{L}'(t) \leq -k_1 \zeta(t) E(t) - 2k_2 E'(t), \quad \forall t \geq t_0,$$

which can be rewritten as

$$(\zeta(t) \mathcal{L}(t) + 2k_2 E(t))' - \zeta'(t) \mathcal{L}(t) \leq -k_1 \zeta(t) E(t), \ \forall t \geq t_0.$$

Using the fact that $\zeta'(t) \leq 0, \forall t \geq 0$, we have

$$(\zeta(t) \mathcal{L}(t) + 2k_2 E(t))' \leq -k_1 \zeta(t) E(t), \quad \forall t \geq t_0.$$

By exploiting (3.2), it can easily be shown that

$$\mathcal{R}(t) = \zeta(t) \mathcal{L}(t) + 2k_2 E(t) \sim E(t). \tag{3.7}$$

Consequently, for some positive constant λ_2, we obtain

$$\mathcal{R}'(t) \leq -\lambda_2 \zeta(t)\mathcal{R}(t), \quad \forall t \geq t_0. \tag{3.8}$$

A simple integration of (3.8) over (t_0, t) leads to

$$\mathcal{R}(t) \leq \mathcal{R}(t_0)e^{-\lambda_2 \int_{t_0}^{t} \zeta(s)\mathrm{d}s}, \quad \forall t \geq t_0. \tag{3.9}$$

Using (3.7) for some positive constant $\tilde{\lambda}_1$, we obtain,

$$E(t) \leq \tilde{\lambda}_1 e^{-\lambda_2 \int_{t_0}^{t} \zeta(s)\mathrm{d}s}, \quad \forall t \geq t_0. \tag{3.10}$$

Consequently, (3.3) is established by virtue of the continuity and boundedness of E and ζ. In other words, since $E(t) \leq E(t_0) \leq E(0), \ \forall t \geq t_0 > 0$, we get

$$E(t) \leq \tilde{\lambda}_1 E(0)e^{\lambda_2 \int_0^{t_0} \zeta(s)\mathrm{d}s} e^{-\lambda_2 \int_0^{t} \zeta(s)\mathrm{d}s}, \quad \forall t \geq t_0 > 0.$$

Consequently, by taking $\lambda_1 = \tilde{\lambda}_1 E(0)e^{\lambda_2 \int_0^{t_0} \zeta(s)\mathrm{d}s}$ we obtain (3.3). □

Remark 3.3 The result given by Theorem 3.2 shows that the memory term together with the heat effect is strong enough to uniformly stabilize the system without imposing the condition of equal wave of speeds. This same result was obtained by Apalara [27] for equivalent Timoshenko system.

3.1 Example

Now, we give two examples to illustrate the energy decay rates obtained by Theorem 3.2. Given that $\tau, \gamma > 0$ with $\tau < \gamma\delta$.

(1) If $g(t) = \tau e^{-\gamma t}$, then

$$E(t) \leq c_0 e^{-\gamma c_1 t}, \quad \forall t \geq 0.$$

(2) If $g(t) = \frac{\tau}{(1+t)^{\gamma+1}}$, then

$$E(t) \leq \frac{c_0}{(1+t)^{(\gamma+1)c_1}}, \quad \forall t \geq 0.$$

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