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# A Survey of Fixed Point Theorems Under Pata-Type Conditions

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**Abstract**
We present a survey of various variants of fixed point results for single- and multivalued mappings, under conditions of the type first used by Pata (J Fixed Point Theory Appl 10:299–305, 2011). A number of examples are given, showing the effectiveness of these results. Some recent misinterpretations of the use of Pata-type conditions are commented.

**Keywords** Fixed point · Common fixed point · Coupled fixed point · Pata-type condition

**Mathematics Subject Classification** 47H10 · 54H25

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## 1 Introduction

Investigation of fixed point and related problems has been a wide area of research of many mathematicians in the last few decades. Hundreds of results have been obtained for mappings in various settings and under various conditions.

Pata-type inequalities are a recent introduction in fixed point theory. It was initiated in a paper of V. Pata [33] in which he proved a result that appeared to be stronger than Banach Contraction Principle, even (in some cases) than the well-known Boyd–Wong fixed point theorem. The inequality in this case is not a single one; instead, it is a group of inequalities which is obtained by varying a parameter over a certain range. At the same time, any single inequality obtained by fixing the parameter to a certain value is not sufficient to imply the existence of a fixed point. A new methodology was introduced in [33] to ensure that fixed points of such mappings exist. This result inspired several researchers to obtain further fixed point, cycled-type fixed point, common fixed point, coupled fixed point and other types of fixed point theorems, both for single-valued and for multivalued mappings in various types of spaces.

In this paper, we present some of the mentioned results (mostly skipping the proofs which can be found in respective papers), including examples showing that they are stronger than some other known ones. Comments on some misinterpretations of Pata's result are also included.

## 2 Basic Pata-Type Results

Throughout this paper (except Sect. 7), $(X, d)$ will be a complete metric space and a point $x_0 \in X$ will be fixed (sometimes it will be called "the zero of $X$"). For $x \in X$, we will denote $\|x\| = d(x, x_0)$. It will be clear that the obtained results do not depend on the particular choice of point $x_0$. Also, $\Psi$ will denote the set of increasing functions $\psi : [0,\ 1] \to [0,\ \infty)$, continuous at zero, with $\psi(0) = 0$.

In the paper [33], Pata obtained the following interesting refinement of the classical Banach Contraction Principle.

**Theorem 1** [33, Theorem 1] *Let $\Lambda \geq 0$, $\alpha \geq 1$, and $\beta \in [0,\ \alpha]$ be some constants and $\psi \in \Psi$. Let $f : X \to X$ be such that for every $\varepsilon \in [0,\ 1]$ and all $x,\ y \in X$,*

$$d(fx, fy) \leq (1-\varepsilon)d(x, y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\|\big]^{\beta}. \tag{1}$$

*Then $f$ has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to $z$.*

Chakraborty and Samanta [9], resp. Kadelburg and Radenović [26], resp. Jacob et al. [21] extended Theorem 1 to the case of Kannan-type, resp. Chatterjea-type, resp. Zamfirescu-type contractive condition, as follows.

**Theorem 2** [9,21,26] *Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be some constants and $\psi \in \Psi$. Let $f : X \to X$ be such that the inequality*

$$d(fx, fy) \leq \frac{1-\varepsilon}{2}\big[d(x, fx) + d(y, fy)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\big]^{\beta}, \tag{2}$$

*resp.*

$$d(fx, fy) \leq \frac{1-\varepsilon}{2}\big[d(x, fy) + d(y, fx)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\big]^{\beta}, \tag{3}$$

*resp.*

$$d(fx, fy) \leq (1-\varepsilon)\max\Big\{d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)]\Big\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\big]^{\beta}, \tag{4}$$

*is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$. Then $f$ has a unique fixed point $z \in X$.*

As a sample, we provide the proof of the second mentioned statement.

***Proof*** [26] Starting from the given $x_0$, we introduce the sequences

$$x_n = fx_{n-1} = f^n x_0 \quad \text{and} \quad c_n = \|x_n\|.$$

The sequence $d(x_{n+1}, x_n)$ is non-increasing, that is,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \leq \cdots \leq d(x_1, x_0), \tag{5}$$

for all $n \in \mathbb{N}$. Indeed, putting $\varepsilon = 0$, $x = x_n$, $y = x_{n-1}$ in (3), we obtain (5).

Further, the sequence $\{c_n\}$ is bounded. In order to prove this, using (5), we deduce the following estimate

$$c_n = d(x_n, x_0) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_1) + d(x_1, x_0) \leq d(x_{n+1}, x_1) + 2c_1 = d(fx_n, fx_0) + 2c_1.$$

Therefore, we infer from (3) that

$$c_n \leq \frac{1-\varepsilon}{2}\big[d(x_n, x_1) + d(x_{n+1}, x_0)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x_n\| + \|x_{n+1}\| + \|x_1\|\big]^{\beta} + 2c_1.$$

Using $d(x_n, x_1) \le d(x_n, x_0) + d(x_0, x_1)$, $d(x_{n+1}, x_0) \le d(x_{n+1}, x_n) + d(x_n, x_0)$ and (5), as $\beta \le \alpha$, the previous inequality implies that

$$c_n \le (1-\varepsilon)(c_n + c_1) + \Lambda\varepsilon^\alpha \psi(\varepsilon)[1 + 2c_n + 2c_1]^\alpha + 2c_1.$$

Now,

$$[1 + 2c_n + 2c_1]^\alpha \le (1 + 2c_n)^\alpha (1 + 2c_1)^\alpha \le 2^\alpha c_n^\alpha (1 + 2c_1)^\alpha,$$

which implies that

$$c_n \le (1-\varepsilon)c_n + a\varepsilon^\alpha \psi(\varepsilon) c_n^\alpha + b,$$

for some $a, b > 0$. Hence,

$$\varepsilon c_n \le a\varepsilon^\alpha \psi(\varepsilon) c_n^\alpha + b.$$

If there existed a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $c_{n_k} \to \infty$ as $k \to \infty$, taking $\varepsilon = \varepsilon_k = (1+b)/c_{n_k}$ would lead to a contradiction

$$1 \le a(1+b)^\alpha \psi(\varepsilon_k) \to 0.$$

Now, we will show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
For all $\varepsilon \in (0, 1]$ and for $x = x_n$, $y = x_{n-1}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\ &\le \frac{1-\varepsilon}{2}\big[d(x_n, x_n) + d(x_{n-1}, x_{n+1})\big] \\ &\quad + \Lambda\varepsilon^\alpha \psi(\varepsilon)\big[1 + 2\|x_n\| + \|x_{n-1}\| + \|x_{n+1}\|\big]^\beta \\ &\le \frac{1-\varepsilon}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] + K\varepsilon\psi(\varepsilon), \quad K > 0. \end{aligned} \tag{6}$$

If $\lim_{n\to\infty} d(x_{n+1}, x_n) = d^* > 0$, it follows from (6) that

$$d^* \le K\psi(\varepsilon),$$

which implies that $d^* = 0$, a contradiction.

In order to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, suppose the contrary. In this case, similarly as in [34, Lemma 2.1], we can chose $\delta > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers, such that the following sequences tend to $\delta$ when $k \to \infty$:

$$\begin{aligned} &d(x_{2m(k)}, x_{2n(k)}), \quad d(x_{2m(k)}, x_{2n(k)+1}), \quad d(x_{2m(k)-1}, x_{2n(k)}), \\ &d(x_{2m(k)-1}, x_{2n(k)+1}), \quad d(x_{2m(k)+1}, x_{2n(k)+1}). \end{aligned}$$

Putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$ in (3), we obtain

$$d(x_{2m(k)}, x_{2n(k)+1}) \leq \frac{1-\varepsilon}{2}\big[d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2m(k)}, x_{2n(k)})\big] + K\varepsilon\psi(\varepsilon), \tag{7}$$

where $d(x_{2m(k)}, x_{2n(k)+1}) \to \delta$, $d(x_{2m(k)-1}, x_{2n(k)+1}) \to \delta$ and $d(x_{2m(k)}, x_{2n(k)}) \to \delta$. Letting $k \to \infty$ in (7), we obtain

$$\delta \leq K\psi(\varepsilon),$$

which implies that $\delta = 0$, a contradiction.

Taking into account the completeness of $(X, d)$, we can now guarantee the existence of some $z \in X$ to which $\{x_n\}$ converges. It remains to show that $z$ is a fixed point for $f$.

For this, we observe that, for all $n \in \mathbb{N}$ and for $\varepsilon = 0$,

$$\begin{aligned} d(fz, z) &\leq d(fz, x_{n+1}) + d(x_{n+1}, z) = d(fz, fx_n) + d(x_{n+1}, z) \\ &\leq \frac{1}{2}\big[d(z, x_{n+1}) + d(fz, x_n)\big] + d(x_{n+1}, z). \end{aligned}$$

Hence, $d(fz, z) \leq \frac{1}{2}d(fz, z)$, that is, $fz = z$, which is the required result.

Finally, we prove the uniqueness of the fixed point. For any two fixed $u, v \in X$, we can write (3) in the form

$$d(fu, fv) \leq \frac{1-\varepsilon}{2}\big[d(u, fv) + d(v, fu)\big] + K\varepsilon\psi(\varepsilon), \qquad K > 0.$$

If $fu = u$ and $fv = v$, then

$$d(u, v) \leq K\psi(\varepsilon),$$

for all $\varepsilon \in (0, 1]$, which implies that $d(u, v) = 0$, that is, $u = v$. □

***Remark 1*** There is a scope of misunderstanding with the inequality (1) in the work of Pata and also in similar other inequalities like (2)–(4) in works incorporating the ideas of Pata. Berinde noted in [7] that if the condition (1) is satisfied, not for all $\varepsilon \in [0, 1]$, but just for some specific values, the conclusion of Theorem 1 might not hold. For example, if (1) holds just for $\varepsilon = 0$, then one has just the non-expansive condition

$$d(fx, fy) \leq d(x, y), \quad x, y \in X,$$

which obviously does not imply the existence of fixed point. Similarly, if $\varepsilon = 1$, (1) reduces to

$$d(fx, fy) \leq L\big[1 + \|x\| + \|y\|\big]^{\beta},$$

with some constant $L$, which is also known to be insufficient for the existence of a fixed point of $f$.

Similar conclusions hold for conditions (2)–(4).

From this observation, Berinde concludes that the Pata-type result is incorrect. But this is not so. There is no contradiction between the above observations and conclusions of the Pata-type theorems for the following reasons. As we have already noted in the introduction, the Pata-type results are obtained for functions satisfying a family of inequalities and any single inequality from the above-mentioned family will not provide us with a sufficient condition for the existence of a fixed point. Had it been so, then there is no need of considering a family of inequalities. Thus, the argument of Berinde is not tenable.

***Remark 2*** It was shown already in [33] that Theorem 1 is strictly stronger than the Banach Contraction Principle, even stronger than the Boyd–Wong fixed point result [8] in the case when the underlying space $(X, d)$ is unbounded (on the bounded space, Pata's and Boyd–Wong's results are equivalent).

Indeed, suppose that

$$d(fx, fy) \leq \lambda\, d(x, y)$$

holds for some $\lambda \in [0, 1)$ and all $x, y \in X$. Further, we follow the procedure as in [33, §3], only we write it with some details that were skipped in [33].

First of all, for arbitrary $\varepsilon \in [0, 1]$, the last inequality implies that

$$\begin{aligned} d(fx, fy) &\leq (1-\varepsilon)d(x, y) + (\lambda + \varepsilon - 1)d(x, y) \\ &\leq (1-\varepsilon)d(x, y) + (\lambda + \varepsilon - 1)(\|x\| + \|y\|). \end{aligned}$$

We want to prove that there are some $\gamma \geq 0$ and $\Lambda \geq 0$ such that

$$(\lambda + \varepsilon - 1)(\|x\| + \|y\|) \leq \Lambda \varepsilon^{1+\gamma}(1 + \|x\| + \|y\|)$$

holds for each $\varepsilon \in [0, 1]$ and all $x, y \in X$. Indeed, this will be the case if one can find $\Lambda \geq 0$ such that

$$\Lambda \geq \frac{\lambda + \varepsilon - 1}{\varepsilon^{1+\gamma}}$$

holds for some $\gamma \geq 0$ and each $\varepsilon \in [0, 1]$. By a routine procedure, it is easy to show that this is the case if we chose $\gamma$ such that $\frac{\gamma}{1+\gamma} > 1 - \lambda$ and then

$$\Lambda = \frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \frac{1}{(1-\lambda)^{\gamma}}. \tag{8}$$

Hence, the condition (1) is fulfilled with $\alpha = \beta = 1$, $\gamma \geq 0$ such that $\frac{\gamma}{1+\gamma} > 1 - \lambda$, $\psi(\varepsilon) = \varepsilon^{\gamma}$ and $\Lambda$ defined by (8).

On the other hand, the following example shows that there exists a mapping satisfying Pata's condition and not satisfying Boyd–Wong's one. (Recall that the Boyd–Wong's result guarantees the existence of fixed point of the mapping $f : X \to X$ if there is an increasing continuous function $\varrho : [0, \infty) \to [0, \infty)$ such that $\varrho(t) < t$ for each $t > 0$ and

$$d(fx, fy) \leq \varrho(d(x, y)) \tag{9}$$

for all $x \neq y$.)

***Example 1*** [33] Let $X = [1, \infty)$ and let $f : X \to X$ be defined by

$$fx = -2 + x - 2\sqrt{x} + 4\sqrt[4]{x}.$$

It has a unique fixed point $z = 1$. For any given $r > 0$ and $x \geq 1$, if

$$F(x, r) = 2[\sqrt{x + r} - \sqrt{x}] - 4[\sqrt[4]{x + r} - \sqrt[4]{x}],$$

then

$$|f(x + r) - f(x)| = r - F(x, r)$$

holds for all $r$ and $x$. If there existed $\varrho$ satisfying the condition (9), then choosing any $r > 0$ one would have a contradiction

$$r > \varrho(r) \geq \lim_{x \to +\infty} [r - F(x, r)] = r.$$

On the other hand, for every $\varepsilon \in [0, 1]$, one can prove that

$$-\varepsilon r + \varepsilon^2(2x + r)^{3/2} + F(x, r) \geq F(x, r) - \frac{r^2}{4(r + 2x)^{3/2}} \geq 0.$$

It follows that

$$|f(x + r) - f(x)| = r - F(x, r) \leq (1 - \varepsilon)r + \varepsilon^2(2x + r)^{3/2},$$

and the condition (1) of Theorem 1 is fulfilled.

Similar conclusions as presented in Remark 2 and Example 1 hold for other Pata-type conditions.

## 3 Cyclic Fixed Point Results

Cyclic versions of the results of previous section were treated in [2,3,10,24]. As a sample, we state the following theorem for cyclic generalized contractions from [24].

Recall that $Y = \bigcup_{i=1}^{p} A_i$ is said to be a cyclic representation of $Y \subseteq X$ w.r.t. the mapping $f : Y \to Y$ if $A_i$, $i = 1, 2, \dots, p$, are non-empty closed subsets of $Y$ and $f(A_i) \subseteq A_{i+1}$ for $i = 1, 2, \dots, p$ (for $j \in \mathbb{N}$, $j > p$, we always put $A_j := A_i$, where $j \equiv i \pmod p$ and $1 \le i \le p$).

**Theorem 3** [24] *Let $f : Y \to Y$, with $Y = \bigcup_{i=1}^{p} A_i$ being a cyclic representation of $Y$ w.r.t. $f$. Assume that there exist $\psi \in \Psi$ and constants $\Lambda \ge 0$, $\alpha \ge 1$ and $\beta \in [0, \alpha]$ such that*

$$d(fx, fy) \le (1-\varepsilon) \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\} + \Lambda \varepsilon^{\alpha} \psi(\varepsilon)\left[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\right]^{\beta} \tag{10}$$

*holds for all $\varepsilon \in [0,\ 1]$, $i = 1, 2, \dots, p$, $x \in A_i$ and $y \in A_{i+1}$. Then $f$ has a unique fixed point $z$, it belongs to $\bigcap_{i=1}^{p} A_i$, and the Picard iteration sequence $\{f^n x_1\}_{n\in\mathbb{N}}$ converges to $z$ for any initial point $x_1 \in Y$.*

***Example 2*** [24] Let $X = \mathbb{R}$ be equipped with the standard metric and let

$$A_1 = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\} \cup \{0\}, \quad A_2 = \left\{-\frac{1}{2n-1} : n \in \mathbb{N}\right\} \cup \{0\}, \quad Y = A_1 \cup A_2.$$

Define $f : Y \to Y$ by

$$fx = \begin{cases} -\dfrac{x}{x+4}, & x \in A_1, \\ -\dfrac{x}{4}, & x \in A_2. \end{cases}$$

It is easy to see that $Y = A_1 \cup A_2$ is a cyclic representation of $Y$ w.r.t. $f$ (with $p = 2$). We will show that $f$ satisfies the contractive condition of Theorem 3.

Indeed, let $x \in A_1$ and $y \in A_2$. Then

$$\begin{aligned} d(fx, fy) &= \left|\frac{x}{x+4} - \frac{y}{4}\right| \le \frac{1}{4}(x + |y|) = \frac{1}{4} d(x, y) \\ &\le \frac{1}{4} \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\} \\ &=: \frac{1}{4}\mathcal{F}(x, y). \end{aligned}$$

The rest of the procedure is similar to the one in Remark 2. It is enough to choose $\gamma$ such that $\frac{\gamma}{1+\gamma} > 1 - \frac{1}{4}$ and

$$\Lambda = \frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \frac{1}{(1-\frac{1}{4})^{\gamma}}.$$

Then we have

$$\Lambda \geq \frac{\frac{1}{4} + \varepsilon - 1}{\varepsilon^{1+\gamma}}$$

and it easily follows that, for the chosen $\gamma$ and $\Lambda$,

$$d(fx, fy) \leq (1-\varepsilon) \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\} + \Lambda\varepsilon^{1+\gamma}\left[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\right],$$

for each $\varepsilon > 0$ and all $x \in A_1$, $y \in A_2$ or $x \in A_2$, $y \in A_1$. Thus, the conditions of Theorem 3 are fulfilled (with $\alpha = \beta = 1$ and $\psi(\varepsilon) = \varepsilon^\gamma$), and the mapping $f$ has a unique fixed point (which is $z = 0$).

## 4 Pata-Type Fixed Point Results in Ordered Metric Spaces

Fixed point theory in metric spaces along with a partial order has developed rapidly in recent years. Although an early result appeared in the work of Turinici [39] in 1986, the actual development of this line of research took place following the publications of the works of Ran and Reurings [35] and Nieto and Rodriguez-López [31]. For the existence of fixed points, the various types of contractive inequality conditions to be satisfied by the operators for pairs of points collected from the metric space can be restricted to those pairs which are related by the partial order, and still not disturbing the conclusions of the theorems. As a kind of compensation for this restrictive use of contractive conditions, some condition is usually added to the mapping itself (e.g., its monotonicity).

Following the above-mentioned ideas, many Pata-type results can be adapted to ordered metric spaces (see, e.g., [23,26]). The following are some notions and definitions.

Throughout the section, $(X, \preceq, d)$ denotes a partially ordered metric space, i.e., a triple where $(X, \preceq)$ is a partially ordered set and $(X, d)$ is a metric space.

For $x, y \in X$, $x \asymp y$ will denote that $x$ and $y$ are comparable, i.e., either $x \preceq y$ or $y \preceq x$ holds.

Recall that the space $(X, \preceq, d)$ is said to be regular if it has the following properties:

1. if for a non-decreasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all n;
2. if for a non-increasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \succeq x$ for all n.

**Theorem 4** [23, Theorem 3.1] *Let $(X, \preceq, d)$ be a complete ordered metric space and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be some constants and $\psi \in \Psi$. Let $f : X \to X$ be a non-decreasing map such that there exists $x_0$ satisfying $x_0 \preceq fx_0$ and suppose that the inequality (1) is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$ with $x \asymp y$. If $f$ is continuous or $(X, d, \preceq)$ is regular, then $f$ has a fixed point $z \in X$. Moreover,*

(1) *the set of fixed points of $f$ is a singleton if and only if it is totally ordered;*

(2) *the set of fixed points of* $f$ *is a singleton if for every two points* $u, v \in X$ *there exists* $w \in X$*, comparable with* $u, v$ *and* $fw$.

***Remark 3*** Theorem 4 is strictly stronger than Theorem 2.1 in [35]. On the one side, the hypotheses of Theorem 2.1 [35] imply those of Theorem 4, which follows in the same way as in Remark 2. On the other side, the example of function

$$f : [1, \infty) \to [1, \infty); \qquad f(x) = -2 + x - 2\sqrt{x} + 4\sqrt[4]{x}$$

(see Example 1) shows that condition (1) can be satisfied when Banach's condition is not. It is also an example of the situation when condition (2) for the uniqueness of fixed point (in the previous theorem) is fulfilled (since the given space is totally ordered).

As was already shown, some of the generalizations of Banach Contraction Principle have their Pata-type versions. We shall present here a result of Pata-type for so-called generalized contractions, in the "ordered" version.

**Theorem 5** [23, Theorem 3.2] *Let* $(X, \preceq, d)$ *be a complete ordered metric space and let* $\Lambda \geq 0$*,* $\alpha \geq 1$ *and* $\beta \in [0, \alpha]$ *be some constants and* $\psi \in \Psi$*. Let* $f : X \to X$ *be a non-decreasing map such that there exists* $x_0$ *satisfying* $x_0 \preceq fx_0$ *and suppose that the inequality* (10) *is satisfied for every* $\varepsilon \in [0, 1]$ *and all* $x, y \in X$ *with* $x \asymp y$*. If* $f$ *is continuous or* $(X, d, \preceq)$ *is regular, then* $f$ *has a fixed point* $z \in X$*. Moreover,*

(1) *the set of fixed points of* $f$ *is a singleton if and only if it is totally ordered;*
(2) *the set of fixed points of* $f$ *is a singleton if for every two points* $u, v \in X$ *there exists* $w \in X$*, comparable with* $u, v$ *and* $fw$.

***Remark 4*** Similarly as in the classical situation, treated in [36], it can be proved that Theorem 5 contains as special cases several other Pata-type results in their ordered versions. In particular, this includes Kannan, Chatterjea, Reich, Zamfirescu and Hardy–Rogers results. The exact formulations and proofs are obvious.

## 5 Coupled and Tripled Fixed Point Results

Coupled fixed points are relatively new concepts in the study of fixed point theory which originated in the work of Guo and Lakshmikantham [19]. It was after the publication of the coupled contraction mapping principle by Gnana Bhaskar and Lakshmikantham [17] that the interest in these problems gained momentum.

Recall the following notions.

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$.

1. $F$ is said to have the *mixed monotone property* if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in X)\ x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y),$$
$$(\forall x, y_1, y_2 \in X)\ y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2).$$

2. A point $(x, y) \in X \times X$ is said to be a *coupled fixed point* of $F$ if $F(x, y) = x$ and $F(y, x) = y$.

Coupled fixed point results under Pata-type contractive conditions were firstly obtained in [14]. The basic result was the following.

**Theorem 6** [14] *Let $(X, \preceq, d)$ be a complete ordered metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$; for $x, y \in X$, denote $\|x, y\| = d(x, x_0) + d(y, y_0)$. Let there exist $\psi \in \Psi$ and constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ such that the inequality*

$$d(F(x, y), F(u, v)) \leq \frac{(1-\varepsilon)}{2}\big[d(x, u) + d(y, v)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x, y\| + \|u, v\|\big]^{\beta}$$

*is satisfied for every $\varepsilon \in [0, 1]$ and all $(x, y), (u, v) \in X \times X$ with $u \preceq x$, $y \preceq v$. If*

1. *$F$ is continuous, or*
2. *The space $(X, \preceq, d)$ is regular,*

*then $F$ has a coupled fixed point in $X \times X$.*

A new approach to coupled fixed point problems was initiated by Berinde in [6] and further developed, e.g., in [1,18,22]. The basic idea is to exploit results for mappings with one variable and apply them to mappings defined on products of spaces. Using this approach, an improved version of the above result is obtained in [26].

**Theorem 7** [26] *Let $(X, \preceq, d)$ be a complete ordered metric space and $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$; for $x, y \in X$, denote $\|x, y\| = d(x, x_0) + d(y, y_0)$. Let there exist $\psi \in \Psi$ and constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ such that the inequality*

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\qquad \leq (1-\varepsilon)[d(x, u) + d(y, v)] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x, y\| + \|u, v\|\big]^{\beta} \end{aligned}$$

*is satisfied for every $\varepsilon \in [0, 1]$ and all $(x, y), (u, v) \in X \times X$ with $u \preceq x$, $y \preceq v$. Then $F$ has a coupled fixed point in $X \times X$.*

***Example 3*** [26] Let $X = \mathbb{R}$ be equipped with the usual metric and order. The mapping $F : X \times X \to X$ defined by $F(x, y) = \frac{1}{6}(x - 4y)$ is obviously mixed monotone. It is easy to obtain that

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &= \left|\frac{x-4y}{6} - \frac{u-4v}{6}\right| + \left|\frac{y-4x}{6} - \frac{v-4u}{6}\right| \end{aligned}$$

$$\leq \frac{1}{6}|x-u| + \frac{4}{6}|y-v| + \frac{1}{6}|y-v| + \frac{4}{6}|x-u|$$
$$= \frac{5}{6}[d(x,u) + d(y,v)].$$

The further procedure is similar to the one in Remark 2. The mapping $F$ has a unique coupled fixed point (which is $(0, 0)$).

Note that the same conclusion cannot be made using the results from [14].

We present also an example of a tripled fixed point result.

**Theorem 8** [23] *Let $(X, \preceq, d)$ be a complete ordered metric space and $F : X^3 \to X$ be a non-decreasing mapping w.r.t. each variable, and suppose that there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \preceq F(y_0, x_0, z_0)$, $z_0 \preceq F(z_0, y_0, x_0)$. Let, for some $\psi \in \Psi$ and some constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$d(F(x,y,z), F(u,v,w)) + d(F(y,x,z), F(v,u,w)) + d(F(z,y,x), F(w,v,u))$$
$$\leq (1-\varepsilon)(d(x,u) + d(y,v) + d(z,w)) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x,y,z\| + \|u,v,w\|]^{\beta}$$

*holds for all $\varepsilon \in [0, 1]$ and all $x, y, z, u, v, w \in X$ with ($x \preceq u$, $y \preceq v$ and $z \preceq w$) or ($x \succeq u$, $y \succeq v$ and $z \succeq w$). Finally, suppose that $F$ is continuous or that the space is regular. Then there exists $(x^*, y^*, z^*) \in X^3$ such that $F(x^*, y^*, z^*) = z^*$, $F(y^*, x^*, z^*) = y^*$, $F(z^*, y^*, x^*) = z^*$.*

## 6 Pata-Type Results for Multivalued Mappings

Fixed point results for multivalued mappings under Pata-type conditions were proved in [11,12,28]. As a sample, we state here the following theorems.

In the fixed point theory of set-valued maps, two types of distances are generally used. One is the Hausdorff–Pompeiu distance, and the other is $\delta$-distance. The following standard notations will be used. $N(X)$ is the family of all non-empty subsets of $X$, $B(X)$ is the family of all non-empty bounded subsets of $X$, $CB(X)$ is the family of all non-empty closed and bounded subsets of $X$, $C(X)$ is the family of all non-empty compact subsets of $X$ and

$$D(x, B) = \inf\{d(x,y) : y \in B\} \text{where } x \in X \text{and } B \in B(X),$$
$$\delta(A, B) = \sup\{d(x,y) : x \in A, \ y \in B\} \text{ where } A, B \in B(X),$$
$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A)\} \text{ where } A, B \in CB(X).$$

$H$ is known as the Hausdorff–Pompeiu metric induced by the metric $d$ on $CB(X)$ [29]. The $\delta$-distance [15] is not metric as the Hausdorff–Pompeiu distance but shares most of the properties of a metric.

Let $T : X \to N(X)$ be a multivalued mapping. An element $x \in X$ is called a fixed point of $T$ if $x \in Tx$. An element $x \in X$ is called an endpoint of $T$ if $Tx = \{x\}$. Obviously, each endpoint of $T$ is a fixed point of it, but the converse is not true.

The following theorem deals with two types of inequalities with Hausdorff–Pompeiu and $\delta$-distances, respectively, in two different situations.

**Theorem 9** [11] *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ be a multivalued mapping. Suppose that for some $\Lambda \geq 0$, $L \geq 0$, $\eta > 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and $\psi \in \Psi$, every $\varepsilon \in [0,\ 1]$ and all $x, y \in X$,*

$$H(Tx, Ty) \leq (1-\varepsilon)\big[M(x, y) + L\,N(x, y)\big] \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|\big]^{\beta}$$

*holds whenever $d(x, y) \geq \eta$ and*

$$\delta(Tx, Ty) \leq (1-\varepsilon)\big[M(x, y) + L\,N(x, y)\big] \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|\big]^{\beta} \tag{11}$$

*holds whenever $d(x, y) < \eta$, where*

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)\,d(y, Ty)}{1 + d(x, y)}\right\},$$
$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

*$\|x\| = d(x, x_0)$ and $\|Tx\| = D(x_0, Tx)$. Also, suppose that $Tx^*$ is a singleton for some $x^* \in X$. Then $T$ has a fixed point.*

The following result, where the inequality involving $\delta$-distance only is considered, is contained in Theorem 9.

**Theorem 10** [11, Corollary 3.1] *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ be a multivalued mapping. Suppose that for some $\Lambda \geq 0$, $L \geq 0$, $\alpha \geq 1$, $\beta \in [0,\ \alpha]$ and $\psi \in \Psi$, every $\varepsilon \in [0,\ 1]$, and for all $x, y \in X$, the inequality (11) is satisfied, where $M(x, y)$, $N(x, y)$, $\|x\|$ and $\|Tx\|$ are same as defined in Theorem 9. Also, suppose that $Tx^*$ is a singleton for some $x^* \in X$. Then $T$ has a fixed point.*

***Example 4*** [11] Let $X = [0,\ 1] \cup \{3, 6\}$ be equipped with the usual metric $d$ and let $T : X \to C(X)$ be given as

$$Tx = \begin{cases} \{x - \frac{x^2}{2}\}, & \text{if } \ 0 \leq x \leq 1, \\ \{0, 1\}, & \text{if } \ x = 3, \\ \{1, 3\}, & \text{if } \ x = 6. \end{cases}$$

Consider $\psi \in \Psi$ defined by

$$\psi(t) = \begin{cases} t, & \text{if } \ 0 \leq t \leq \frac{1}{100}, \\ 4t, & \text{otherwise,} \end{cases}$$

and $\eta = \frac{1}{1000}$, $\Lambda = 100$, $\alpha = 2$, $\beta = 1$. Then the conditions of Theorem 9 are fulfilled and $T$ has a fixed point (which is 0).

***Remark 5*** [11] In Example 4, when $x = 6$, $y = 3$ and $\varepsilon = 10^{-6}$, the inequality of Theorem 10, that is, (11) is not satisfied. So Example 4 is not applicable to Theorem 10, and hence, Theorem 9 properly contains Theorem 10. Further, it is observed that for any $\varepsilon > 0$, the example does not satisfy the inequality $H(Tx, Ty) \leq (1-\varepsilon)\big[M(x, y) + LN(x, y)\big]$ or $\delta(Tx, Ty) \leq (1-\varepsilon)\big[M(x, y) + LN(x, y)\big]$ for all $x, y \in X$. This indicates that the second term $\Lambda\varepsilon^{\alpha}\,\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|\big]^{\beta}$ is essential for the theorem which is the spirit of Pata-type results.

The following results are proved using $\delta$-distance under two alternative sets of assumptions, namely, partial-order assumptions and admissibility conditions. The following class of function is also used in these results.

Let $\Phi$ denote the family of all functions $\varphi : [0, \infty)^4 \to [0, \infty)$ such that $(i_\varphi)$ $\varphi$ is non-decreasing in each coordinate and continuous; $(ii_\varphi)$ $\varphi(t, t, t, t) \leq t$ for $t \geq 0$.

The following ordering between subsets of a partially ordered set will be considered.

**Definition 1** [5] Let $A$ and $B$ be two non-empty subsets of a partially ordered set $(X, \preceq)$. We say that $A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

**Theorem 11** [12] *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to B(X)$ be a multivalued mapping. Suppose that (i) for $x, y \in X$, $x \preceq y$ implies $Tx \prec_1 Ty$; (ii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$; (iii) if $\{x_n\}$ is a sequence in $X$ whose consecutive terms are comparable and $x_n \to x$, then $x_n$ and $x$ are comparable for all $n$; and (iv) there exist $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and $\psi \in \Psi$, $\varphi \in \Phi$ such that for every $\varepsilon \in [0, 1]$ and for all comparable $x, y \in X$,*

$$\delta(Tx, Ty) \leq (1-\varepsilon)\,\varphi\Big(d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\Big) + \Lambda\,\varepsilon^{\alpha}\,\psi(\varepsilon)\big[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|\big]^{\beta}, \tag{12}$$

*where $\|x\| = d(x, x_0)$ and $\|Tx\| = D(x_0, Tx)$. Then $T$ has an endpoint.*

***Example 5*** [12] Let $X = \{(0, 0), (-\frac{1}{4}, -\frac{1}{4}), (0, -1)\}$ be a subset of $\mathbb{R}^2$ with the order $\preceq$ defined as: for $(x, y), (u, v) \in X$, $(x, y) \preceq (u, v)$ if and only if $x \leq u$, $y \leq v$. Let $d : X \times X \to \mathbb{R}$ be given as $d(p, q) = \max\{|x - u|, |y - v|\}$, for $p = (x, y)$, $q = (u, v) \in X$. Let $T : X \to B(X)$ be given as

$$Tx = \begin{cases} \{(0, 0)\}, & \text{if } x = (0, 0), \\ \{(0, 0), (-\frac{1}{4}, -\frac{1}{4})\}, & \text{if } x = (0, -1), \\ \{(0, 0)\}, & \text{if } x = (-\frac{1}{4}, -\frac{1}{4}). \end{cases}$$

Define $\varphi : [0, \infty)^4 \to [0, \infty)$ and $\psi : [0, 1] \to [0, \infty)$, respectively, as

$$\varphi(x_1, x_2, x_3, x_4) = \frac{x_1 + x_2 + x_3 + x_4}{4}, \text{ where } (x_1, x_2, x_3, x_4) \in [0, \infty)^4,$$

and

$$\psi(t) = \begin{cases} t^2, & \text{if } 0 \le t \le \dfrac{1}{3}, \\ 4t, & \text{otherwise.} \end{cases}$$

Let $\Lambda > 16$, $\alpha = 2$ and $\beta = 1$. The conditions of Theorem 11 are fulfilled and $(0, 0)$ is an endpoint of $T$.

It was shown by Samet et al. [38] that the purpose of introducing the partial order can be also served by a set of conditions called admissibility conditions. Later, several other types of admissibilities have been introduced and used in the domain of fixed point theory. These conditions, rather than introducing a new structure like partial order in metric spaces, are requirements on the operator under consideration.

**Definition 2** [12] Let $T : X \to N(X)$ and $\theta, \eta : X \to [0, \infty)$. $T$ is said to be a cyclic $(\theta, \eta)$-admissible mapping if (i) $\theta(x) \ge 1$ for some $x \in X$ implies $\eta(u) \ge 1$ for all $u \in Tx$, (ii) $\eta(x) \ge 1$ for some $x \in X$ implies $\theta(v) \ge 1$ for all $v \in Tx$.

**Theorem 12** [12] *Let $(X, d)$ be a complete metric space, $T : X \to B(X)$ and $\theta, \eta : X \to [0, \infty)$. Suppose that (i) $T$ is cyclic $(\theta, \eta)$-admissible; (ii) there exists $x_0 \in X$ such that $\theta(x_0) \ge 1$ and $\eta(x_0) \ge 1$; (iii) if $\{x_n\}$ is a sequence in $X$ with $\theta(x_n) \ge 1$ $\big(\text{or } \eta(x_n) \ge 1\big)$ and $x_n \to x$, then $\theta(x) \ge 1$ or $\eta(x) \ge 1$; and (iv) there exist $\Lambda \ge 0$, $\alpha \ge 1$, $\beta \in [0, \alpha]$ and $\psi \in \Psi$, $\varphi \in \Phi$ such that (12) is satisfied for every $\varepsilon \in [0, 1]$ and for all $x, y \in X$ with $\theta(x)\,\eta(y) \ge 1$. Then $T$ has an endpoint.*

***Example 6*** [12] Let $X = [0, \infty)$ be equipped with the usual metric $d$ and order and let $T : X \to B(X)$ be given as

$$Tx = \begin{cases} \{\frac{x}{8}\}, & \text{if } 0 \le x \le 1 \\ \left[x + \frac{1}{x} - \frac{1}{n},\ n\right], & \text{if } n - 1 < x \le n \text{ with } n \ge 2. \end{cases}$$

Define $\theta, \eta : X \to [0, \infty)$, respectively, as

$$\theta(x) = \begin{cases} e^x, & \text{if } 0 \le x \le 1 \\ \frac{1}{4}, & \text{if } x > 1 \end{cases} \qquad \eta(x) = \begin{cases} x + 2, & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x > 1. \end{cases}$$

Take the same functions $\psi$ and $\varphi$ as in Example 5. Let $\Lambda > 2$, $\alpha = 2$ and $\beta = 1$. Then the conditions of Theorem 12 are fulfilled and $\{0, 2, 3, 4, n, \ldots\}$ is the set of endpoints of $T$.

***Remark 6*** One can treat $T : X \to X$ as a multivalued mapping even in the case when $Tx$ is a singleton for every $x \in X$. In Theorem 11, considering $\psi(x_1, x_2, x_3, x_4) = \max\{x_1, x_2, x_3, x_4\}$ and if $T$ is single valued, Theorem 5 (i.e., [23, Theorem 3.2], see also [12]) is obtained.

In Theorem 12, taking $\psi(x_1, x_2, x_3, x_4) = \max\{x_1, x_2, x_3, x_4\}$, $T$ to be single valued and $\theta(x) = 1$ and $\eta(x) = 1$ for all $x \in X$, the following "non-cycling" variant of Theorem 3 can again be obtained.

**Theorem 13** [12] *Suppose that for some $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and $\psi \in \Psi$, every $\varepsilon \in [0,\ 1]$ and all $x, y \in X$, the condition (10) is fulfilled. Then $T$ has a fixed point.*

## 7 Some Results in More General Settings

There have been large efforts for generalizing metric spaces by changing the form and interpretation of the metric function. Fixed point and common fixed point problems under Pata-type conditions were also investigated in spaces more general than metric ones. We will shortly present here some results in $b$-metric and $b$-rectangular metric spaces from [25], as well as in modular spaces from [32].

**Definition 3** [4,13] Let $X$ be a non-empty set, $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0,\ \infty)$ is called a $b$-metric with parameter $s$ if

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$ ($b$-triangular inequality).

Then $(X, d)$ is called a $b$-metric space.

**Theorem 14** [25] *Let $(X, d)$ be a complete b-metric space with parameter $s > 1$ and $f, g : X \to X$ be two self-mappings such that $fX \subseteq gX$. Suppose that for some $\psi \in \Psi$, and constants $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$,*

$$d(fx, fy) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(gx, gy)}{2s}, d(gx, fx), d(gy, fy)\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|gx\| + \|gy\| + \|fx\| + \|fy\|\right]^{\beta}, \quad (13)$$

*holds for all $x, y \in X$ and each $\varepsilon \in [0,\ 1]$. Then $f$ and $g$ have a unique point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then they have a unique common fixed point.*

***Example 7*** [25] Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to [0,\ \infty)$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd} \\ & \quad (\text{and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then $d$ is a $b$-metric on $X$ which is not continuous (see [20]). Consider the mapping $f : X \to X$, $fx = \begin{cases} 100, & x \leq 100, \\ 4, & \text{otherwise} \end{cases}$. The only non-trivial case to be considered is when $x \in \{1, 2, \ldots, 100\}$ and $y \in \{101, 102, \ldots, \infty\}$. Then

$$d(fx, fy) = d(100, 4) = \left|\frac{1}{100} - \frac{1}{4}\right| = \frac{24}{100}.$$

We have to check that

$$\frac{24}{100} \leq \frac{2(1-\varepsilon)}{5} \max\left\{\frac{d(x, y)}{5}, d(x, 100), d(y, 4)\right\} + \varepsilon^2$$

(which is condition (13) with $g = I_X$, $\Lambda = 1$, $\psi(\varepsilon) = \varepsilon$, $\alpha = 1$, $\beta = 0$).

Now

$$\max \frac{d(x, y)}{5} = \max \begin{cases} \frac{1}{5}\left|\frac{1}{x} - \frac{1}{y}\right|, & \text{both } x \text{ and } y \text{ are even or one is even and the other is } \infty, \\ \frac{1}{5} \cdot 5, & \text{both } x \text{ and } y \text{ are odd or one is odd the other is } \infty, \\ \frac{1}{5} \cdot 2, & \text{otherwise} \end{cases}$$
$$\leq 1,$$
$$\max d(x, 100) = \begin{cases} \left|\frac{1}{x} - \frac{1}{100}\right|, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \leq 2,$$
$$\max d(y, 4) = \begin{cases} \left|\frac{1}{y} - \frac{1}{100}\right|, & \text{if } y \text{ is even or } \infty, \\ 2, & \text{if } y \text{ is odd or } \infty \end{cases} \leq 2.$$

Then

$$\frac{24}{100} \leq \frac{2(1-\varepsilon)}{5} \cdot 2 + \varepsilon^2,$$

which is fulfilled for each $\varepsilon \in \mathbb{R}$, a fortiori for $\varepsilon \in [0, 1]$.

**Definition 4** [16,37] Let $X$ be a non-empty set and $s \geq 1$ be a given real number. Let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from $x$ and $y$,

(i) $d(x, y) = 0$ iff $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ ($b$-rectangular inequality)

hold. Then $(X, d)$ is called a $b$-rectangular metric space with parameter $s$.

**Theorem 15** [25] *Let $(X, d)$ be a complete b-rectangular metric space with parameter $s > 1$ and $f, g : X \to X$ be such that $fX \subseteq gX$. Suppose that for some $\psi \in \Psi$, and constants $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$,*

$$d\left(fx,fy\right)\leq\frac{1-\varepsilon}{s}\max\left\{\frac{d\left(gx,gy\right)}{2s},d\left(gx,fx\right),d(gy,fy)\right\}$$
$$+\Lambda\varepsilon^{\alpha}\psi\left(\varepsilon\right)\left[1+\|gx\|+\|gy\|\right]^{\beta},$$

*holds for all* $x, y \in X$ *and each* $\varepsilon \in [0,1]$. *Then* $f$ *and* $g$ *have a unique point of coincidence. Moreover, if* $f$ *and* $g$ *are weakly compatible, then they have a unique common fixed point.*

***Example 8*** [25] Let $X = A \cup B$, where $A = \left\{\frac{1}{n} : n \in \{2,3,4,5\}\right\}$ and $B = [1,2]$. Define $d : X \times X \to [0,\infty)$ so that $d(x,y) = d(y,x)$ for all $x, y \in X$, and

$$d\left(\frac{1}{2},\frac{1}{3}\right)=d\left(\frac{1}{4},\frac{1}{5}\right)=0.03;\quad d\left(\frac{1}{2},\frac{1}{5}\right)=d\left(\frac{1}{3},\frac{1}{4}\right)=0.02;$$
$$d\left(\frac{1}{2},\frac{1}{4}\right)=d\left(\frac{1}{5},\frac{1}{3}\right)=0.06;\quad d\left(x,y\right)=\left(x-y\right)^{2}\ \text{otherwise.}$$

Then $(X,d)$ is a $b$-rectangular metric space with coefficient $s = 3$, but $(X,d)$ is neither a metric space nor a rectangular metric space (see [25] for details). Let $f, g : X \to X$ be defined as:

$$f\left(x\right)=\begin{cases}\frac{1}{3}, & \text{if}\quad x\in A\\ \frac{1}{5}, & \text{if}\quad x\in B,\end{cases}\qquad g\left(x\right)=x.$$

We have to check the condition

$$d\left(fx,fy\right)\leq\frac{1-\varepsilon}{3}\max\left\{\frac{d\left(x,y\right)}{6},d\left(x,fx\right),d\left(y,fy\right)\right\}+\varepsilon^{2}\left[1+\|x\|+\|y\|\right],$$

with $x_0 = \frac{1}{5}$ (i.e., $\|x\| = d(x,\frac{1}{5})$). It is enough to consider the case when $x \in A$, $y \in B$. Then,

$d\left(fx,fy\right)=d\left(\frac{1}{3},\frac{1}{5}\right)=0.06$, $\max\frac{d(x,y)}{6}=\max\frac{|x-y|^2}{6}=\frac{\left(2-\frac{1}{5}\right)^2}{6}=\frac{27}{50}$,
$\max d\left(x,fx\right)=\max d\left(x,\frac{1}{3}\right)=0.06$, $\max d\left(y,fy\right)=\max d\left(y,\frac{1}{5}\right)=\left(2-\frac{1}{5}\right)^2=\frac{81}{25}$.

Hence, $\max\left\{\frac{d(x,y)}{6},d\left(x,fx\right),d\left(y,fy\right)\right\}=\frac{81}{25}$ when $x \in A$, $y \in B$. Then it is to be checked that

$$0.06\leq\frac{1-\varepsilon}{3}\cdot\frac{81}{25}+\varepsilon^{2}\left(1+\|x\|+\|y\|\right).$$

Since $\min\{1+\|x\|+\|y\|\}=1+0+\left(1-\frac{1}{5}\right)^2=1+\frac{16}{25}=\frac{41}{25}$, it is required to check the inequality

$$0.06\leq\frac{1-\varepsilon}{3}\cdot\frac{81}{25}+\frac{41}{25}\varepsilon^{2},$$

which is satisfied for all $\varepsilon \in \mathbb{R}$, a fortiori for $\varepsilon \in [0, 1]$.

Thus, it is proved that all the conditions of Theorem 15 are fulfilled and $f$ and $g$ have a unique common fixed point (which is $\frac{1}{3}$).

**Definition 5** [27,30] Let $X$ be an arbitrary vector space over the field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$.

(a) A function $\rho : X \to [0, \infty)$ is called modular if for all $x, y \in X$

  (i) $\rho(x) = 0$ if and only if $x = 0$;
  (ii) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$;
  (iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \ge 0$, $\beta \ge 0$.

(b) If (iii) is replaced by (iv) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \ge 0$, $\beta \ge 0$, then $\rho$ is called convex modular.
(c) A modular function $\rho$ defines the corresponding modular space, i.e., the vector space $X_\rho$ given by $X_\rho = \left\{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\right\}$.

The following is a Pata-type extension of the contraction mapping principle in modular spaces.

**Theorem 16** [32] *Let $X_\rho$ be a modular function space, $C$ be a non-empty, $\rho$-complete and $\rho$-bounded subset of $X_\rho$. Let $\alpha \ge 1$, $\beta > 0$ and $k \ge 0$ be fixed constants and $\psi \in \Psi$. Let $f : C \to C$ be such that for all $x, y \in C$ and each $\varepsilon \in [0, 1]$,*

$$\rho(fx - fy) \le (1 - \varepsilon)\, \rho(x - y) + \varepsilon^\alpha \psi(\varepsilon) \left[\rho(x - y) + k\right]^\beta,$$

*holds. Then $f$ has a unique fixed point.*

## 8 An Open Question

The concept in the Pata-type inequality can be combined with several other metric inequalities known already in the literatures to yield Pata-versions of these mappings. It is perceived that investigations of fixed points of those mappings are worthy of being studied because they may yield effective generalizations of many established fixed point theorems. As an instance, we pose the following (possible) generalization of Ćirić's quasicontraction principle (see [36]).

***Question 1*** Prove or disprove the following. Let $f : X \to X$ and let there exist $\psi \in \Psi$ and constants $\Lambda \ge 0$, $\alpha \ge 1$ and $\beta \in [0, \alpha]$ such that the inequality

$$\begin{aligned} d(fx, fy) \le{}& (1 - \varepsilon) \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\} \\ &+ \Lambda\varepsilon^\alpha \psi(\varepsilon)[1 + \|x\| + \|y\|]^\beta \end{aligned}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$. Then $f$ has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to $z$.

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