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Pell and Pell–Lucas Numbers as Sums of Two Repdigits

Chèfiath Adegbindin[1] · Florian Luca[2] · Alain Togbé[3]

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Abstract
In this paper, we find all Pell and Pell–Lucas numbers expressible as sums of two base 10 repdigits.

Keywords Pell equation · Repdigit · Linear forms in complex and p-adic logarithms

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1 Introduction

Let $g \geq 2$ be an integer. A natural number N is called a *base g repdigit* if all of its base g-digits are equal; that is, if

$$N = a\left(\frac{g^m - 1}{g - 1}\right), \quad \text{for some} \quad m \geq 1 \text{ and } a \in \{1, 2, \ldots, g - 1\}.$$

When $g = 10$, we omit the base and simply say that N is a repdigit. Diophantine equations involving repdigits were also considered in several papers which found all

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✉ Alain Togbé
atogbe@pnw.edu

Chèfiath Adegbindin
adegbindinchefiath@gmail.com

Florian Luca
florian.luca@wits.ac.za

1 Institut de Mathématiques et de Sciences Physiques, Dangbo, Bénin

2 School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa

3 Department of Mathematics, Statistics and Computer Science, Purdue University Northwest, 1401 S, U.S. 421, Westville, IN 46391, USA

repdigits which are perfect powers, or Fibonacci numbers, or generalized Fibonacci numbers, and so on (see [1,2,4,9,11–13] for a sample of such results).

Let $\{P_m\}_{m\geq 0}$ be the Pell sequence given by

$$P_{m+2} = 2P_{m+1} + P_m, \tag{1}$$

for $m \geq 0$, where $P_0 = 0$ and $P_1 = 1$. A few terms of this sequence are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, \ldots$$

We let $\{Q_m\}_{m\geq 0}$ be the companion Lucas sequence of the Pell sequence given by $Q_{m+2} = 2Q_{m+1} + Q_m$, for $m \geq 0$, where $Q_0 = 2$ and $Q_1 = 2$. Its first few terms are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, 551614, \ldots$$

In [8], it was shown that there are no Pell or Pell–Lucas numbers larger than 10 with only one distinct digit.

Here, we extend this and prove the following results.

Theorem 1.1 *The largest Pell number which is a sum of two repdigits is*

$$P_6 = 70 = 4 + 66. \tag{2}$$

Theorem 1.2 *The largest Pell–Lucas number which is a sum of two repdigits is*

$$Q_6 = 198 = 99 + 99. \tag{3}$$

We organize this paper as follows: In Sect. 2, we recall some elementary properties of Pell and Pell–Lucas numbers, a result due to Matveev concerning a lower bound for a linear form in logarithms of algebraic numbers, as well as a variant of a reduction result due to Baker and Davenport reduction. The proofs of Theorems 1.1 and 1.2 are achieved in Sects. 3, 4, respectively. We start with some elementary considerations.

2 Preliminaries

2.1 Some Properties of Pell and Pell–Lucas Numbers

In this subsection, we recall some properties of Pell sequence. Binet's formula says that

$$P_m = \frac{\alpha^m - \beta^m}{2\sqrt{2}}, \tag{4}$$

for all $m \geq 0$, where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ are the two roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell sequence.

The sequence of Pell–Lucas numbers $\{Q_m\}_{m\geq 0}$ starts with $Q_0 = 2,\ \ Q_1 = 2$ and obeys the same recurrence relation

$$Q_{m+2} = 2Q_{m+1} + Q_m, \tag{5}$$

for all $m \geq 0$ as Pell sequence. Its Binet formula is

$$Q_m = \alpha^m + \beta^m, \quad \text{for all} \quad m \geq 0. \tag{6}$$

2.2 Linear Forms in Logarithms

We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 of [3], which is a modified version of a result of Matveev [14]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log |\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is Theorem 9.4 in [3].

Theorem 2.1 *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

2.3 The Baker–Davenport Lemma

Here, we present a variant of the reduction method of Baker and Davenport due to de Weger [6].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{7}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{8}$$

$$X \leq X_0. \tag{9}$$

When $\beta = 0$ in (7), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.1 (See Lemma 3.2 in [6]) *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (8) *and* (9) *hold for* x_1, x_2 *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{10}$$

When $\beta \neq 0$ in (7), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we let $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.2 (See Lemma 3.3 in [6]) *Suppose that*

$$\parallel q\psi \parallel > \frac{2X_0}{q}.$$

Then, the solutions of (8) *and* (9) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

3 The Proof of Theorem 1.1

3.1 An Elementary Estimate

We assume that

$$P_n = d_1\left(\frac{10^{m_1}-1}{9}\right) + d_2\left(\frac{10^{m_2}-1}{9}\right), \tag{11}$$

for some integers $m_1 \le m_2$ and $d_1, d_2 \in \{1, 2, \ldots, 9\}$. A quick computation with Maple reveals no solutions in the interval $n \in [7, 1000]$. So, from now on, we may assume that $n > 1000$. For this computation, we first note that P_{1000} has 383 digits. Thus, we generated the list of all repdigits with at most 383 digits; let us call it $\mathcal{A}$. Then, for every $n \in [7, 1000]$, we computed $M := \lfloor \log P_n / \log 10 \rfloor + 1$ (the number of digits of P_n) and then checked whether $P_n - d(10^m - 1)/9$ is a member of $\mathcal{A}$, for some digit $d \in \{1, \ldots, 9\}$ and some $m \in \{M-1, M\}$. This computation took a few minutes.

Lemma 3.1 *All solutions of Eq.* (11) *satisfy*

$$m_2 \log 10 - 3 < n \log \alpha < m_2 \log 10 + 3.$$

Proof The proof follows easily from the fact that $\alpha^{n-2} < P_n < \alpha^{n-1}$, for $n \ge 2$. One can see that

$$\alpha^{n-2} < P_n < 2 \cdot 10^{m_2}.$$

Taking the logarithm of all sides, we get $(n-2)\log\alpha < \log 2 + m_2 \log 10$, which leads to

$$n \log \alpha < 2\log\alpha + \log 2 + m_2 \log 10 < m_2 \log 10 + 3.$$

The lower bound follows similarly. □

We first return to Eq. (11) and use the Binet formula (4) to get

$$\frac{\alpha^n - \beta^n}{2\sqrt{2}} = d_1\left(\frac{10^{m_1}-1}{9}\right) + d_2\left(\frac{10^{m_2}-1}{9}\right);$$

i.e.,

$$\frac{9}{2\sqrt{2}}(\alpha^n - \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} = -(d_1 + d_2). \tag{12}$$

We examine (12) in two different steps as follows.
Step 1 Equation (12) gives

$$\frac{9}{2\sqrt{2}}\alpha^n - d_2 10^{m_2} = d_1 10^{m_1} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2),$$

which we rewrite as

$$\left|\frac{9}{2\sqrt{2}}\alpha^n - d_2 10^{m_2}\right| = \left|d_1 10^{m_1} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2)\right| < 30 \cdot 10^{m_1}.$$

Thus, dividing both sides by $d_2 10^{m_2}$, we get

$$\left|\left(\frac{9}{2\sqrt{2}d_2}\right)\alpha^n 10^{-m_2} - 1\right| < \frac{30}{10^{m_2 - m_1}}. \tag{13}$$

Put

$$\Gamma := \left(\frac{9}{2\sqrt{2}d_2}\right)\alpha^n 10^{-m_2} - 1. \tag{14}$$

We compare this upper bound with the lower bound on the quantity Γ given by Theorem 2.1. Observe first that Γ is not zero, for if it were, then $\sqrt{2} = q\alpha^n$, with some $q \in \mathbb{Q}$, and hence $\alpha^{2n} = 2q^{-2} \in \mathbb{Q}$, which is false for any $n > 0$. We take

$$\eta_1 = \frac{9}{2\sqrt{2}d_2},\ \ \eta_2 = \alpha,\ \ \eta_3 = 10,\ \ d_1 = 1,\ \ d_2 = n,\ \ d_3 = -m_2,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $d_1,\ d_2,\ d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}} = 2$.

Since $10^{m_2 - 1} < P_n < \alpha^{n-1}$, we have that $m_2 < n$. Therefore, we can take $D = n$. We note also that the conjugates of $\eta_1,\ \eta_2$, and η_3 are $\eta_1' = -\eta_1,\ \eta_2' = \beta,\ \eta_3' = \eta_3$. Furthermore, $\eta_1,\ \eta_2$ are algebraic integers, while the minimal polynomial of η_1 over $\mathbb{Q}$ is

$$(X - \eta_1)(X - \eta_1') = X^2 - \frac{81}{8d_2^2}.$$

Hence, we get

$$h(\eta_1) \le h(9) + h(2d_2\sqrt{2}) \le h(9) + h(18) + h(\sqrt{2}),$$

which implies that

$$2h(\eta_1) < 10.9.$$

Thus, we can take

$$A_1 := 10.9.$$

Clearly,

$$h(\eta_2) = \frac{1}{2}\log\alpha, \quad h(\eta_3) = \log(10).$$

We have

$$\max\{2h(\eta_2), |\log\eta_2|, 0.16\} = \log(\alpha) < 0.9 := A_2.$$

$$\max\{2h(\eta_3), |\log\eta_3|, 0.16\} = 2\log(10) < 4.7 := A_3.$$

Theorem 2.1 tells us that

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (13) leads to

$$(m_2 - m_1)\log(10) < \log(30) + 5 \cdot 10^{13}(1 + \log n)$$

giving

$$(m_2 - m_1)\log(10) < 5.1 \cdot 10^{13}(1 + \log n).$$

Thus, we obtain

$$m_2 - m_1 < 2.21 \cdot 10^{13}(1 + \log n). \tag{15}$$

Step 2 Equation (12) becomes

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{d_1 10^{m_1} + d_2 10^{m_2}}{9} = \frac{\beta^n}{2\sqrt{2}} - \frac{d_1 + d_2}{9},$$

which we rewrite as

$$\left|\frac{\alpha^n}{2\sqrt{2}} - 10^{m_2}\left(\frac{d_1 10^{m_1 - m_2} + d_2}{9}\right)\right| = \left|\frac{\beta^n}{2\sqrt{2}} - \frac{d_1 + d_2}{9}\right| < 3.$$

Thus, dividing both sides by $\alpha^n/2\sqrt{2}$, we get

$$\left|1-\alpha^{-n}10^{m_2}\left(\frac{2\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right)\right| < \frac{3\alpha^2}{\alpha^n} < \frac{1}{\alpha^{n-3.25}}. \tag{16}$$

Put

$$\Gamma' := 1-\alpha^{-n}10^{m_2}\left(\frac{2\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right). \tag{17}$$

Suppose that $\Gamma' = 0$. We then have

$$\alpha^n = 2\sqrt{2}\left(\frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9}\right).$$

Conjugating in $\mathbb{Q}(\sqrt{2})$, we get

$$\beta^n = -2\sqrt{2}\left(\frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9}\right).$$

Consequently, we obtain

$$\frac{4\sqrt{2}\cdot 10^{m_1}}{9} \le 2\sqrt{2}\left(\frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9}\right) = |\beta|^n < 1,$$

which leads to a contradiction as $m_1 \ge 1$. Thus, $\Gamma' \ne 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \left(\frac{2\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right),\ \eta_2=\alpha,\ \eta_3=10,\ d_1=1,\ d_2=-n,\ d_3=m_2,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $d_1,\ d_2,\ d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}} = 2$.

Since $1 \le m_1 \le m_2$ and $m_2 < n$, we can take $D = n$. We now discuss the A_j's for $j = 1, 2, 3$. We have

$$\begin{aligned} h(\eta_1) &\le h\left(2\sqrt{2}\left(\frac{d_1 10^{m_1-m_2}+d_2}{9}\right)\right) \\ &\le h(\sqrt{2})+h(18)+h(d_1 10^{m_1-m_2}+d_2) \\ &\le \frac{1}{2}\log 2+h(18)+h(d_1)+h(d_2)+(m_2-m_1)h(10)+\log 2 \\ &\le 8.32+2.3(m_2-m_1); \end{aligned}$$

i.e.,

$$2h(\eta_1) < 16.64 + 4.6(m_2 - m_1).$$

Thus, we can take

$$A_1 := 16.64 + 4.6(m_2 - m_1).$$

Clearly, we get

$$h(\eta_2) = \frac{1}{2}\log\alpha, \quad h(\eta_3) = \log(10).$$

We have

$$\max\{2h(\eta_2), |\log\eta_2|, 0.16\} = \log(\alpha) < 0.9 := A_2.$$
$$\max\{2h(\eta_3), |\log\eta_3|, 0.16\} = 2\log(10) < 4.7 := A_3.$$

Theorem 2.1 implies that

$$\log|\Gamma_2| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (16) leads to

$$(n - 3.25)\log(\alpha) < 5 \cdot 10^{12}(16.64 + 4.6(m_2 - m_1))(1 + \log n).$$

Hence, using inequality (15), we obtain

$$n\log(\alpha) - \log(\alpha^{3.25}) < 5 \cdot 10^{12}(16.64 + 4.6(2.21 \cdot 10^{13}(1 + \log n)))(1 + \log n).$$

The above inequality gives us

$$n < 3 \cdot 10^{30}.$$

Lemma 3.1 implies

$$m_1 \le m_2 < 1.2 \cdot 10^{30}.$$

We summarize what we have proved so far in the following lemma.

Lemma 3.2 *All solutions of Eq.* (11) *satisfy*

$$m_1 \le m_2 < 1.2 \cdot 10^{30}, \quad n < 3 \cdot 10^{30}.$$

3.2 Bound Reduction

To lower this bound, we return to Eq. (11). We rewrite it into the form

$$P_n = \frac{d_2 10^{m_2}}{9} + \left(d_1 \frac{10^{m_1} - 1}{9} - \frac{d_2}{9}\right).$$

Observe that the term in parentheses is always positive or zero and is zero only when $d_1 = m_1 = 1$ and $d_2 = 9$. In this last case, we get $P_n = 10^{m_2}$, but such a relation is not possible for $n > 1000$, because by the primitive divisor theorem (see [5]), the Pell number P_n has a prime factor $\geq n - 1$, for all values of $n > 12$. Thus, the number appearing in parentheses is $\geq 1/9$. Hence,

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{d_2 10^{m_2}}{9} = \left(d_1 \frac{10^{m_1} - 1}{9} - \frac{d_2}{9}\right) + \frac{\beta^n}{2\sqrt{2}} \geq \frac{1}{9} - \frac{1}{2\sqrt{2}\alpha^{1000}} > 0.$$

Hence, from the above notations given by (13) and (14), we have

$$0 < \eta_1 \eta_2^n \eta_3^{-m_2} - 1 < \frac{30}{10^{m_2 - m_1}}.$$

Put

$$\Lambda = -m_2 \log 10 + n \log \alpha + \log\left(\frac{9}{2d_2\sqrt{2}}\right).$$

We obtain that

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{d_2 10^{m_2}}{9} = \frac{d_2 10^{m_2}}{9}\left(e^{\Lambda} - 1\right) > 0,$$

so

$$0 < \Lambda < e^{\Lambda} - 1 = \Gamma < \frac{30}{10^{m_2 - m_1}},$$

which implies that

$$0 < \log\left(\frac{9}{2d_2\sqrt{2}}\right) + m_2(-\log 10) + n \log \alpha < \frac{30}{10^{m_2 - m_1}}$$
$$< 10^{1.5} \exp(-2.3 \cdot (m_2 - m_1)).$$

Thus, we see that

$$\Lambda < 10^{1.5} \exp(-2.3Y)$$

holds with $Y := m_2 - m_1 < m_2 < 1.2 \cdot 10^{30}$.

We also have

$$\frac{\Lambda}{\log 10} = \frac{\log(9/(2d_2\sqrt{2}))}{\log 10} + n\frac{\log\alpha}{\log 10} - m_2.$$

Thus, we take

$$c = 10^{1.5},\ \delta = 2.3,\ X_0 = 1.2\cdot 10^{30},\ \psi = \frac{\log(9/(2d_2\sqrt{2}))}{\log 10},$$
$$\vartheta = -\frac{\log\alpha}{\log 10},\ \vartheta_1 = \log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(9/2d_2\sqrt{2}).$$

The smallest value of $q > X_0$ is $q = q_{69}$. We find that q_{72} satisfies the hypothesis of Lemma 2.2 for $d_2 = 1, \ldots, 9$. Applying it, we get $m_2 - m_1 = Y \leq 36$. We now take $0 \leq m_2 - m_1 \leq 36$.

Put

$$\Lambda' = -n\log\eta_2 + m_2\log\eta_3 + \log\eta_1.$$

From Eq. (12), we have that

$$\frac{\alpha^n}{2\sqrt{2}}(1 - e^{\Lambda'}) = \frac{\beta^n}{2\sqrt{2}} - (d_1 + d_2)/9 = -\left((d_1 + d_2)/9 - \frac{(-1)^n}{2\sqrt{2}\alpha^n}\right).$$

Furthermore, we obtain

$$\frac{d_1 + d_2}{9} - \frac{(-1)^n}{2\sqrt{2}\alpha^n} > \frac{2}{9} - \frac{1}{2\sqrt{2}\alpha^{1000}} > 0.$$

Thus, one can see that

$$e^{\Lambda'} - 1 > 0.$$

Hence, $\Lambda' > 0$, and so from (23) we see that

$$0 < \Lambda' < e^{\Lambda'} - 1 = |\Gamma'| < \frac{1}{\alpha^{n-3.25}},$$

which implies that

$$0 < \log\left(\frac{2\sqrt{2}(d_2 10^{m_1 - m_2} + d)}{9}\right) + m_2\log 10 + n(-\log\alpha) < \frac{1}{\alpha^{n-3.25}}$$
$$< \alpha^{3.25}\exp(-0.88\cdot n).$$

We consider $X_0 = 3 \cdot 10^{30}$

$$\psi' = \frac{\log(2\sqrt{2}(d_1 10^{m_1-m_2} + d_2)/9)}{\log 10}, \ c = \alpha^{3.25}, \ \delta = 0.88,$$
$$\vartheta = \frac{\log \alpha}{\log 10}, \ \vartheta_1 = -\log \alpha, \ \vartheta_2 = \log 10, \ \beta = \log(2\sqrt{2}(d_1 10^{m_1-m_2} + d_2)/9).$$

We get $q = q_{78} > X_0$, and by Lemma 2.2, we obtain $n \leq 108$. This contradicts the assumption that $n > 1000$. Therefore, the theorem is proved.

4 The Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We may sometimes omit some details.

4.1 An Elementary Estimate

We assume that

$$Q_n = d_1 \left(\frac{10^{m_1} - 1}{9} \right) + d_2 \left(\frac{10^{m_2} - 1}{9} \right), \tag{18}$$

for some integers $m_1 \leq m_2$ and $d_1, d_2 \in \{1, 2, \ldots, 9\}$. A quick computation with Maple reveals no solutions in the interval $n \in [7, 1000]$. So, from now on, we may assume that $n > 1000$. For this computation, we first note that Q_{1000} has 383 digits. Thus, we generated the list of all repdigits with at most 383 digits; let us call it $\mathcal{A}$. Then, for every $n \in [7, 1000]$, we computed $M := \lfloor \log Q_n / \log 10 \rfloor + 1$ (the number of digits of Q_n) and then checked whether $Q_n - d\frac{10^m-1}{9}$ is a member of $\mathcal{A}$, for some digit $d \in \{1, \ldots, 9\}$ and some $m \in \{M-1, M\}$. This computation took a few minutes.

Lemma 4.1 *All solutions of Eq.* (18) *satisfy*

$$m_2 \log 10 - 4 < n \log \alpha < m_2 \log 10 + 2.$$

Proof The proof follows easily from the fact that $\alpha^{n-1} < Q_n < \alpha^{n+1}$, for $n \geq 1$. One can see that

$$\alpha^{n-1} < Q_n < 2 \cdot 10^{m_2}.$$

Taking the logarithm of all sides, we get $(n-1)\log \alpha < \log 2 + m_2 \log 10$, which leads to

$$n \log \alpha < \log \alpha + \log 2 + m_2 \log 10 < m_2 \log 10 + 2.$$

Similarly, we get the lower bound. □

Next, we return to Eq. (18) and use Binet's formula (6) to get

$$\alpha^n + \beta^n = d_1\left(\frac{10^{m_1}-1}{9}\right) + d_2\left(\frac{10^{m_2}-1}{9}\right);$$

i.e.,

$$9(\alpha^n + \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} = -(d_1 + d_2). \tag{19}$$

We examine (19) in two different steps as follows.
Step 1 Equation (19) gives

$$9\alpha^n - d_2 10^{m_2} = d_1 10^{m_1} - 9\beta^n - (d_1 + d_2),$$

which we rewrite as

$$\left|9\alpha^n - d_2 10^{m_2}\right| = \left|d_1 10^{m_1} - 9\beta^n - (d_1 + d_2)\right| < 36 \cdot 10^{m_1}.$$

Thus, dividing both sides by $d_2 10^{m_2}$, we get

$$\left|\left(\frac{9}{d_2}\right)\alpha^n 10^{-m_2} - 1\right| < \frac{36}{10^{m_2-m_1}}. \tag{20}$$

Put

$$\Gamma_1 := \left(\frac{9}{d_2}\right)\alpha^n 10^{-m_2} - 1. \tag{21}$$

We apply Theorem 2.1 to Γ_1. Observe first that Γ_1 is not zero, for if it were, then $\alpha^n = \frac{d_2 10^{m_2}}{9}$. We get that $\alpha^n \in \mathbb{Q}$, which is false, for any $n > 0$. We take

$$\eta_1 = \frac{9}{d_2},\ \eta_2 = \alpha,\ \eta_3 = 10,\ d_1 = 1,\ d_2 = n,\ d_3 = -m_2,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\alpha)$ and $d_1,\ d_2,\ d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\alpha)$ is $d_{\mathbb{L}} = 2$.

Since $10^{m_2-1} < Q_m < \alpha^{n+1}$, we have that $m_2 < n$. Therefore, we can take $D = n$. We note also that the conjugates of η_1, η_2, and η_3 are $\eta_1' = \eta_1, \eta_2' = \beta, \eta_3' = \eta_3$. Furthermore, η_2 is algebraic integer, while the minimal polynomial of η_1 over $\mathbb{Q}$ is

$$(X - \eta_2)(X - \eta_2') = X^2 - 2X - 1.$$

Hence, we have $h(\eta_1) \leq h(9) + h(d_2) \leq h(9) + h(9) \leq 2h(9)$. This implies that $2h(\eta_1) < 8.8$. Thus, as in the previous section we can take

$$A_1 := 8.8,\ A_2 := 0.9,\ A_3 := 4.7.$$

From Theorem 2.1, we obtain

$$\log|\Gamma_1| > -1.4\cdot 30^{l+3}l^{4.5}d_{\mathbb{L}}^2(1+\log d_{\mathbb{L}})(1+\log D)A_1A_2A_3.$$

Comparing this last inequality with (20) leads to

$$(m_2-m_1)\log 10 < \log(36)+4\cdot 10^{13}(1+\log n),$$

giving

$$m_2-m_1 < 1.74\cdot 10^{13}(1+\log n). \tag{22}$$

Step 2 Equation (19) becomes

$$\alpha^n-\left(d_1 10^{m_1}+d_2 10^{m_2}\right)/9 = -\beta^n-\frac{d_1+d_2}{9},$$

which we rewrite as

$$\left|\alpha^n-10^{m_2}\left(d_1 10^{m_1-m_2}+d_2\right)/9\right| = \left|-\beta^n-(d_1+d_2)/9\right| < 3.$$

Thus, dividing both sides by α^n, we get

$$\left|1-\alpha^{-n}10^{m_2}(d_1 10^{m_1-m_2}+d_2)/9\right| < \frac{3}{\alpha^n} < \frac{1}{\alpha^{n-1.3}}. \tag{23}$$

Put

$$\Gamma_1' := 1-\alpha^{-n}10^{m_2}(d_1 10^{m_1-m_2}+d_2)/9. \tag{24}$$

Suppose that $\Gamma_1'=0$. Then, we have

$$\alpha^n = \frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9}.$$

Conjugating in $\mathbb{Q}(\alpha)$, we get

$$\beta^n = \frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9}.$$

Consequently, we obtain

$$\frac{2\cdot 10^{m_1}}{9} \le \frac{d_1 10^{m_1}}{9}+\frac{d_2 10^{m_2}}{9} = |\beta|^n < 1,$$

which is impossible for any $m_1 \ge 1$. Thus, $\Gamma_1' \neq 0$. To apply Theorem 2.1, we take

$$\eta_1 = \frac{d_1 10^{m_1-m_2}+d_2}{9},\ \eta_2=\alpha,\ \eta_3=10,\ d_1=1,\ d_2=-n,\ d_3=m_2,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\alpha)$ and $d_1,\ d_2,\ d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} = \mathbb{Q}(\alpha)$ is $d_{\mathbb{L}} = 2$.

As $1 \le m_1 \le m_2$ and $m_2 < n$, we can take $D = n$. Furthermore, we have

$$\begin{aligned} h(\eta_1) &\le h\left(\frac{d_1 10^{m_1-m_2} + d_2}{9}\right) \\ &\le h(9) + h(d_1 10^{m_1-m_2} + d_2) \\ &\le h(9) + h(d_1) + h(d_2) + (m_2 - m_1)h(10) + \log 2 \\ &\le 7.28 + 2.3(m_2 - m_1); \end{aligned}$$

i.e.,

$$2h(\eta_1) < 14.56 + 4.6(m_2 - m_1).$$

Thus, as before we take

$$A_1 := 14.56 + 4.6(m_2 - m_1),\ \ A_2 := 0.9,\ \ A_3 := 4.7.$$

Applying Theorem 2.1 to Γ_1', we get

$$\log|\Gamma_1'| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (23) leads to

$$n \log \alpha - \log \alpha^{1.3} < 5 \cdot 10^{12}(14.56 + 4.6(m_2 - m_1))(1 + \log n).$$

Hence, using inequality (15), we obtain

$$n \log \alpha - \log \alpha^{1.3} < 5 \cdot 10^{12}(14.56 + 4.6(1.74 \cdot 10^{13}(1 + \log n)))(1 + \log n).$$

The above inequality gives us

$$n < 2.3 \cdot 10^{30}.$$

Lemma 4.1 implies

$$m_1 \le m_2 < 8.9 \cdot 10^{29}.$$

We summarize what we have proved so far in the following lemma.

Lemma 4.2 *All solutions of Eq.* (18) *satisfy*

$$m_1 \le m_2 < 8.9 \cdot 10^{29},\ \ n < 2.3 \cdot 10^{30}.$$

4.2 Bound Reduction

To lower the above bounds, we return to Eq. (18). We rewrite it into the form

$$Q_n = \frac{d_2 10^{m_2}}{9} + \left(d_1 \frac{10^{m_1} - 1}{9} - \frac{d_2}{9}\right).$$

Observe that the term in parentheses is always positive or zero and is zero only when $d_1 = m_1 = 1$ and $d_2 = 9$. In this last case, we get $Q_n = 10^{m_2}$, but such a relation is not possible for $n > 1000$, because by the primitive divisor theorem (see [5]), the Pell–Lucas number Q_n has a prime factor $\geq n - 1$, for all values of $n > 12$. Thus, the number appearing in parentheses is $\geq 1/9$. Hence, one can see that

$$\alpha^n - \frac{d_2 10^{m_2}}{9} = \left(d_1 \frac{10^{m_1} - 1}{9} - \frac{d_2}{9}\right) - \beta^n \geq \frac{1}{9} - \frac{1}{\alpha^{1000}} > 0.$$

From Γ_1 given by (21), we have

$$0 < \eta_1 \eta_2^n \eta_3^{-m_2} - 1 < \frac{36}{10^{m_2 - m_1}}.$$

Let

$$\Lambda_1 = -m_2 \log 10 + n \log \alpha + \log\left(\frac{9}{d_2}\right).$$

We get

$$\alpha^n - \frac{d_2 10^{m_2}}{9} = \frac{d_2 10^{m_2}}{9}\left(e^{\Lambda_1} - 1\right) > 0,$$

so

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \Gamma_1 < \frac{36}{10^{m_2 - m_1}},$$

which implies that

$$\begin{aligned} 0 &< \log\left(\frac{9}{d_2}\right) + m_2(-\log 10) + n \log \alpha < \frac{36}{10^{m_2 - m_1}} \\ &< 10^{1.56} \exp(-2.3 \cdot (m_2 - m_1)). \end{aligned}$$

Thus, one can see that

$$\Lambda_1 < 10^{1.56} \exp(-2.3Y),$$

with $Y := m_2 - m_1 < n < 2.3 \cdot 10^{30}$. We also have that

$$\frac{\Lambda_1}{\log 10} = \frac{\log(9/d_2)}{\log 10} + n\frac{\log\alpha}{\log 10} - m_2.$$

Therefore, we take

$$c = 10^{1.56},\ \delta = 2.3,\ X_0 = 2.3 \cdot 10^{30},\ \psi = \frac{\log(9/d_2)}{\log 10}$$
$$\vartheta = -\frac{\log\alpha}{\log 10},\ \vartheta_1 = \log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(9/d_2).$$

The smallest value of $q > X_0$ is $q = q_{69}$. We find that q_{71} satisfies the hypotheses of Lemma 2.2 for $d_2 = 1, \ldots, 8$ (over all the values of $d_2 \neq 9$). Applying Lemma 2.2, we get $m_2 - m_1 = Y \leq 35$ for $d_2 \neq 9$.

When $d_2 = 9$, we get that $\beta = 0$. The largest partial quotient a_k for $0 \leq k \leq 146$ is $a_{120} = 561$. Applying Lemma 2.1, $m_2 - m_1 = Y < m_2 \leq X_0 := 2.3 \times 10^{30}$ that

$$m_2 - m_1 < \frac{1}{2.3}\log\left(\frac{10^{1.56}(561+2)\cdot 2.3\cdot 10^{30}}{|\log 10|}\right),$$

We obtain $m_2 - m_1 \leq 34$, so we get the same conclusion as before, namely that $m_2 - m_1 \leq 35$.

Taking $1 \leq d_1, d_2 \leq 9$ and $0 \leq m_2 - m_1 \leq 35$, we let

$$\Lambda_1' = -n\log\eta_2 + m_2\log\eta_3 + \log\eta_1.$$

From Eq. (19), we have that

$$\alpha^n(1 - e^{\Lambda_1'}) = -\beta^n - \frac{d_1+d_2}{9} = -\left(\beta^n + \frac{d_1+d_2}{9}\right).$$

Furthermore, one can see that

$$\beta^n + \frac{d_1+d_2}{9} > -\frac{1}{\alpha^n} + \frac{2}{9} > -\frac{1}{\alpha^{1000}} + \frac{2}{9} > 0.$$

Thus, we get

$$e^{\Lambda_1'} - 1 > 0.$$

Hence, $\Lambda_1' > 0$ and so from (23) we see that

$$0 < \Lambda_1' < e^{\Lambda_1'} - 1 = |\Gamma_1'| < \frac{3}{\alpha^n} < \frac{1}{\alpha^{n-1.3}},$$

which implies that

$$0 < \log\left(\frac{d_2 10^{m_2-m_1}+d_1}{9}\right) + m_2 \log 10 + n(-\log\alpha) < \frac{1}{\alpha^{n-1.3}}$$
$$< \alpha^{1.3}\exp(-0.88\cdot n).$$

We keep the same values for X_0 and only change ψ to

$$\psi' = \frac{\log(d_2 10^{m_1-m_2}+d_1/9)}{\log 10},\ \delta = 0.88,\ c = \alpha^{1.3},\ \beta = \log(d_2 10^{m_2-m_1}+d_1/9),$$
$$\vartheta = \frac{\log\alpha}{\log 10},\ \vartheta_1 = -\log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(d_1 10^{m_1-m_2}+d_2/9).$$

We get $q = q_{75} > X_0$. By Lemma 2.2, over all the possibilities for the digits $d_1, d_2 \in \{1, \ldots, 9\}$ and $m_2 - m_1 \in \{0, \ldots, 33\}$ except for $m_1 = m_2$ and $d_1 + d_2 = 9$, we get $n \le 101$.

When $m_2 = m_1$ and $d_1 + d_2 = 9$, we have $\beta = 0$. The largest partial quotient a_k for $0 \le k \le 146$ is $a_{119} = 561$. Applying Lemma 2.1, we get

$$n < \frac{1}{0.88}\cdot\log\left(\frac{\alpha^{1.3}(561+2)\cdot 2.3\cdot 10^{30}}{|\log 10|}\right).$$

We obtain $n < 86$, so we get the same conclusion as before, namely that $n \le 101$. This contradicts the assumption that $n > 1000$. Hence, the theorem is proved.

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