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p-Adic Denseness of Members of Partitions of $\mathbb{N}$ and Their Ratio Sets

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Abstract

The *ratio set* of a set of positive integers A is defined as $R(A) := \{a/b : a, b \in A\}$. The study of the denseness of $R(A)$ in the set of positive real numbers is a classical topic, and, more recently, the denseness in the set of p-adic numbers $\mathbb{Q}_p$ has also been investigated. Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. We prove that for all prime numbers p but at most $\lfloor \log_2 k \rfloor$ exceptions at least one of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_p$. Moreover, we show that for all prime numbers p but at most $k - 1$ exceptions at least one of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_p$. Both these results are optimal in the sense that there exist partitions $A_1, \ldots, A_k$ having exactly $\lfloor \log_2 k \rfloor$, respectively, $k - 1$, exceptional prime numbers; and we give explicit constructions for them. Furthermore, as a corollary, we answer negatively a question raised by Garcia et al.

Keywords Denseness · p-Adic topology · Partition · Quotient set · Ratio set

Mathematics Subject Classification 11A07 · 11B05

1 Introduction

The *ratio set* (or *quotient set*) of a set of positive integers A is defined as

$$R(A) := \{a/b : a, b \in A\}.$$

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The study of the denseness of $R(A)$ in the set of positive real numbers $\mathbb{R}_+$ is a classical topic. For example, Strauch and Tóth [10] (see also [11]) showed that $R(A)$ is dense in $\mathbb{R}_+$ whenever A has lower asymptotic density at least equal to $1/2$. Furthermore, Bukor et al. [3] proved that if $\mathbb{N} = A \cup B$ for two disjoint sets A and B, then at least one of $R(A)$ or $R(B)$ is dense in $\mathbb{R}_+$. On the other hand, Brown et al. [1] showed that there exist pairwise disjoint sets $A, B, C \subseteq \mathbb{N}$ such that $\mathbb{N} = A \cup B \cup C$ and none of $R(A)$, $R(B)$, $R(C)$ is dense in $\mathbb{R}_+$. See also [2,4,7,8] for other related results.

More recently, the study of when $R(A)$ is dense in the p-adic numbers $\mathbb{Q}_p$, for some prime number p, has been initiated. Garcia and Luca [6] proved that the ratio set of the set of Fibonacci numbers is dense in $\mathbb{Q}_p$, for all prime numbers p. Their result has been generalized by Sanna [9], who proved that the ratio set of the k-generalized Fibonacci numbers is dense in $\mathbb{Q}_p$, for all integers $k \geq 2$ and prime numbers p. Furthermore, Garcia et al. [5] gave several results on the denseness of $R(A)$ in $\mathbb{Q}_p$. In particular, they studied $R(A)$ when A is the set of values of Lucas sequences, the set of positive integers which are sum of k squares, respectively, k cubes, or the union of two geometric progressions.

In this paper, we continued the study of the denseness of $R(A)$ in $\mathbb{Q}_p$.

2 Denseness of Members of Partitions of $\mathbb{N}$

Motivated by the results on partitions of $\mathbb{N}$ mentioned in Introduction, the authors of [5] showed that for each prime number p there exists a partition of $\mathbb{N}$ into two sets A and B such that neither $R(A)$ nor $R(B)$ is dense in $\mathbb{Q}_p$ [5, Example 3.6]. Then, they asked the following question [5, Problem 3.7]:

Question 2.1 Is there a partition of $\mathbb{N}$ into two sets A and B such that $R(A)$ and $R(B)$ are dense in no $\mathbb{Q}_p$?[1]

We show that the answer to Question 2.1 is negative. In fact, we will prove even more. Our first result is the following:

Theorem 2.1 *Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $k-1$ exceptions, at least one of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_p$.*

Then, from Theorem 2.1 it follows the next corollary, which gives a strong negative answer to Question 2.1.

Corollary 2.1 *Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $k-1$ exceptions, at least one of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_p$.*

Proof It is easy to prove that if A_j is dense in $\mathbb{Z}_p$ then $R(A_j)$ is dense in $\mathbb{Q}_p$. Hence, the claim follows from Theorem 2.1. □

The proof of Theorem 2.1 requires just a couple of easy preliminary lemmas. For positive integers a and b, define $a + b\mathbb{N} := \{a + bk : k \in \mathbb{N}\}$.

[1] Actually, in [5, Problem 3.7] it is erroneously written "such that A and B are dense in no $\mathbb{Q}_p$," so that the answer is obviously: "Yes, pick any partition into two sets!". Question 2.1 is the intended question.

Lemma 2.2 *Suppose that $(a + b\mathbb{N}) \subseteq A \cup B$ for some positive integers a, b and some disjoint sets $A, B \subseteq \mathbb{N}$. If p is a prime number such that $p \nmid b$ and A is not dense in $\mathbb{Z}_p$, then there exist positive integers c and j such that $(c + bp^j\mathbb{N}) \subseteq B$.*

Proof Since A is not dense in $\mathbb{Z}_p$, there exist positive integers d, j such that $(d + p^j\mathbb{N}) \cap A = \varnothing$. Hence, $(a + b\mathbb{N}) \cap (d + p^j\mathbb{N}) \subseteq B$. The claim follows by the Chinese remainder theorem, which implies that $(a + b\mathbb{N}) \cap (d + p^j\mathbb{N}) = c + bp^j\mathbb{N}$, for some positive integer c. □

Lemma 2.3 *Let a and b be positive integers. Then, $a + b\mathbb{N}$ is dense in $\mathbb{Z}_p$ for all prime numbers p such that $p \nmid b$.*

Proof It follows from the Chinese remainder theorem and the fact that $\mathbb{N}$ is dense in $\mathbb{Z}_p$. □

We are now ready for the proof of Theorem 2.1.

Proof of Theorem 2.1 For the sake of contradiction, suppose that $p_1, \ldots, p_k$ are k pairwise distinct prime numbers such that none of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_{p_i}$ for $i = 1, \ldots, k$. Since A_1 is not dense in $\mathbb{Z}_{p_1}$, there exist positive integers c_1 and j_1 such that $(c_1 + p_1^{j_1}\mathbb{N}) \cap A_1 = \varnothing$. Hence, $(c_1 + p_1^{j_1}\mathbb{N}) \subseteq A_2 \cup \cdots \cup A_k$ and, thanks to Lemma 2.2, there exist positive integers c_2 and j_2 such that $(c_2 + p_1^{j_1} p_2^{j_2}\mathbb{N}) \subseteq A_3 \cup \cdots \cup A_k$. Continuing this process, we get that $(c_{k-1} + p_1^{j_1} \cdots p_{k-1}^{j_{k-1}}\mathbb{N}) \subseteq A_k$, for some positive integers $c_{k-1}, j_1, \ldots, j_{k-1}$. By Lemma 2.3, this last inclusion implies that A_k is dense in $\mathbb{Z}_{p_k}$, but this contradicts the hypotheses. □

Remark 2.1 In fact, Theorem 2.1 can be strengthened in the following way: For each partition $A_1, \ldots, A_k$ of $\mathbb{N}$, there exists a member A_j of this partition which is dense in $\mathbb{Z}_p$ for all but at most $k - 1$ prime numbers p.

Indeed, for the sake of contradiction, suppose that each member A_j of the partition $A_1, \ldots, A_k$ of $\mathbb{N}$ has at least k prime numbers p such that A_j is not dense in $\mathbb{Z}_p$. Then, we can choose prime numbers $p_1, \ldots, p_k$ such that for each $j \in \{1, \ldots, k\}$ the set A_j is not dense in $\mathbb{Z}_{p_j}$. Next, we provide the reasoning from the proof of Theorem 2.1 to reach a contradiction.

The next result shows that the quantity $k - 1$ in Theorem 2.1 cannot be improved.

Theorem 2.4 *Let $k \geq 2$ be an integer, and let $p_1, \ldots, p_{k-1}$ be pairwise distinct prime numbers. Then, there exists a partition $A_1, \ldots, A_k$ of $\mathbb{N}$ such that none of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_{p_i}$ for $i = 1, \ldots, k - 1$.*

Proof Let $e_1, \ldots, e_{k-1}$ be positive integers such that $p_i^{e_i} \geq k$ for $i = 1, \ldots, k - 1$, and put

$$V := \left\{0, \ldots, p_1^{e_1} - 1\right\} \times \cdots \times \left\{0, \ldots, p_{k-1}^{e_{k-1}} - 1\right\}.$$

We shall construct a partition $R_0, \ldots, R_{k-1}$ of V (note that the indices of R_i start from 0) such that if $(r_1, \ldots, r_{k-1}) \in R_j$ then none of the components $r_1, \ldots, r_{k-1}$ is equal to j. Then, we define

$$A_j := \left\{n \in \mathbb{N} : \exists (r_1, \ldots, r_{k-1}) \in R_{j-1},\ \forall i = 1, \ldots, k - 1, \quad n \equiv r_i \pmod{p_i^{e_i}}\right\},$$

for $j = 1, \ldots, k$. At this point, it follows easily that $A_1, \ldots, A_k$ is a partition of $\mathbb{N}$ and that none of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_{p_i}$, since A_{j+1} misses the residue class j (mod $p_i^{e_i}$).

The construction of $R_0, \ldots, R_{k-1}$ is algorithmic. We start with $R_0, \ldots, R_{k-1}$ all empty. Then, we pick a vector $\mathbf{x} \in V$ which is not already in $R_0 \cup \cdots \cup R_{k-1}$. It is easy to see that there exists some $j \in \{0, \ldots, k-1\}$ such that j does not appear as a component of $\mathbf{x}$. We thus throw $\mathbf{x}$ into R_j. We continue this process until all the vectors in V have been picked.

Now, by the construction it is clear that $R_0, \ldots, R_{k-1}$ is a partition of V satisfying the desired property. □

3 Denseness of Ratio Sets of Members of Partitions of $\mathbb{N}$

The result in Corollary 2.1 is not optimal. Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and write $\log_2$ for the base 2 logarithm. Our next result is the following:

Theorem 3.1 *Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $\lfloor \log_2 k \rfloor$ exceptions, at least one of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_p$.*

Before proving Theorem 3.1, we need to introduce some notations. For a prime number p and a positive integer w, we identify the group $(\mathbb{Z}/p^w\mathbb{Z})^*$ with $\{a \in \{1, \ldots, p^w\} : p \nmid a\}$. Moreover, for each $a \in (\mathbb{Z}/p^w\mathbb{Z})^*$ we define

$$(a)_{p^w} := \left\{x \in \mathbb{Q}_p^* : x/p^{\nu_p(x)} \equiv a \bmod p^w\right\},$$

where, as usual, ν_p denotes the p-adic valuation. Note that the family of sets

$$(a)_{p^w} \cap \nu_p^{-1}(s) = \left\{\left(a + rp^w\right) p^s : r \in \mathbb{Z}_p\right\}$$

where w is a positive integer, $a \in (\mathbb{Z}/p^w\mathbb{Z})^*$, and $s \in \mathbb{Z}$, is a basis of the topology of $\mathbb{Q}_p^*$. Finally, for all integers $t \leq m$ and for each set $X \subseteq \mathbb{N}$, we define

$$V_{p^w,t,m} := \left\{(a)_{p^w} \cap \nu_p^{-1}(s) : a \in \left(\mathbb{Z}/p^w\mathbb{Z}\right)^*,\ s \in \mathbb{Z} \cap [t, m-1]\right\}$$

and

$$V_{p^w,t,m}(X) := \left\{I \in V_{p^w,t,m} : X \cap I \neq \varnothing\right\}.$$

Note that the following trivial upper bound holds

$$\#V_{p^w,t,m}(X) \leq \#V_{p^w,t,m} = (m-t)\varphi\left(p^w\right),$$

where φ is the Euler's totient function.

Now, we are ready to state a lemma that will be crucial in the proof of Theorem 3.1.

Lemma 3.2 *Fix a prime number p, two positive integers w, t, a real number $c > 1/2$, and a set $X \subseteq \mathbb{N}$. Suppose that $\#V_{p^w,0,m}(X) \geq cm\,\varphi(p^w)$ for some positive integer $m > t/(2c-1)$. Then, the ratio set $R(X)$ intersects nontrivially with each set in $V_{p^w,0,t}$.*

Proof Given $(a_0)_{p^w} \cap \nu_p^{-1}(s_0) \in V_{p^w,0,t}$, we have to prove that $R(X) \cap (a_0)_{p^w} \cap \nu_p^{-1}(s_0) \neq \varnothing$. For the sake of convenience, define $A := V_{p^w,t,m}(X)$ and

$$B := \left\{(a_0 a)_{p^w} \cap \nu_p^{-1}(s_0 + s) : (a)_{p^w} \cap \nu_p^{-1}(s) \in V_{p^w,t-s_0,m-s_0}(X)\right\}.$$

We have

$$\#A = \#V_{p^w,0,m}(X) - \#V_{p^w,0,t}(X) \geq (cm - t)\varphi\left(p^w\right) > \frac{1}{2}(m-t)\varphi\left(p^w\right), \quad (1)$$

where we used the inequality $m > t/(2c-1)$. Similarly,

$$\begin{aligned}\#B &= \#V_{p^w,0,m}(X) - \#V_{p^w,0,t-s_0}(X) - \#V_{p^w,m-s_0,m}(X) \\ &\geq (cm - (t - s_0) - s_0)\varphi\left(p^w\right) > \frac{1}{2}(m-t)\varphi\left(p^w\right). \quad (2)\end{aligned}$$

Now, A and B are both subsets of $V_{p^w,t,m}$, while $\#V_{p^w,t,m} = (m-t)\varphi(p^w)$. Therefore, (1) and (2) imply that $A \cap B \neq \varnothing$. That is, there exist $(a_1)_{p^w} \cap \nu_p^{-1}(s_1) \in A$ and $(a_2)_{p^w} \cap \nu_p^{-1}(s_2) \in V_{p^w,t-s_0,m-s_0}(X)$ such that $a_1/a_2 \equiv a_0 \pmod{p^w}$ and $s_1 - s_2 = s_0$, so that $R(X) \cap (a_0)_{p^w} \cap \nu_p^{-1}(s_0) \neq \varnothing$, as claimed. □

Proof of Theorem 3.1 For the sake of contradiction, put $\ell := \lfloor \log_2 k \rfloor + 1$ and suppose that $p_1, \ldots, p_\ell$ are ℓ pairwise distinct prime numbers such that none of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_{p_i}$ for $i = 1, \ldots, \ell$. Hence, there exist positive integers w and t such that for each $i \in \{1, \ldots, k\}$ and each $j \in \{1, \ldots, \ell\}$ we have $R(A_i) \cap (a_{i,j})_{p_j^w} \cap \nu_{p_j}^{-1}(s_{i,j}) = \varnothing$, for some $a_{i,j} \in (\mathbb{Z}/p_j^w\mathbb{Z})^*$ and some $s_{i,j} \in \{-(t-1), \ldots, t-1\}$. Clearly, since ratio sets are closed under taking reciprocals, we can assume $s_{i,j} \geq 0$. Put $c := 1/\sqrt[\ell]{k}$, so that $c > 1/2$, and pick a positive integer $m > t/(2c-1)$. There are

$$N := m^\ell \prod_{j=1}^{\ell} \varphi\left(p_j^w\right)$$

sets of the form

$$\bigcap_{j=1}^{\ell} \left((a_j)_{p_j^w} \cap \nu_{p_j}^{-1}(s_j)\right), \quad (3)$$

where $a_j \in (\mathbb{Z}/p_j^w\mathbb{Z})^*$ and $s_j \in \{0, \ldots, m-1\}$. Therefore, there exists $i_0 \in \{1, \ldots, k\}$ such that A_{i_0} intersects nontrivially with at least N/k of the sets of form (3). Consequently, there exists $j_0 \in \{1, \ldots, \ell\}$ such that A_{i_0} intersects nontrivially with at least $cm\varphi(p_{j_0}^w)$ sets of the form $(a)_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s)$, where $a \in (\mathbb{Z}/p_{j_0}^w\mathbb{Z})^*$ and $s \in \{0, \dots, m-1\}$. In other words, $\#V_{p_{j_0}^w,0,m}(A_{i_0}) \geq cm\varphi(p_{j_0}^w)$. Hence, by Lemma 3.2, the set $R(A_{i_0})$ intersects nontrivially with all the sets of the form $(a)_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s)$, where $a \in (\mathbb{Z}/p_{j_0}^w\mathbb{Z})^*$ and $s \in \{0, \dots, t-1\}$, but this is in contradiction with the fact that $R(A_{i_0}) \cap (a_{i_0,j_0})_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s_{i_0,j_0}) = \varnothing$. □

The bound $\lfloor \log_2 k \rfloor$ in Theorem 3.1 is sharp in the following sense:

Theorem 3.3 *Let $k \geq 2$ be an integer, and let $p_1 < \cdots < p_\ell$ be $\ell := \lfloor \log_2 k \rfloor$ pairwise distinct prime numbers. Then, there exists a partition of $\mathbb{N}$ into k sets $A_1, \dots, A_k$ such that none of $R(A_1), \dots, R(A_k)$ is dense in $\mathbb{Q}_{p_i}$ for $i = 1, \dots, \ell$.*

Proof We give two different constructions. Put $h := 2^\ell$ and let $S_1, \dots, S_h$ be all the subsets of $\{1, \dots, \ell\}$. For $j = 1, \dots, h$, define

$$B_j := \{n \in \mathbb{N} : \forall i = 1, \dots, \ell \quad \nu_{p_i}(n) \equiv \chi_{S_j}(i) \pmod{2}\},$$

where χ_{S_j} denotes the characteristic function of S_j. It follows easily that $B_1, \dots, B_h$ is a partition of $\mathbb{N}$ and that none of $R(B_1), \dots, R(B_h)$ is dense in $\mathbb{Q}_{p_i}$, for $i = 1, \dots, \ell$, since each $R(B_j)$ contains only rational numbers with even p_i-adic valuations. Finally, since $h \leq k$, the partition $B_1, \dots, B_h$ can be refined to obtain a partition $A_1, \dots, A_k$ satisfying the desired property.

The second construction is similar. For $j = 1, \dots, h$, define

$$C_j = \left\{ n \in \mathbb{N} : \left(\frac{n/p_i^{\nu_{p_i}(n)}}{p_i} \right) = (-1)^{\chi_{S_j}(i)} \text{ for each } i \in \{1, \dots, \ell\} \right\},$$

where $\left(\frac{a}{p}\right)$ means the Legendre symbol and in case of $p_1 = 2$ we put $\left(\frac{a}{2}\right) = a \pmod{4}$. It follows easily that $C_1, \dots, C_h$ is a partition of $\mathbb{N}$, and that none of $R(C_1), \dots, R(C_h)$ is dense in $\mathbb{Q}_{p_i}$, for $i = 1, \dots, \ell$, since each $R(C_j)$ contains only products of powers of p_i and quadratic residues modulo p_i (in case of $p_1 = 2$ we have only products of powers of 2 and numbers congruent to 1 modulo 4). Finally, since $h \leq k$, the partition $C_1, \dots, C_h$ can be refined to obtain a partition $A_1, \dots, A_k$ satisfying the desired property. □

In light of Remark 2.1, it is worth to ask the following question.

Question 3.1 Let us fix a positive integer k. What then is the least number $m = m(k)$ such that for each partition $A_1, \dots, A_k$ of $\mathbb{N}$ there exists a member A_j of this partition such that $R(A_j)$ is dense in $\mathbb{Q}_p$ for all but at most m prime numbers p?

In virtue of Remark 2.1, we know that $m(k)$ exists and $m(k) \leq k - 1$. On the other hand, by Theorem 3.3 the value $m(k)$ is not less than $\lfloor \log_2 k \rfloor$.

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