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Extensions of a Minimal Third-Order Formally Symmetric Operator

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Abstract
In this paper, we consider some regular boundary value problems generated by a third-order differential equation and some boundary conditions. In particular, we construct maximal self-adjoint, maximal dissipative and maximal accumulative extensions of the minimal operator. Further using Lax–Phillips scattering theory and Sz.-Nagy–Foiaş characteristic function theory we prove a completeness theorem.

Keywords Third-order operator · Extension · Dilation · Spectral analysis

Mathematics Subject Classification Primary 34B05 · 47A20; Secondary 47A40 · 47A48

1 Introduction

The main purpose of this paper is to introduce a method for describing the self-adjoint and non-self-adjoint (dissipative and accumulative) boundary conditions for the solutions of a regular formally symmetric third-order differential equation and investigate the spectral properties of the maximal dissipative boundary value problem with the aid of the characteristic function theory introduced by Sz.-Nagy–Foiaş [1]. But before introducing these problems and the results, we shall give background information about odd-order differential equations and some related results.

An nth-order differential expression can be introduced as follows:

$$\ell_n(y) = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y,$$

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where the coefficients p_r, $0 \leq r \leq n$, are complex-valued functions. Some basic information has been introduced in the literature on this differential expression (for example, see [2–5]). Integration by parts procedure implies that the adjoint differential expression is found as

$$\ell_n^*(y) = (-1)^n \overline{p_n} y^{(n)} + (-1)^{n-1} \overline{p_{n-1}} y^{(n-1)} + \cdots - \overline{p_1} y' + \overline{p_0} y.$$

Moreover, Lagrange's formula can be introduced as follows

$$\int_a^b \left[\ell_n(y)\overline{z} - y\overline{\ell_n^*(z)} \right] \mathrm{d}x = P(y, z) \mid_a^b,$$

where $P(y, z) \mid_a^b$ is a kind of bilinear form obtained from the remainders in the process of integration by parts.

It is well known that the differential expression ℓ_n is formally symmetric provided that it is of the form

$$\ell_n(y) = \sum_{j=0}^{[n/2]} (-1)^j (a_j(x) y^{(j)})^{(j)} + i \sum_{j=0}^{[(n-1)/2]} \left[\left(b_j(x) y^{(j)} \right)' + b_j(x) y^{(j+1)} \right]^{(j)},$$

where a_j and b_j are real-valued functions and $[n/2]$ is the integer part of $[n/2]$. If the formal differential expression ℓ_n is real then the second term disappears, and thus if $\ell_n = \ell_{2k}$ then

$$\ell_{2k}(y) = \sum_{j=0}^{k} (-1)^j \left(a_j(x) y^{(j)} \right)^{(j)},$$

where each a_j is real-valued.

The literature has a huge number of works on even-order differential equations. However, the same is not true for odd-order differential equations even if there are some works on odd-order equations [6–12]. It should be noted that in [9–12] the formally symmetric differential equations are studied with the aid of quasi-derivatives and Shin–Zettl matrices. In particular, in [6] some algebraic properties for the formally symmetric boundary conditions:

$$\sum_{s=1}^{n} \left(M_{rs} y^{(s-1)}(a) + N_{rs} y^{(s-1)}(b) \right) = 0,$$

have been introduced, where $1 \leq r \leq n$, $M = [M_{rs}]$, $N = [N_{rs}]$ are the matrices having complex elements such that

$$\operatorname{rank}[M \mid N] = n.$$

At this stage, to stress the difficulties of working on such problems it is better to share the following quotation [6].

> The odd-order case presents special difficulties since for these equations it is impossible to find separated self-adjoint boundary conditions and this entails a further complication in the analysis of this case.

The same is true for non-self-adjoint boundary conditions. However, we share a method to cope with this difficulty.

In this paper, we introduce the maximal self-adjoint, maximal dissipative and maximal accumulative extensions of the minimal regular third-order differential operator with the aid of the boundary value mappings. Then we investigate some spectral properties of the maximal dissipative extension of the minimal operator using the equivalence of Lax–Phillips scattering function and Sz.-Nagy–Foiaş characteristic function. For this purpose we first construct the scattering function in the dilation space. In fact, on the basis of Lax–Phillips scattering theory [13] there exists a decomposition of the main Hilbert space $\boldsymbol{H}$ as

$$\boldsymbol{H} = D_- \oplus H \oplus D_+,$$

where H is a Hilbert space and D_-, D_+ are subspaces, the so-called incoming and outgoing subspaces, respectively. These subspaces together with a group of unitary operators $\{U(t);\ -\infty < t < \infty\}$ have some certain properties. To construct the scattering function it is applied the Fourier transformations to the functions $f \in \boldsymbol{H}$ to obtain the spectral representations. Then as was stated in [13] to each vector $f \in \boldsymbol{H}$ there are incoming translation representer k_- and outgoing translation representer k_+ and the mapping

$$S : k_- \to k_+$$

is then called the scattering operator. To investigate the spectral properties of the scattering function they use the semigroup $\{Z(t)\}$ as

$$Z(t) = P_+ U(t) P_-,\ t \geq 0,$$

where P_+ is the orthogonal projection of $\boldsymbol{H}$ onto the orthogonal complement of D_+ and P_- is the orthogonal projection of $\boldsymbol{H}$ onto the orthogonal complement of D_-. However, such an approximation has been introduced independently by Sz.-Nagy and Foiaş for the contraction operators with the help of characteristic operator function [1]

$$\Theta_T(\mu) = \Big[-T + \mu D_{T^*}(I - \mu T^*)^{-1} D_T\Big] \mid \mathcal{D}_T,$$

where T is a contraction on the Hilbert space $\mathfrak{H}$, $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$, $\mathcal{D}_T = \overline{D_T \mathfrak{H}}$, $\mathcal{D}_{T^*} = \overline{D_{T^*} \mathfrak{H}}$, and μ is the complex parameter such that $I - \mu T^*$ is boundedly invertible. They use dilation theory and functional model theory for the given contraction, and as they stated at the end of Chap. VI in [1], there exists a connection between Lax–Phillips scattering function and Sz.-Nagy–Foiaş characteristic function.

We should note that such an equivalence has been used for the second-order operators, Dirac operators and even-order Hamiltonian operators (for example, see [14–24]).

However, it seems that such an equivalence has not been used previously for the third-order differential operator. Therefore, this will be the first work on odd-order scalar differential equations together with appropriate dissipative boundary conditions.

2 Third-Order Operators

Throughout the paper, we consider the following differential expression

$$\ell(y) = \frac{1}{w}\left\{-i\left(q_0\left(q_0 y'\right)'\right)' - (p_0 y')' + i\left[q_1 y' + (q_1 y)'\right] + p_1 y\right\}$$

on the interval $[a, b]$. The basic assumptions are as follows.

(i) q_0, q_1, p_0, p_1 and w are continuous, real-valued functions on $[a, b]$,
(ii) $-\infty < a < b < \infty$,
(iii) $q_0 > 0$ (or $q_0 < 0$), $w > 0$ on $[a, b]$.

Let H denote the Hilbert space consisting of all functions y satisfying

$$\int_a^b |y|^2 \, w\mathrm{d}x < \infty$$

with the usual inner product

$$(y, z) = \int_a^b y\bar{z}w\mathrm{d}x.$$

The rth quasi-derivative $y^{[r]}$ of y is defined as follows [8]

$$y^{[0]} = y,\ y^{[1]} = -\tfrac{1+i}{\sqrt{2}} q_0 y',\ y^{[2]} = iq_0(q_0 y')' + p_0 y' - iq_1 y.$$

Consider the equation

$$-i\left(q_0\left(q_0 y'\right)'\right)' - (p_0 y')' + i\left[q_1 y' + (q_1 y)'\right] + p_1 y = \lambda w y + w f, \quad x \in [a, b], \tag{2.1}$$

where $f \in H$. (2.1) can be considered as

$$Y' = [\lambda P + Q] Y + F, \tag{2.2}$$

where

$$Y = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \end{bmatrix}, \quad P = \begin{bmatrix} & & \\ & & \\ -w & & \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{(1+i)q_0} & 0 \\ -\frac{1+i}{\sqrt{2}}\frac{q_1}{q_0} & i\frac{p_0}{q_0^2} & -\frac{\sqrt{2}}{(1+i)q_0} \\ p_1 & -\frac{1+i}{\sqrt{2}}\frac{q_1}{q_0} & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ wf \end{bmatrix}.$$

and all the other entries of P are zero. Using (2.2) one may obtain that there exists a unique solution y of (2.1) satisfying the initial conditions

$$y^{[j]}(c, \lambda) = k_j,$$

where $c \in [a, b]$, $j = 0, 1, 2$, and k_j are arbitrary complex numbers. The function $y^{[j]}(., \lambda)$ is entire in λ.

With a similar discussion as in [4, p. 58], we may infer that the dimension of solutions of (2.1) is 3.

Consider the subspace D of H consisting of all functions $y \in H$ such that $y^{[r]}$, $0 \leq r \leq 2$, are locally absolutely continuous on $[a, b]$ and $\ell(y) \in H$. We define the maximal operator L on D as follows

$$Ly = \ell(y), \ y \in D, \quad x \in [a, b].$$

For $y, z \in D$ one obtains the following Lagrange's identity

$$(Ly, z) - (y, Lz) = [y, \overline{z}] \mid_a^b, \tag{2.3}$$

where $[y, z] \mid_a^b = [y, z](b) - [y, z](a)$ and

$$[y, z] := yz^{[2]} - y^{[2]}z + iy^{[1]}z^{[1]}.$$

(2.3) particulary implies that the value $[y, \overline{z}]$ at the end points a and b exists and is finite for $y, z \in D$.

Now consider the subspaces

$$D_0 = \left\{ y \in D, y^{[r]} = 0, \ \ell(y) \in H \right\}.$$

We define the operator L_0 as the restriction of L to the subspace D_0. Then one obtains L_0 and L are densely defined, closed operators in H, L_0 is symmetric with deficiency indices (3, 3) and $L_0^* = L$, $L^* = L_0$ [8,9,12].

3 Boundary Value Mappings

Now we describe all maximal self-adjoint, maximal dissipative and maximal accumulative extensions of the minimal operator L_0 with the aid of the boundary values. Remind that a triple (S, Γ_1, Γ_2) is a boundary value space of the operator A if for any $f, g \in D(A^*)$ [25]

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_2 g)_S - (\Gamma_2 f, \Gamma_1 g)_S$$

and for any $F_1, F_2 \in S$ there exists a vector $f \in D(A^*)$ such that

$$\Gamma_1 f = F_1, \quad \Gamma_2 f = F_2,$$

where A is a closed symmetric operator on H with equal, finite or infinite deficiency indices.

Now we define the following mappings for $y \in D$

$$\Gamma_1 y = \begin{bmatrix} y^{[2]}(a) \\ \frac{1}{2} y^{[1]}(a) + \frac{i}{2} y^{[1]}(b) \\ y(b) \end{bmatrix}, \; \Gamma_2 y = \begin{bmatrix} y(a) \\ i y^{[1]}(a) + y^{[1]}(b) \\ y^{[2]}(b) \end{bmatrix}.$$

Then we may introduce the following.

Theorem 3.1 $(\mathbb{C}^3, \Gamma_1, \Gamma_2)$ *is a boundary value space of* L_0.

Proof Naimark's patching lemma implies that for any $F_1, F_2 \in \mathbb{C}^3$ there exists a function $y \in D$ such that $\Gamma_1 y = F_1$ and $\Gamma_2 y = F_2$.

Now consider the following

$$\begin{aligned} &(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^3} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^3} \\ &\quad = y^{[2]}(a)\overline{z(a)} + \left(\frac{1}{2} y^{[1]}(a) + \frac{i}{2} y^{[1]}(b)\right)\left(-i\overline{z^{[1]}(a)} + \overline{z^{[1]}(b)}\right) \\ &\qquad + y(b)\overline{z^{[2]}(b)} - \left[y(a)\overline{z^{[2]}(a)} + \left(i y^{[1]}(a) + y^{[1]}(b)\right)\left(\frac{1}{2}\overline{z^{[1]}(a)} - \frac{i}{2}\overline{z^{[1]}(b)}\right) \right. \\ &\qquad \left. + y^{[2]}(b)\overline{z(b)} \right] = [y, \overline{z}] \mid_a^b . \end{aligned} \tag{3.1}$$

Moreover, we have

$$(L_0^* y, z) - (y, L_0^* z) = (Ly, z) - (y, Lz) = [y, \overline{z}] \mid_a^b . \tag{3.2}$$

(3.1) and (3.2) completes the proof. □

Now we may introduce the following with the result given in [25, p. 156].

Theorem 3.2 *Let K be a contraction on $\mathbb{C}^3$. Then the restriction of L_0^* to the set of vectors $f \in D_0^*$ satisfying*

$$(K - I)\Gamma_1 f + i(K + I)\Gamma_2 f = 0,$$

or

$$(K - I)\Gamma_1 f - i(K + I)\Gamma_2 f = 0$$

gives the maximal dissipative, respectively, maximal accumulative extension of L_0. If K is chosen as an isometric operator then

$$(\cos C)\Gamma_2 f - (\sin C)\Gamma_1 f = 0$$

gives a self-adjoint extension of L_0, where C is a self-adjoint operator on $\mathbb{C}^3$.

Since we will investigate the spectral properties of the dissipative extension of the minimal operator we shall introduce the following.

Corollary 3.3 *All maximal dissipative extensions of L_0 can be described by*

$$\begin{aligned}
&y(a) + h_1 y^{[2]}(a) = 0, \quad Imh_1 \geq 0,\\
&iy^{[1]}(a) + y^{[1]}(b) + \widetilde{h}_2\left(\tfrac{1}{2}y^{[1]}(a) + \tfrac{i}{2}y^{[1]}(b)\right) = 0, \quad Im\widetilde{h}_2 \geq 0,\\
&y^{[2]}(b) + h_3 y(b) = 0, \quad Imh_3 \geq 0.
\end{aligned}$$

The second condition can be replaced by the following.

Corollary 3.4 *All maximal dissipative extensions of L_0 can be described by*

$$\begin{aligned}
&y(a) + h_1 y^{[2]}(a) = 0, \quad Imh_1 = 0,\\
&(i + h_2)y^{[1]}(a) + (1 + ih_2)y^{[1]}(b) = 0, \quad Imh_2 > 0, h_2 \neq i,\\
&y^{[2]}(b) + h_3 y(b) = 0, \quad Imh_3 > 0,
\end{aligned} \tag{3.3}$$

where $h_2 = \widetilde{h}_2/2$.

One of our aims is to investigate the spectral properties of the problem generated by the equation

$$\ell(y) = \lambda y,\ y \in D,\ x \in [a, b], \tag{3.4}$$

and boundary conditions (3.3), where λ is the complex parameter. For this purpose, we define the operator T with the rule

$$Ty = \ell(y),\ y \in D(T),\ x \in [a, b],$$

where $D(T)$ is the domain of T consisting of all functions $y \in D$ satisfying conditions (3.3). Therefore, the eigenvalue problem of T coincides with the eigenvalue problem of (3.4), (3.3).

A direct calculation shows that the adjoint operator T^* of T is obtained by the rule

$$Ty = \ell(y),\ y \in D(T^*),\ x \in [a, b],$$

where $D(T^{*})$ is the domain of T^{*} consisting of all functions $y \in D^{*}$ satisfying the conditions

$$\begin{array}{l} y(a) + h_1 y^{[2]}(a) = 0, \quad \mathrm{Im} h_1 = 0, \\ (-i + \overline{h}_2) y^{[1]}(a) + (1 - i\overline{h}_2) y^{[1]}(b) = 0, \quad \mathrm{Im} h_2 > 0, h_2 \neq i, \\ y^{[2]}(b) + \overline{h}_3 y(b) = 0, \quad \mathrm{Im} h_3 > 0. \end{array}$$

Note that T is maximal dissipative in H.

Theorem 3.5 *T is totally dissipative (simple) in H.*

Proof It is enough to show that there does not exist a subspace H_s in H such that $H_s \neq \{0\}$.

Let us assume that T has a self-adjoint part on a subspace H_s of H. Then for $y \in H_s \cap D(T)$ one has

$$\begin{aligned} (Ty, y) - (y, Ty) &= y(b)\overline{y^{[2]}(b)} - y^{[2]}(b)\overline{y(b)} + iy^{[1]}(b)\overline{y^{[1]}(b)} \\ &\quad - \left(y(a)\overline{y^{[2]}(a)} - y^{[2]}(a)\overline{y(a)} + iy^{[1]}(a)\overline{y^{[1]}(a)} \right) \\ &= 2i \mathrm{Im} h_3 \left| y(b) \right|^2 + i \left[\tfrac{i+h_2}{1+ih_2} \tfrac{-i+\overline{h_2}}{1-i\overline{h_2}} - 1 \right] \left| y^{[1]}(a) \right|^2 \\ &= 2i \mathrm{Im} h_3 \left| y(b) \right|^2 + \tfrac{4i \mathrm{Im} h_2}{|1+ih_2|^2} \left| y^{[1]}(a) \right|^2 = 0. \end{aligned}$$

The last equation implies that $y(b) = 0$, $y^{[1]}(a) = 0$ ($y^{[1]}(b) = 0$) and $y^{[2]}(b) = 0$. Consequently, $y \equiv 0$ and this completes the proof. □

4 Self-Adjoint Dilation

In this section, first of all, we want to construct the Lax–Phillips scattering function. For this purpose, we consider the following direct sum Hilbert space

$$\boldsymbol{H} = L^2(\mathbb{R}_-; \mathbb{C}^2) \oplus H \oplus L^2(\mathbb{R}_+; \mathbb{C}^2),$$

where $\mathbb{R}_- : (-\infty, 0]$, $\mathbb{R}_+ := [0, \infty)$ and $L^2(\mathbb{R}_\pm; \mathbb{C}^2)$ are the spaces consisting of all vector functions with ranges in $\mathbb{C}^2$ that are squarely integrable on $\mathbb{R}_\pm$.

Let $D(\mathcal{L})$ be the set in $\boldsymbol{H}$ consisting of all functions $f = \langle \varphi_-, y, \varphi_+ \rangle$ such that

$$\varphi_- = \begin{bmatrix} \varphi_-^1 \\ \varphi_-^2 \end{bmatrix} \in W_2^1(\mathbb{R}_-; \mathbb{C}^2), \; \varphi_+ = \begin{bmatrix} \varphi_+^1 \\ \varphi_+^2 \end{bmatrix} \in W_2^1(\mathbb{R}_+; \mathbb{C}^2),$$

where W_2^1 is the Sobolev space satisfying

$$\varphi_-(0) = \begin{bmatrix} \varphi_-^1(0) \\ \varphi_-^2(0) \end{bmatrix} = \begin{bmatrix} y^{[1]}(a) \\ \frac{1}{\sqrt{2 \mathrm{Im} h_3}} \left(y^{[2]}(b) + h_3 y(b) \right) \end{bmatrix}$$

and

$$\varphi_+(0) = \begin{bmatrix} \varphi_+^1(0) \\ \varphi_+^2(0) \end{bmatrix} = \begin{bmatrix} y^{[1]}(b) \\ \frac{1}{\sqrt{2\mathrm{Im}h_3}} \left(y^{[2]}(b) + \overline{h_3} y(b) \right) \end{bmatrix}.$$

Consider the differential expression

$$\eta(f) = \left\langle i \frac{\mathrm{d}\varphi_-}{\mathrm{d}r}, \ell(y), i \frac{\mathrm{d}\varphi_+}{\mathrm{d}s} \right\rangle.$$

Now we construct the operator

$$\mathcal{L}f = \eta(f), \;\; f \in D(\mathcal{L}).$$

Note that the adjoint operator $\mathcal{L}^*$ is defined by

$$\mathcal{L}^* f = \eta(f), \;\; f \in D(\mathcal{L}^*),$$

where

$$\eta(f) = \left\langle i \frac{\mathrm{d}\varphi_-}{\mathrm{d}r}, \ell(y), i \frac{\mathrm{d}\varphi_+}{\mathrm{d}s} \right\rangle$$

and $\varphi_- \in W_2^1(\mathbb{R}_-; \mathbb{C}^2),\; \varphi_+ \in W_2^1(\mathbb{R}_+; \mathbb{C}^2),\; y \in D$.

Theorem 4.1 *$\mathcal{L}$ is self-adjoint in $\boldsymbol{H}$.*

Proof First of all, we shall show that $D(\mathcal{L}) \subset D(\mathcal{L}^*)$. For $f = \langle \varphi_-, y, \varphi_+ \rangle, g = \langle \psi_-, z, \psi_+ \rangle \in \boldsymbol{H}$ we obtain

$$\begin{aligned} \langle \mathcal{L}f, g \rangle - \langle f, \mathcal{L}g \rangle &= i\,(\varphi_-(0), \psi_-(0)) - i\,(\varphi_+(0), \psi_+(0)) + [y, \overline{z}]\,|_a^b \\ &= i y^{[1]}(a)\overline{z^{[1]}(a)} + \frac{1}{2\mathrm{Im}h_3} \Big[(y^{[2]}(b) + h_3 y(b))(\overline{z^{[2]}(b)} + \overline{h_3 z(b)}) \\ &\quad \times (y^{[2]}(b) + \overline{h_3} y(b))(\overline{z^{[2]}(b)} + h_3 \overline{z(b)}) \Big] - i y^{[1]}(b)\overline{z^{[1]}(b)} + [y, \overline{z}]\,|_a^b = 0. \end{aligned}$$

This implies that $\mathcal{L}$ is formally symmetric in $\boldsymbol{H}$.

Now we shall show that $D(\mathcal{L}^*) \subset D(\mathcal{L})$. For this purpose, first of all we consider $f = \langle \varphi_-, 0, \varphi_+ \rangle$ such that $\varphi_-(0) = \psi_-(0) = 0$ and let $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathcal{L}^*)$. Then

$$\begin{aligned} \langle \mathcal{L}f, g \rangle &= \left\langle \left\langle i \frac{\mathrm{d}\varphi_-}{\mathrm{d}r}, 0, i \frac{\mathrm{d}\varphi_+}{\mathrm{d}s} \right\rangle, \langle \psi_-, z, \psi_+ \rangle \right\rangle \\ &= \left\langle \langle \varphi_-, 0, \varphi_+ \rangle, \left\langle i \frac{\mathrm{d}\psi}{\mathrm{d}r}_-, z^*, i \frac{\mathrm{d}\psi_+}{\mathrm{d}s} \right\rangle \right\rangle, \end{aligned}$$

where $\psi_- \in W_2^1(\mathbb{R}_-; \mathbb{C}^2)$, $\psi_+ \in W_2^1(\mathbb{R}_+; \mathbb{C}^2)$, $z^* \in D(\mathcal{L}^*)$ and $\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}g \rangle$. Therefore, we get

$$\begin{aligned} i\,(\varphi_+(0), \psi_+(0)) - i\,(\varphi_-(0), \psi_-(0)) &= y(b)\overline{z^{[2]}(b)} - y^{[2]}(b)\overline{z(b)} + i y^{[1]}(b)\overline{z^{[1]}(b)} \\ &\quad - i y^{[1]}(a)\overline{z^{[1]}(a)} \end{aligned}$$

or

$$\begin{aligned}
&i\varphi_+^1(0)\overline{\psi_+^1(0)} + i\varphi_+^2(0)\overline{\psi_+^2(0)} - i\varphi_-^1(0)\overline{\psi_-^1(0)} - i\varphi_-^2(0)\overline{\psi_-^2(0)} \\
&\quad = i\varphi_+^2(0)\left[\frac{\overline{z^{[2]}(b)} + h_3\overline{z(b)}}{\sqrt{2\mathrm{Im}h_3}}\right] - i\varphi_-^2(0)\left[\frac{\overline{z^{[2]}(b)} + \overline{h_3 z(b)}}{\sqrt{2\mathrm{Im}h_3}}\right] \\
&\qquad + i\varphi_+^1(0)\overline{z^{[1]}(b)} - i\varphi_-^1(0)\overline{z^{[1]}(a)}. \qquad (4.1)
\end{aligned}$$

Comparing the coefficients of the same factors in (4.1), one obtains $D(\mathcal{L}^*) \subset D(\mathcal{L})$ and this completes the proof. □

Now we may infer that semigroup $Z(t)$ defined by

$$Z(t) = PU(t)P_1, \; 0 \le t < \infty,$$

is a strongly continuous semigroup of completely non-unitary contractions on $\boldsymbol{H}$, where

$$U(t) = \exp(i\mathcal{L}t), \quad -\infty < t < \infty,$$

is the unitary group and

$$\begin{aligned}
P &: \boldsymbol{H} \to H, & P_1 &: H \to \boldsymbol{H}, \\
\langle \varphi_-, y, \varphi_+ \rangle &\to y, & y &\to \langle 0, y, 0 \rangle .
\end{aligned}$$

Consider the operator

$$G = \lim_{t \to 0^+} \frac{Z(t) - I}{it}.$$

G is the generator of $Z(t)$, and $\mathcal{L}$ is called the self-adjoint dilation of G.

Note that G is maximal dissipative.

Theorem 4.2 *$\mathcal{L}$ is self-adjoint dilation of T.*

Proof This fact will be proved by the following

$$(T - \lambda I)^{-1} = (G - \lambda I)^{-1}.$$

For this purpose, we shall consider the following

$$(\mathcal{L} - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle ,$$

where $y \in H$, $g \in D(\mathcal{L})$ and $\mathrm{Im}\lambda < 0$. A direct calculation shows that

$$\ell(z) - \lambda z = y, \; \psi_-(r) = \psi_-(0)e^{-i\lambda r}, \; \psi_+(r) = \psi_+(0)e^{-i\lambda s}.$$

Since $\psi_- \in L^2(\mathbb{R}_-; \mathbb{C}^2)$ and a value λ with $\mathrm{Im}\lambda < 0$ cannot be an eigenvalue of T one obtains

$$(\mathcal{L} - \lambda I)^{-1} P_1 y = \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, (T - \lambda I)^{-1} y, \begin{bmatrix} y^{[1]}(b) \\ \frac{1}{\sqrt{\mathrm{Im} h_3}} \left(y^{[2]}(b) + \overline{h_3} y(b) \right) \end{bmatrix} \right\rangle. \quad (4.2)$$

Applying the mapping P to (4.2) we have

$$P(\mathcal{L} - \lambda I)^{-1} P_1 y = (T - \lambda I)^{-1} y. \quad (4.3)$$

On the other hand, for $\mathrm{Im}\lambda < 0$ the equality

$$P(\mathcal{L} - \lambda I)^{-1} P_1 = -iP \int_0^\infty U(t) e^{-i\lambda t} \mathrm{d}t P_1 = -i \int_0^\infty Z(t) e^{-i\lambda t} \mathrm{d}t = (G - \lambda I)^{-1} \quad (4.4)$$

holds and therefore (4.3) and (4.4) prove that $T = G$. The proof is therefore completed. □

Now we consider the spaces $D_- = \left\langle L^2(\mathbb{R}_-; \mathbb{C}^2), 0, 0 \right\rangle$ and $D_- = \left\langle 0, 0, L^2(\mathbb{R}_+; \mathbb{C}^2) \right\rangle$. Then we have the following.

Theorem 4.3 *The following properties are satisfied:*

(i) $U(t)D_+ \subset D_+$, $t \geq 0$; $U(t)D_- \subset D_-$, $t \leq 0$,
(ii) $\bigcap_{t \leq 0} U(t)D_- = \bigcap_{t \geq 0} U(t)D_+ = \{0\}$,
(iii) $\overline{\bigcup_{t \geq 0} U(t)D_-} = \overline{\bigcup_{t \leq 0} U(t)D_+} = \mathbf{H}$,
(iv) $D_- \perp D_+$.

Proof For $\mathrm{Im}\lambda < 0$ and $f = \langle 0, 0, \psi_+ \rangle \in D_+$ we get

$$(\mathcal{L} - \lambda I)^{-1} f = \left\langle 0, 0, -e^{-i\lambda s} \int_0^s e^{i\lambda t} \psi_+(t) \mathrm{d}t \right\rangle.$$

Hence, $(\mathcal{L} - \lambda I)^{-1} f \in D_+$. For $g \perp D_+$ we get for $\mathrm{Im}\lambda < 0$ that

$$0 = \left\langle (\mathcal{L} - \lambda I)^{-1} f, g \right\rangle = -i \int_0^\infty e^{-i\lambda t} \langle U(t) f, g \rangle \, \mathrm{d}t.$$

This implies that $\langle U(t) f, g \rangle = 0$, $t \geq 0$, and therefore, $U(t)D_+ \subset D_+$, $t \geq 0$. Similarly, one may show that $U(t)D_- \subset D_-$, $t \leq 0$.

Now consider the semigroup of isometries $U_+(t) = \wp U(t) \wp_1$, $t \geq 0$, where

$$\begin{array}{ll} \wp : \boldsymbol{H} \to H, & \wp_1 : H \to \boldsymbol{H}, \\ \langle \varphi_-, y, \varphi_+ \rangle \to \varphi_-, & \varphi \to \langle 0, 0, \varphi \rangle . \end{array}$$

The generator G_1 of the semigroup of isometries is

$$G_1\varphi = \wp\mathcal{L}\wp_1\varphi = \wp\mathcal{L}\langle 0, 0, \varphi\rangle = \wp\left\langle 0, 0, i\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right\rangle = i\frac{\mathrm{d}\varphi}{\mathrm{d}s},$$

where $\varphi \in W_2^1(\mathbb{R}_+; \mathbb{C}^2)$ and $\varphi(0) = 0$. On the other hand, the generator of the semigroup of one-sided shift, say $\widetilde{U}_+(t)$ in $L^2(\mathbb{R}_+; \mathbb{C}^2)$ is the differential operator $id/\mathrm{d}x$ with the boundary condition $\varphi(0) = 0$. Since a semigroup is uniquely determined by its generator, we have $U_+(t) = \widetilde{U}_+(t)$ and hence

$$\bigcap_{t\geq 0} U_+(t)D_+ = \left\langle 0, 0, \bigcap_{t\geq 0} \widetilde{U}_+(t)L^2(\mathbb{R}_+; \mathbb{C}^2)\right\rangle = \{0\}\,.$$

This implies that (ii) holds. A similar proof can be given for D_-.

Now let

$$H^< = \overline{\bigcup_{t\geq 0} U(t)D_-},\ H^> = \overline{\bigcup_{t\leq 0} U(t)D_+}.$$

Our assertion is that $H^< + H^> = H$. Otherwise, there would be a non-trivial subspace $\widetilde{H} = H \ominus (H^< + H^>)$ which would be invariant relative to group $U(t)$ and the restrictions of $U(t)$ to $\widetilde{H}$ were unitary and the restrictions of $U(t)$ to $\widetilde{H}$ were unitary and the restrictions of T on $\widetilde{H}$ were self-adjoint.

Let $\chi(x, \lambda)$ be a solution of (3.4) satisfying the conditions

$$\chi(a, \lambda) = -h_1,\ \chi^{[1]}(a, \lambda) = 0,\ \chi^{[2]}(a, \lambda) = 1.$$

This solution belongs to D. According to Theorem 8 in [9] or Lemma 2.4 in [12] we may infer that χ can be considered as satisfying the conditions

$$\chi^{[1]}(b, \lambda) = 0,$$

and arbitrary values $\chi(b, \lambda)$ and $\chi^{[2]}(b, \lambda)$.

Now we set the vectors

$$V^< = \left\langle \frac{\overline{\chi^{[2]}(b,\lambda) + \overline{h_3}\chi(b,\lambda)}}{\overline{\chi^{[2]}(b,\lambda) + h_3\chi(b,\lambda)}} \begin{bmatrix} e^{-i\lambda r} \\ e^{-i\lambda r} \end{bmatrix}, \frac{1}{\overline{\chi^{[2]}(b,\lambda) + h_3\chi(b,\lambda)}}\chi(x,\lambda), \begin{bmatrix} e^{-i\lambda s} \\ e^{-i\lambda s} \end{bmatrix} \right\rangle \tag{4.5}$$

and

$$V^> = \left\langle \begin{bmatrix} e^{-i\lambda r} \\ e^{-i\lambda r} \end{bmatrix}, \frac{1}{\overline{\chi^{[2]}(b,\lambda) + \overline{h_3}\chi(b,\lambda)}}\chi(x,\lambda), \frac{\chi^{[2]}(b,\lambda) + h_3\chi(b,\lambda)}{\chi^{[2]}(b,\lambda) + \overline{h_3}\chi(b,\lambda)} \begin{bmatrix} e^{-i\lambda s} \\ e^{-i\lambda s} \end{bmatrix} \right\rangle. \tag{4.6}$$

For $f = \langle \varphi_-, y, \varphi_+\rangle$ we define the Fourier transformations

$$F_< : f \to k_-(\lambda) = \frac{1}{\sqrt{2\pi}}\langle f, V^<\rangle$$

and

$$F_{>}: f \to k_{+}(\lambda) = \frac{1}{\sqrt{2\pi}} \left\langle f, V^{>} \right\rangle,$$

where φ_{-}, y and φ_{+} are smooth, compactly supported functions. These definitions imply that $H^{<}$ and $H^{>}$ are isometrically identical with $L^2(\mathbb{R}; \mathbb{C}^2)$. Indeed, for $f = \langle \varphi_{-}, 0, 0 \rangle \in D_{-}$ the equation

$$k_{-}(\lambda) = \frac{1}{\sqrt{2\pi}} \left\langle f, V^{<} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi_{-}(t) e^{i\lambda t} \mathrm{d}t$$

holds. Hence, $k_{-} \in H_{-}^2$, where $H_{\pm}^2$ denote the Hardy class in $L^2(\mathbb{R}; \mathbb{C}^2)$ consisting of all vector-valued functions analytically extendible to the upper (lower) half-planes. Now consider the dense set $H_0^{<}$ in $H^{<}$ consisting of all vectors f such that f is compactly supported in D_{-} and $f \in H_0^{<}$ if $f = U(T) f_0$, $f_0 = \langle \varphi_{-}, 0, 0 \rangle$, $\varphi_{-} \in C_0^{\infty}(\mathbb{R}_{-}; \mathbb{C}^2)$, where $T = T_f$ is a non-negative number. Then if $f, g \in H^{<}$ we get for $T > T_f$ and $T > T_g$ that $U(-T) f$, $U(-T) g \in D_{-}$ and their first components belong to $C_0^{\infty}(\mathbb{R}_{-}; \mathbb{C}^2)$. Therefore,

$$\begin{aligned} \langle f, g \rangle &= \langle U(-T) f, U(-T) g \rangle = \left\langle e^{-i\lambda t} U(-T) f, e^{-i\lambda t} U(-T) g \right\rangle_{L^2} \\ &= \langle F_{<} f, F_{<} g \rangle_{L^2} . \end{aligned}$$

Thus, Parseval's equation holds for the whole space $H^{<}$. Further, the inversion formula

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_{-}(\lambda) V_{<} \mathrm{d}\lambda$$

follows from the Parseval's equation if all the integrals are taken as limits in the mean of integrals. Finally, we have

$$F_{<} H^{<} = \overline{\bigcup_{t \geq 0} F_{<} U(t) D_{-}} = \overline{\bigcup_{t \geq 0} e^{-i\lambda t} H_{-}^2} = L^2(\mathbb{R}; \mathbb{C}^2).$$

A similar proof can be given for $H^{>}$. Therefore, $H^{<} = H^{>} = H$.

Finally, D_{-} is orthogonal to D_{+}. □

Remark 4.4 Throughout the proof of Lemma 4.3, we have obtained that $F_{<}$ is the incoming spectral representation and $F_{>}$ is the outgoing spectral representation for the group $\{U(t)\}$. Moreover, $U(t)$ is transformed into $e^{i\lambda t}$.

Now we set the function $S(\lambda)$ as

$$S(\lambda) = \frac{\chi^{[2]}(b, \lambda) + h_3 \chi(b, \lambda)}{\chi^{[2]}(b, \lambda) + \overline{h_3} \chi(b, \lambda)}. \tag{4.7}$$

Since $\chi(b,\lambda)$ and $\chi^{[2]}(b,\lambda)$ are entire functions of λ, the poles of S are discrete. One has the following

$$1 - S(\lambda)S^{*}(\lambda) = \frac{2\text{Im}h_3\left(2\text{Im}\lambda \int\limits_a^b |\chi|^2 w\text{d}x\right)}{\left|\chi^{[2]}(b,\lambda) + \overline{h_3}\chi(b,\lambda)\right|^2}. \tag{4.8}$$

From (4.8), we obtain for real λ that $|S(\lambda)| = 1$ except the poles of $S(\lambda)$ and for $\text{Im}\lambda > 0$ we get $|S(\lambda)| < 1$. Consequently, $S(\lambda)$ is an inner function for $\text{Im}\,\lambda \geq 0$.

Using (4.5)–(4.7) one has

$$V^{<} = \overline{S(\lambda)}V^{>}. \tag{4.9}$$

Consequently, (4.9) gives

$$k_{-} = S(\lambda)k_{+}.$$

According to the Lax–Phillips scattering theory the scattering function is the coefficient by which the $F_{>}$ representation must be multiplied for getting $F_{<}$ representation. Hence, we have proved the following.

Theorem 4.5 *$S(\lambda)$ is the scattering function of the group of $U(t)$.*

5 Completeness of the System of Root Functions

For giving a complete spectral analysis for the operator T, first of all, we shall give the following nice connection between dissipative operators and related contraction operators [1].

Lemma 5.1 (i) *Assume the operator L_0 is dissipative. Then the operator $T_0 = K(L_0) = (L_0 - iI)(L_0 + iI)^{-1}$ is a contraction from $(L_0 + iI)D(L_0)$ onto $(L_0 - iI)D(L_0)$ and $L_0 = i(I + T_0)(I - T_0)^{-1}$. For each contraction T_0 such that $1 \notin \sigma_p(T_0)$ (the point spectrum of the operator), operator $L_0 = K^{-1}(T_0)$, $D(L_0) = (I - T_0)D(T_0)$, is dissipative.*

(ii) *Each dissipative operator L_0 has a maximal dissipative extension L. A maximal dissipative operator is closed.*

(iii) *A maximal dissipative operator is maximal dissipative if and only if $T = K(L)$ is a contraction such that $D(T) = H$ and $1 \notin \sigma_p(T)$.*

(iv) *If L is a maximal dissipative operator, $L = K^{-1}(T)$, then $-L^*$ is also maximal dissipative, $L^* = -K^{-1}(T^*)$.*

(v) *If L is a maximal dissipative operator, then $\sigma(T) \subset \overline{\mathbb{C}}_+$, $\left\|(L - \lambda I)^{-1}\right\| \leq |Im\lambda|^{-1}$, $\lambda \in \mathbb{C}_-$.*

Now let us consider the Cayley transform $\mathcal{C}$ of the dissipative operator T as follows

$$\mathcal{C} = (T - iI)(T + iI)^{-1}.$$

Note that the domain of $\mathcal{C}$ is the whole Hilbert space H because T is maximal dissipative [1]. Then we may introduce the following important result.

Theorem 5.2 $\|\mathcal{C}\| < 1$.

Proof Let us construct the function

$$y = (T + i\mathbf{1})^{-1} f, \tag{5.1}$$

where $y \in D(T)$ and $f \in H$. Since T is simple using (5.1) one obtains

$$\|(N - i\mathbf{1})y\|_H^2 < \|(N + i\mathbf{1})y\|_H^2$$

and this completes the proof. □

One of the important types of contractions on a Hilbert space is the *completely non-unitary* (c.n.u.) contractions. Recall that a contraction acting on a Hilbert space H is called c.n.u. if for no nonzero reducing subspace $\mathcal{R}$ for T is $T \mid \mathcal{R}$ a unitary operator. A special class consisting of c.n.u. contractions is the class C_0. The class C_0 consists of those c.n.u. contractions T for which there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$, where H^p $(0 < p \leq \infty)$ denotes the Hardy class of functions u, holomorphic on the unit disc and the corresponding norm

$$\|u\|_p = \begin{cases} \sup\limits_{0<r<1} \left[\frac{1}{2\pi} \int\limits_0^{2\pi} \left| u(re^{it}) \right|^p \mathrm{d}t \right]^{1/p}, & 0 < p < \infty, \\ \sup\limits_{\lambda \in \boldsymbol{D}} |u(\lambda)|, \ p = \infty, \end{cases}$$

is finite.

Corollary 5.3 *$\mathcal{C}$ is c.n.u. contraction on H, and 1 does not belong to the point spectrum of $\mathcal{C}$.*

Now we shall turn to the functional model construction.

As we have seen that the transformation $F_<$ implies

$$\boldsymbol{H} = D_- \oplus H \oplus D_+ \to L^2(\mathbb{R}; \mathbb{C}^2) = H_-^2 \oplus H \oplus S(\lambda) H_+^2.$$

Hence,

$$H = H_+^2 \ominus S(\lambda) H_+^2.$$

The subspace H is not a trivial subspace. Since $U(t)$ is unitary equivalent under the transformation $F_<$ to $e^{i\lambda t} k_-$ it can be concluded that $\widetilde{Z}(t) = P\left[e^{i\lambda t} z(\lambda)\right]$, $t \geq 0$, where P is the orthogonal projection from H_+^2 onto H, is a semigroup of operators. Hence, the generator of $\widetilde{Z}(t)$

$$\widetilde{G} = \lim_{t \to 0^+} \frac{\widetilde{Z}(t) - I}{it}$$

is a maximal dissipative operator on H. Here $\widetilde{G}$ is the model operator, and therefore, $S(\lambda)$ is its characteristic function. However, since the characteristic functions of unitary equivalent operators coincide we obtain the following.

Theorem 5.4 *$S(\lambda)$ is the characteristic function of T.*

For the simple maximal dissipative operator T and its Cayley transform $\mathcal{C}$, the connection between the characteristic function S of T and characteristic function Θ of $\mathcal{C}$ is given by the following

$$S(\lambda) = \Theta\left(\frac{\lambda - i}{\lambda + i}\right), \quad \text{Im}\lambda > 0.$$

Therefore, we have the following.

Theorem 5.5 *The characteristic function $\Theta(\mu)$ of $\mathcal{C}$ is*

$$S(\lambda) = \Theta(\mu), \ \mu = \frac{\lambda - i}{\lambda + i}, \quad Im\lambda > 0.$$

Remark 5.6 The spectrum of $\mathcal{C}$ coincides with those μ that belong to the disc $\boldsymbol{D} = \{\mu : |\mu| < 1\}$ for which the operator $\Theta(\mu)$ is not boundedly invertible, together with those $\mu \in \boldsymbol{C} = \{\mu : |\mu| = 1\}$ not lying on any of the open arcs of $\boldsymbol{C}$ on which $\Theta(\mu)$ is a unitary operator-valued analytic function of μ and point spectrum of C coincides with those $\mu \in \boldsymbol{D}$ for which $\Theta(\mu)$ is not invertible at all. Since the zeros of $\chi^{[2]}(b, \lambda) + h_3\chi(b, \lambda)$, $\text{Im}\lambda > 0$, are eigenvalues of T, and 1 does not belong to the point spectrum of C, $\lambda = i(1 + \mu)/(1 - \mu)$ for $\lambda = is$, $\lim_{s\to\infty}(is) =: \lambda_\infty$ cannot be a zero of $\chi^{[2]}(b, \lambda) + h_3\chi(b, \lambda)$ or equivalently an eigenvalue of T.

Theorem 5.7 *$S(\lambda)$ is a Blaschke product in the upper half-plane.*

Proof For $\text{Im}\lambda > 0$, $S(\lambda)$ has a factorization

$$S(\lambda) = B(\lambda)e^{i\lambda b}, \quad b > 0, \tag{5.2}$$

where $B(\lambda)$ is a Blaschke product. Hence, one has

$$|S(\lambda)| \le e^{-b\text{Im}\lambda}, \quad \text{Im}\lambda > 0.$$

For $\lambda_s := is$ we obtain from (5.2) that $\chi^{[2]}(b, \lambda) + h_3\chi(b, \lambda) \to 0$ as $s \to \infty$. This implies that λ_∞ is an eigenvalue of T. However, from Remark 5.6 this is not possible. Therefore, this completes the proof. □

Now we may introduce the main result of our paper.

Theorem 5.8 *Root functions of T associated with the points of the spectrum of T in the upper half-plane span the Hilbert space H.*

Corollary 5.9 (i) *Eigenvalues of* (3.4), (3.3) *are countable in the open upper half-plane,*
(ii) *Infinity is the only possible limit point of the eigenvalues of* (3.4), (3.3),
(iii) *Infinity must belong to the spectrum of (3.4),* (3.3) *; however, it may not be an eigenvalue of* (3.4), (3.3),
(iv) *The system of all eigen- and associated functions of problems* (3.4), (3.3) *spans the Hilbert space H.*

6 Conclusion and Remarks

In this paper, our main aim is to give a description of the maximal self-adjoint and maximal non-self-adjoint (dissipative, accumulative) boundary conditions for the solutions of a formally symmetric third-order differential equation and, for this purpose, we have constructed the boundary mappings and used Gorbachuks' theorem on extension. It seems that such an analysis is new in the literature.

As can be seen in (3.3) (Corollary 3.3) we have imposed the self-adjointness boundary condition at left end point, dissipativeness conditions at right end point and at mixed end points and this construction has given rise to a non-standard dilation space in Sect. 4. The conditions in (3.3) may have been given as follows

$$\begin{aligned} & y(a) + h_1 y^{[2]}(a) = 0, \quad \mathrm{Im} h_1 > 0, \\ & (i + h_2) y^{[1]}(a) + (1 + i h_2) y^{[1]}(b) = 0, \quad \mathrm{Im} h_2 > 0, h_2 \neq i, \\ & y^{[2]}(b) + h_3 y(b) = 0, \quad \mathrm{Im} h_3 = 0, \end{aligned} \tag{6.1}$$

where $h_2 = \widetilde{h}_2/2$. Following a similar method we may introduce the following theorem.

Theorem 6.1 (i) *Eigenvalues of* (3.4), (6.1) *are countable in the open upper half-plane,*
(ii) *Infinity is the only possible limit point of the eigenvalues of* (3.4), (6.1)*,*
(iii) *Infinity must belong to the spectrum of* (3.4), (6.1)*; however, it may not be an eigenvalue of* (3.4), (6.1)*,*
(iv) *The system of all eigen- and associated functions of problems* (3.4), (6.1) *spans the Hilbert space* H*.*

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