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On Nonlinear Set-Valued θ-Contractions

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Abstract
In this paper, we introduce and study new fixed point results for nonlinear set-valued θ-contractions. Our results are based on a new approach, which is called set-valued θ-contraction and they extend and generalize many fixed point theorems in the literature.

Keywords Fixed point · Multivalued mappings · Multivalued θ-contraction · Complete metric space

Mathematics Subject Classification Primary 54H25; Secondary 47H10

1 Introduction

Banach established the most famous fundamental fixed point result called the Banach's contraction principle for metric fixed point theory in 1922. This principle has played an important role in various fields of applied mathematical analysis and is one of a very power tests for existence and uniqueness of the solution of considerable problems arising in mathematics. Subsequently, this principle has been remarkably extended and generalized in many ways (see [7,8,17,28,40]). The set-valued version of Banach's principle has been thoroughly proposed by many authors.

For the sake of completeness, we recall some important concepts and results about set-valued mappings.

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Let (X, d) be a metric space. It is well known that $H:\mathcal{CB}(X)\times\mathcal{CB}(X) \to \mathbb{R}$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

is a metric on $\mathcal{CB}(X)$, which is called the Pompeiu–Hausdorff metric, where $\mathcal{CB}(X)$ denotes the class of all nonempty, closed and bounded subsets of X and $d(x, B) = \inf\{d(x, y):y \in B\}$. H also is called generalized Pompeiu–Hausdorff distance on $\mathcal{C}(X)$, which denotes the family of all nonempty closed subsets of X. We also denote by $\mathcal{K}(X)$ the family of all nonempty compact subsets of X.

A fixed point of a set-valued mapping $T:X \to \mathcal{P}(X)$, which denotes the class of all nonempty subsets of X, is an element $x \in X$ such that $x \in Tx$. A function $f:X \to \mathbb{R}$ is lower semi-continuous if for any $\{x_n\} \subseteq X$ and $x \in X$, $x_n \to x$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Nadler [31] in 1969 initiated the idea for set-valued contraction mapping and extended the Banach contraction principle to set-valued mappings and proved the following:

Theorem 1 (Nadler [31]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ set-valued contraction, that is, there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then, T has a fixed point in X.

Later on, several researches were conducted on a variety of generalizations, extensions and applications of this result of Nadler (see [1,5,6,10–14,20–22,25,30,32–39]). Furthermore, Feng and Liu [15] introduced important generalization of this result and thereupon Klim and Wardowski [23] generalized their theorem as follows:

Theorem 2 [15] *Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$. If there exist constants $b, c \in (0, 1)$, $b < c$, such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$d(y, Ty) \leq cd(x, y),$$

where

$$I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\},$$

then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Recently, Klim and Wardowski [23] generalized Theorem 2 and proved the following results.

Theorem 3 [23] *Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$. If there exist $b \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, b)$ satisfying*

$$\limsup_{t\to s^+}\varphi(t) < b \text{ for all } s \in [0, \infty)$$

and for any $x \in X$ there is $y \in I_b^x$ satisfying

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y), \tag{1.1}$$

then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Theorem 4 [23] *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$, which is the family of all nonempty compact subsets of X. If there exists a function $\varphi : [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1 \text{ for all } s \in [0, \infty)$$

and for any $x \in X$ there is $y \in I_1^x$ satisfying

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y), \tag{1.2}$$

then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

In the literature, we can find many interesting results dealing with Feng–Liu's and Klim–Wardowski's fixed point theorems (see [4,9,24,26,27]).

On the other hand, the concept of θ-contraction is introduced by Jleli and Samet [19] in 2014 and so they introduce a new type of contractive mapping. Following their work, many authors recently proved various several fixed point theorems for set-valued mappings (see [2,3,18,29]).

The purpose of this paper is to give some fixed point results for set-valued mappings on complete metric spaces using the concept of set-valued θ -contraction. These results extend and generalize many fixed point theorems including Theorem 2 and Theorem 3.

2 Preliminaries

We recall basic definitions, relevant notions and related result concerning θ-contraction.

Let Θ be the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ and $\lim_{n\to\infty} t_n = 0^+$ are equivalent;

(θ_3) There exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Let (X, d) be a metric space and $\theta \in \Theta$. A mapping $T : X \to X$ is said to be a θ-contraction if there exists $k \in (0, 1)$ such that

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{2.1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

An easy example of such mappings is contraction which can be obtained by taking $\theta(t) = e^{\sqrt{t}}$ in inequality (2.1). Also, by choices of mapping $\theta(t) = e^{\sqrt{te^t}}$ in (2.1), we obtain a contraction-type condition

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2, \tag{2.2}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

It is clear that if a mapping T is contraction, then it satisfies inequality (2.2). In addition, it is easy to see that if T is a θ-contraction, then T is a contractive mapping, i.e., $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Thus, every θ -contraction mapping on a metric space is continuous. Jleli and Samet [19] proved the following fixed point result using concept of θ -contractions on complete metric spaces.

Theorem 5 (Corollary 2.1 of [19]) *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. If T is a θ-contraction, then T has a unique fixed point in X.*

The concept of θ-contraction extended to set-valued mappings by Han çer et al. [16]. Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ and $\theta \in \Theta$. Then, we say that T is a set-valued θ-contraction if there exists $k \in (0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{2.3}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Consequently, they established that mappings of this type possess a fixed point on complete metric spaces as follows:

Theorem 6 [16] *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a set-valued θ-contraction. Then, T has a fixed point.*

Moreover, Hançer et al. [16] showed the following example that we cannot unfortunately replace $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$ in Theorem 6 with the same conditions.

Example 1 Let $X = [0, 2]$. Define a metric on X by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1 + |x - y|$ if $x \neq y$. Then, (X, d) is a complete metric space. Define a mapping $T : X \to \mathcal{CB}(X)$, by $Tx = \mathbb{Q}$ if $x \in X \backslash \mathbb{Q}$ and $Tx = X \backslash \mathbb{Q}$ if $x \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers in X. Then, T is a set-valued θ-contraction with respect to $\theta \in \Theta$ defined by $\theta(t) = e^{\sqrt{t}}$ if $t \leq 1$ and $\theta(t) = 9$ if $t > 1$. However, T has no fixed point.

However, Hançer et al. [16] proved that we can take $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$, by adding the following condition on $\theta : (0, \infty) \to (1, \infty)$:

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Note that if θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is right continuous. Let Ξ be the family of all functions θ satisfying (θ_1) -(θ_4) .

Theorem 7 [16] *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a set-valued θ-contraction. If $\theta \in \Xi$, then T has a fixed point.*

3 Main Results

Let $T : X \to \mathcal{P}(X)$, $\theta \in \Theta$ and $s \in (0, 1]$. Define a set $\theta_s^x \subseteq X$ by

$$\theta_s^x = \{y \in Tx : [\theta(d(x, y))]^s \leq \theta(d(x, Tx))\},$$

$x \in X$ with $d(x, Tx) > 0$.

For the set θ_s^x, we will consider the following three cases (see [3] for more information):

Case 1 If $T : X \to \mathcal{K}(X)$, then we have $\theta_s^x \neq \emptyset$ for all $s \in (0, 1]$ and $x \in X$ with $d(x, Tx) > 0$.

Case 2 If $T : X \to \mathcal{C}(X)$, then θ_s^x may be empty for some $x \in X$ and $s \in (0, 1]$.

Case 3 If $T : X \to \mathcal{C}(X)$ (even if $T : X \to P(X)$) and $\theta \in \Xi$, then we have $\theta_s^x \neq \emptyset$ for all $s \in (0, 1)$ and $x \in X$ with $d(x, Tx) > 0$. We reprove this case using the property of right continuity of θ as different from the paper [3]. Since θ is right continuous, there exists a real number $h > 1$ such that

$$\theta(hd(x, Tx)) \leq [\theta(d(x, Tx))]^{\frac{1}{s}}.$$

Since $h > 1$, there exists $y \in Tx$ such that $d(x, y) \leq hd(x, Tx)$. Then, from (θ_1), we have

$$\theta(d(x, y)) \leq \theta(hd(x, Tx)) \leq [\theta(d(x, Tx))]^{\frac{1}{s}},$$

and so,

$$[\theta(d(x, y))]^s \leq \theta(d(x, Tx)),$$

which implies $y \in \theta_s^x$.

Then, Altun et al. [3] proved the following fixed point theorems. It is easy to see that Theorem 2 is a special case of Theorem 8.

Theorem 8 *Let (X, d) be a complete metric space, $T : X \to \mathcal{C}(X)$ and $\theta \in \Xi$. If there exists $k \in (0, 1)$ such that there is $y \in \theta_s^x$, $s \in (0, 1)$ and $k < s$, satisfying*

$$\theta(d(y, Ty)) \leq [\theta(d(x, y))]^k,$$

for each $x \in X$ with $d(x, Tx) > 0$, then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Theorem 9 *Let (X, d) be a complete metric space, $T : X \to \mathcal{K}(X)$ and $\theta \in \Theta$. If there exists $k \in (0, 1)$ such that there is $y \in \theta_1^x$ satisfying*

$$\theta(d(y, Ty)) \leq [\theta(d(x, y))]^k,$$

for each $x \in X$ with $d(x, Tx) > 0$, then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Inspired with the above results, we give the following theorems, which we called nonlinear form of Theorem 8 and Theorem 9. Note that Theorem 10 is a proper generalization of Theorem 3.

Theorem 10 *Let (X, d) be a complete metric space, $T : X \to \mathcal{C}(X)$ and $\theta \in \Xi$. If there exist $s \in (0, 1)$ and a function $k : [0, \infty) \to [0, s)$ satisfying*

$$\limsup_{t \to r^+} k(t) < s \text{ for all } r \in [0, \infty) \tag{3.1}$$

and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in \theta_s^x$ satisfying

$$\theta(d(y, Ty)) \leq [\theta(d(x, y))]^{k(d(x,y))}, \tag{3.2}$$

then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Proof Suppose that T has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. Since $Tx \in \mathcal{C}(X)$ for every $x \in X$ and $\theta \in \Xi$, the set θ_s^x is nonempty for any $s \in (0, 1)$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in \theta_s^{x_0}$ such that

$$\theta(d(x_1, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))}$$

and for $x_1 \in X$, there exists $x_2 \in \theta_s^{x_1}$ satisfying

$$\theta(d(x_2, Tx_2)) \leq [\theta(d(x_1, x_2))]^{k(d(x_1,x_2))}.$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in \theta_s^{x_n}$ and

$$\theta(d(x_{n+1}, Tx_{n+1})) \leq \left[\theta(d(x_n, x_{n+1}))\right]^{k(d(x_n,x_{n+1}))}. \tag{3.3}$$

We will verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in \theta_s^{x_n}$, we have

$$\left[\theta(d(x_n, x_{n+1}))\right]^s \leq \theta(d(x_n, Tx_n)). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\theta(d(x_{n+1}, Tx_{n+1})) \leq [\theta(d(x_n, Tx_n))]^{\frac{k(d(x_n,x_{n+1}))}{s}} \tag{3.5}$$

and

$$\theta(d(x_{n+1}, x_{n+2})) \leq \left[\theta(d(x_n, x_{n+1}))\right]^{\frac{k(d(x_n,x_{n+1}))}{s}}. \tag{3.6}$$

From (3.5), (3.6) and (θ_1), it follows that the sequences $\{d(x_n, Tx_n)\}$ and $\{d(x_n, x_{n+1})\}$ are decreasing and hence convergent. Now, from (3.1), there exists $w \in [0, s)$ such that

$$\limsup_{n \to \infty} k(d(x_n, x_{n+1})) = w.$$

Therefore, there exists $b \in (w, s)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < b$ for all $n \geq n_0$. Thus, using (3.6), we obtain for all $n \geq n_0$ the following inequalities:

$$1 < \theta(d(x_n, x_{n+1}))$$

$$
\begin{aligned}
&\leq \left[\theta(d(x_{n-1}, x_n))\right]^{\frac{k(d(x_{n-1},x_n))}{s}}\\
&\leq \left[\theta(d(x_{n-2}, x_{n-1}))\right]^{\frac{k(d(x_{n-1},x_n))}{s}\frac{k(d(x_{n-1},x_n))}{s}}\\
&\ \ \vdots\\
&\leq [\theta(d(x_0, x_1))]^{\frac{k(d(x_0,x_1))}{s}\cdots\frac{k(d(x_{n-1},x_n))}{s}\frac{k(d(x_{n-1},x_n))}{s}}\\
&= [\theta(d(x_0, x_1))]^{\frac{k(d(x_0,x_1))}{s}\cdots\frac{k(d(x_{n_0-1},x_{n_0}))}{s}\frac{k(d(x_{n_0},x_{n_0+1}))}{s}\cdots\frac{k(d(x_{n-1},x_n))}{s}\frac{k(d(x_{n-1},x_n))}{s}}\\
&\leq [\theta(d(x_0, x_1))]^{\frac{k(d(x_{n_0},x_{n_0+1}))}{s}\cdots\frac{k(d(x_{n-1},x_n))}{s}\frac{k(d(x_{n-1},x_n))}{s}}\\
&\leq [\theta(d(x_0, x_1))]^{\frac{b^{(n-n_0)}}{s^{(n-n_0)}}}.
\end{aligned}
$$

Thus, we have

$$
1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{\left(\frac{b}{s}\right)^{(n-n_0)}} \tag{3.7}
$$

for all $n \geq n_0$. Letting $n \to \infty$ in (3.7), since $\lim_{n\to\infty}\left(\frac{b}{s}\right)^{(n-n_0)} = 0$, we obtain

$$
\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1. \tag{3.8}
$$

From (θ_2), $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+$ (similarly, we can obtain $\lim_{n\to\infty} d(x_n, Tx_{n+1}) = 0^+$) and so from (Θ_3), there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$
\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{\left[d(x_n, x_{n+1})\right]^r} = l.
$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$
\left|\frac{\theta(d(x_n, x_{n+1})) - 1}{\left[d(x_n, x_{n+1})\right]^r} - l\right| \leq B.
$$

This implies that, for all $n \geq n_1$,

$$
\frac{\theta(d(x_n, x_{n+1})) - 1}{\left[d(x_n, x_{n+1})\right]^r} \geq l - B = B.
$$

Then, for all $n \geq n_1$,

$$
n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\theta(d(x_n, x_{n+1})) - 1\right],
$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{\left[d(x_n, x_{n+1})\right]^r} \geq B.$$

This implies that, for all $n \geq n_1$,

$$n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\theta(d(x_n, x_{n+1})) - 1\right],$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\theta(d(x_n, x_{n+1})) - 1\right].$$

Using (3.7), we obtain, for all $n \geq n_2 = \max\{n_0, n_1\}$,

$$n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\left[\theta(d(x_0, x_1))\right]^{\left(\frac{b}{s}\right)^{(n-n_2)}} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[d(x_n, x_{n+1})\right]^r = 0.$$

Thus, there exits $n_3 \in \mathbb{N}$ such that $n\left[d(x_n, x_{n+1})\right]^r \leq 1$ for all $n \geq n_3$. So, we have, for all $n \geq n_3$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}. \tag{3.9}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_3$. Using the triangular inequality for the metric and from (3.9), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. On the other hand, since

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

and $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \le d(z, Tz) \le \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

This is a contradiction. Hence, T has a fixed point. □

Remark 1 If we take $\mathcal{K}(X)$ instead of $\mathcal{CB}(X)$ in Theorem 10, we can remove the condition (θ_4) on θ. Further, by taking into account Case 1, we can take $s = 1$. Therefore, the proof of the following theorem is easy.

Theorem 11 *Let (X, d) be a complete metric space, $T : X \to \mathcal{K}(X)$ and $\theta \in \Theta$. If there exists a function $k : [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{t\to r^+} k(t) < 1 \textit{ for all } r \in [0, \infty)$$

and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in \theta_1^x$ satisfying

$$\theta(d(y, Ty)) \le [\theta(d(x, y))]^{k(d(x,y))}, \tag{3.10}$$

then T has a fixed point in X provided that function $x \to d(x, Tx)$ is lower semi-continuous.

Proof Suppose that T has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. Since $Tx \in \mathcal{K}(X)$ for every $x \in X$, the set θ_1^x is nonempty. Hence, there exists $y \in \theta_1^x$ for all $x \in X$ such that $x \ne y$ and $d(x, y) = d(x, Tx)$. Let $x_0 \in X$ be any initial point. By (3.10), using the analogous method like in the proof of Theorem 10, we obtain the existence of a Cauchy sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_{n+1} \ne x_n$, satisfying

$$d(x_n, x_{n+1}) = d(x_n, Tx_n),$$
$$\theta(d(x_{n+1}, Tx_{n+1})) \le \left[\theta(d(x_n, x_{n+1}))\right]^{k(d(x_n,x_{n+1}))},$$

and

$$\lim_{n\to\infty} x_n = z.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, we get

$$0 \le d(z, Tz) \le \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

This is a contradiction. Hence, T has a fixed point. □

4 Nontrivial Examples

Now, we give some significant examples showing that there are some multivalued mappings such that our result (Theorem 10) can be applied but Theorem 3 cannot.

Example 2 Consider the complete metric space (X, d), where $X = [0, 1] \cup \{2, 3, \ldots\}$ and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ |x - y|, & \text{if } x, y \in [0, 1] \\ x + y, & \text{if one of } x, y \notin [0, 1] \end{cases}.$$

Define a mapping $T : X \to \mathcal{C}(X)$ by

$$Tx = \begin{cases} \left\{\frac{x}{16}\right\}, & x \in [0, 1] \\ \{x - 1, x + 1, x + 2, \ldots\}, & x > 1 \end{cases}.$$

It is easy to see that

$$D(x, Tx) = \begin{cases} \frac{15x}{16}, & x \in [0, 1] \\ 2x - 1, & x > 1 \end{cases}$$

and the function $x \to D(x, Tx)$ is lower semi-continuous.

Now we show that condition (3.2) of Theorem 10 is satisfied with $\theta(t) = e^{\sqrt{te^t}}$, $s \in \left(\frac{1}{e}, 1\right)$ and $k : [0, \infty) \to [0, s)$ defined by $k(t) = \frac{1}{e}$. Inequality (3.2) also turns to

$$\frac{D(y, Ty)}{d(x, y)} e^{D(y,Ty) - d(x,y)} \le [k(d(x, y))]^2. \tag{4.1}$$

We will show that T satisfies (4.1).

Note that if $D(x, Tx) > 0$, then $x \neq 0$. Therefore, for $x \in (0, 1]$, we have $y = \frac{x}{16} \in \theta_s^x$ for all $s \in \left(\frac{1}{e}, 1\right)$ and

$$\begin{aligned} \frac{D(y, Ty)}{d(x, y)} e^{D(y,Ty) - d(x,y)} &= \frac{\frac{15x}{256}}{\frac{15x}{16}} e^{-\frac{225x}{256}} \\ &\le \frac{1}{16} < \frac{1}{e^2} \\ &= \left[k\left(\frac{15x}{16}\right)\right]^2 \\ &= [k(d(x, y))]^2, \end{aligned}$$

and for $x > 1$, we have $y = x - 1 \in \theta_s^x$ for all $s \in \left(\frac{1}{e}, 1\right)$ and

$$\begin{aligned} \frac{D(y, Ty)}{d(x, y)} e^{D(y,Ty) - d(x,y)} &= \frac{2x - 3}{2x - 1} e^{-2} \\ &\le e^{-2} \\ &= [k(2x - 1)]^2 \\ &= [k(d(x, y))]^2. \end{aligned}$$

Therefore, all the assumptions of Theorem 10 are satisfied and so T has a fixed point.

Now we claim that condition (1.1) of Theorem 3 is not satisfied. Indeed, let $x > 1$, then $Tx = \{x-1, x+1, x+2, \ldots\}$. In this case, if $x+k \in I_b^x$ where $k \in \mathbb{N}$ for all $b \in (0,1)$, then

$$\begin{aligned} D(y, Ty) &= 2x + 2k - 1 \\ &> \varphi(2x+k)(2x+k) \\ &= \varphi(d(x,y))d(x,y), \end{aligned}$$

for all $\varphi : [0,\infty) \to [0,b)$ satisfying

$$\limsup_{t \to s^+} \varphi(t) < b \text{ for all } s \geq 0;$$

if $x - 1 \in I_b^x$ for all $b \in (0,1)$, then, for all such function φ, we get

$$\begin{aligned} D(y, Ty) &= 2x - 3 \\ &\leq \varphi(2x-1)(2x-1) \\ &= \varphi(d(x,y))(2x-1) \\ &< b(2x-1), \end{aligned}$$

that is,

$$\frac{2x-3}{2x-1} < b,$$

which this is not possible after a certain value of $x \in \{2, 3, \ldots\}$.

Example 3 Consider the complete metric space (X, d), where $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and

$$d(x,y) = \begin{cases} 0, & x = y \\ \max\{x,y\}, & x \neq y \end{cases}.$$

Define a mapping $T : X \to \mathcal{K}(X)$ by

$$Tx = \begin{cases} \{x\}, & x \in \{0,1\} \\ \left\{\frac{1}{n+2}, \frac{1}{n+1}\right\}, & x = \frac{1}{n},\ n > 1 \end{cases}.$$

It is easy to see that

$$D(x, Tx) = \begin{cases} 0, & x \in \{0,1\} \\ \frac{1}{n}, & x = \frac{1}{n},\ n > 1 \end{cases}$$

and the function $x \to D(x, Tx)$ is lower semi-continuous.

Now we show that condition (3.10) of Theorem 11 is satisfied with $\theta(t) = e^{\sqrt{te^t}}$ and $k : [0,\infty) \to [0,1)$ defined by

$$k(t) = \begin{cases} \sqrt{e^{\frac{1}{n+1} - \frac{1}{n}}}, & \text{if } t = \frac{1}{n} \text{ for some } n \in \mathbb{N} \text{ with } n > 1 \\ 0, & \text{otherwise} \end{cases}.$$

Since $\limsup_{t\to a^+} k(t) = 0 < 1$ for all $a \in [0, \infty)$ and so k satisfies (3.1). Observe that taking $\theta(t) = e^{\sqrt{te^t}}$,

$$\frac{D(y, Ty)}{d(x, y)} e^{D(y,Ty)-d(x,y)} \leq [k(d(x, y))]^2 .$$

Note that if $D(x, Tx) > 0$, then $x = \frac{1}{n}$ for $n > 1$. In this case, $D(x, Tx) = \frac{1}{n}$ for $n > 1$. Therefore, for $y = \frac{1}{n+1} \in T\frac{1}{n} = \left\{\frac{1}{n+2}, \frac{1}{n+1}\right\}$, we have $y \in \theta_1^x$. Then, we get

$$\begin{aligned} \frac{D(y, Ty)}{d(x, y)} e^{D(y,Ty)-d(x,y)} &= \frac{\frac{1}{n+1}}{\frac{1}{n}} e^{\frac{1}{n+1}-\frac{1}{n}} \\ &= \frac{n}{n+1} e^{\frac{1}{n+1}-\frac{1}{n}} \\ &\leq e^{\frac{1}{n+1}-\frac{1}{n}} \\ &= \left[k\left(\frac{1}{n}\right)\right]^2 \\ &= [k(d(x, y))]^2 . \end{aligned}$$

Therefore, all the assumptions of Theorem 11 are satisfied and so T has a fixed point.

Now we claim that condition (1.2) of Theorem 4 is not satisfied. Indeed, let $x = \frac{1}{n}$ for $n > 2$, then $Tx = \left\{\frac{1}{n+2}, \frac{1}{n+1}\right\}$. In this case, $I_1^x = \left\{\frac{1}{n+2}, \frac{1}{n+1}\right\}$. If $y = \frac{1}{n+1}$, since

$$D(y, Ty) = \frac{1}{n+1} \text{ and } d(x, y) = \frac{1}{n},$$

we obtain

$$\begin{aligned} D(y, Ty) &\leq \varphi(d(x, y))d(x, y) \\ &\Leftrightarrow \frac{1}{n+1} \leq \varphi\left(\frac{1}{n}\right)\frac{1}{n} \\ &\Leftrightarrow \frac{n}{n+1} \leq \varphi\left(\frac{1}{n}\right). \end{aligned}$$

Taking limit supremum as $n \to \infty$ in above, we have

$$1 \leq \limsup_{n\to\infty} \varphi\left(\frac{1}{n}\right) \leq \limsup_{t\to 0^+} \varphi(t) < 1,$$

which is a contradiction. If $y = \frac{1}{n+2}$, since

$$D(y, Ty) = \frac{1}{n+2} \text{ and } d(x, y) = \frac{1}{n},$$

we obtain

$$\begin{aligned} d(Tx, Ty) &\leq \varphi(d(x,y))d(x,y) \\ &\Leftrightarrow \frac{1}{n+2} \leq \varphi\left(\frac{1}{n}\right)\frac{1}{n} \\ &\Leftrightarrow \frac{n}{n+2} \leq \varphi\left(\frac{1}{n}\right). \end{aligned}$$

Taking limit supremum as $n \to \infty$ in above, we have

$$1 \leq \limsup_{n\to\infty} \varphi\left(\frac{1}{n}\right) \leq \limsup_{t\to 0^+} \varphi(t) < 1,$$

which is a contradiction. Therefore, Theorem 4 cannot be applied to this example.

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