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Some Properties of Solutions of a Fourth-Order Parabolic Equation for Image Processing

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Abstract
In this paper, for the IBVP of a fourth-order nonlinear parabolic equation, which is related to image analysis, we studied the existence and uniqueness of weak solutions. Moreover, we also considered the asymptotic behavior and the regularity of solutions of such problem.

Keywords Fourth-order parabolic equation · Existence · Asymptotic behavior · Regularity

Mathematics Subject Classification 35D05 · 35B40 · 35G30 · 35K55

1 Introduction

In this paper, we investigate the following fourth-order parabolic equation

$$u_t + \Delta\left(\ln\left(\Delta u + \sqrt{1+(\Delta u)^2}\right)\right) + \lambda|u|^{p-2}u = 0, \quad \text{in} \quad \Omega_T, \tag{1.1}$$

where $\lambda > 0$, $p > 2$, $\Omega_T = \Omega \times (0, T)$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary.

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On the basis of physical consideration, as usual Eq. (1.1) is supplemented with the natural boundary value conditions

$$u = \Delta u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and the initial value condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Here, inspired by the ideas described in Wei [11], we give a sketch of the formulation of Eq. (1.1) from the image restoration. Wei [11] proposed a real-valued, bounded edge enhancing functional, which leads to a generalized Perona–Malik equation

$$\frac{\partial u}{\partial t} = \mathrm{div}(d(u, |\nabla u|)\nabla u) + e(u, |\nabla u|).$$

In image systems, the distribution of image pixels can be highly inhomogeneous. Hence, the generalized Perona–Malik equation can be made more efficient for image segmentation and noise removing by incorporating an edge sensitive super diffusion operator [11]

$$\frac{\partial u}{\partial t} = \mathrm{div}(d_1(u, |\nabla u|)\nabla u) + \mathrm{div}(d_2(u, |\nabla u|, \Delta u)\nabla \Delta u) + e(u, |\nabla u|). \tag{1.4}$$

Here d_1, d_2 are edge sensitive diffusion functions. The typical cases of d_2 are $d_2(u, |\nabla u|, \Delta u) = -g(|\nabla u|)$ or $= -g(\Delta u)$. The $g(s)$ is a nonincreasing function satisfying the following ([8])

$$g(0) = 1, \quad g(s) > 0, \quad \lim_{s\to\infty} \frac{\mathrm{d}^n}{\mathrm{d}s^n} g(s) = 0, \text{ for each integer } n \geq 0.$$

An example typically used in applications is [3,8]

$$g(s) = \frac{1}{1+s^2}.$$

The $g(s) = \frac{1}{\sqrt{1+s^2}}$ is reasonable for Eq. (1.4). If taking $d_1 = 0$, $e(u, |\nabla u|) = -\lambda|u|^{p-2}u$ and $d_2 = -\frac{1}{\sqrt{1+|\Delta u|^2}}$, we obtain Eq. (1.1). Equation (1.1) is original, which has not been studied by others so far.

Taking $d_1 = 0$, $e(u, |\nabla u|) = 0$ and $d_2 = -\frac{1}{1+|\Delta u|^2}$, Eq. (1.4) becomes the fourth-order Perona–Malik analogue [9]

$$u_t + \nabla\left(\frac{\nabla \Delta u}{1 + |\Delta u|^2}\right) = 0. \tag{1.5}$$

Wang et al. [10] considered the low-curvature equation

$$u_t + \Delta(\arctan \Delta u) = 0,$$

which is exactly the equation of (1.5). They established the existence and uniqueness of weak solutions.

Wei [11] introduced the following equation by taking $d_1 = 0$, $e(u, |\nabla u|) = 0$ and $d_2 = -\frac{1}{1+|\nabla u|^2}$ for highly inhomogeneous images,

$$u_t + \nabla\left(\frac{\nabla\Delta u}{1 + |\nabla u|^2}\right) = 0.$$

Other fourth-order partial differential equations are also proposed in image analysis. You and Kaveh [13] introduced a different form of the fourth-order diffusion,

$$u_t + \Delta[g(\Delta u)\Delta u] = 0.$$

This form is derived from a variational formulation. Osher et al. [7] employed a new model

$$u_t + \frac{1}{2\lambda}\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) = f - u,$$

for image decomposition and image restoration into cartoon and texture. The relevant fourth-order parabolic equations have also been studied in [1,4,6,12].

Now we give the definition of the solution in a weak sense of problems (1.1)–(1.3).

Definition 1.1 A function u is a weak solution of problems (1.1)–(1.3), if the following conditions are satisfied

(1) $u \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ with $\ln(\Delta u + \sqrt{1 + (\Delta u)^2}) \in L^2(0, T; H_0^1(\Omega))$;
(2) For any $\varphi \in C^2(\overline{\Omega}_T)$ with $\varphi(x, T) = 0$ and $\varphi(x, t)\mid_{\partial\Omega} = 0$, we have

$$\begin{aligned} &-\int_\Omega \varphi(x, 0)u_0(x)\mathrm{d}x - \int_0^T\int_\Omega u\varphi_t\mathrm{d}x\mathrm{d}t \\ &+\int_0^T\int_\Omega \ln\left(\Delta u + \sqrt{1 + (\Delta u)^2}\right)\Delta\varphi\mathrm{d}x\mathrm{d}t \\ &+\lambda\int_0^T\int_\Omega |u|^{p-2}u\varphi\mathrm{d}x\mathrm{d}t = 0. \end{aligned} \tag{1.6}$$

In this paper, we will study a general equation as described in (1.1). Our method for investigating the existence of weak solutions is based on the difference and variation methods to construct an approximate solution. By means of the uniform estimates on solutions of the time difference equations, we prove the existence of weak solutions.

Based on a suitable integral equality and the energy techniques, we also obtain the asymptotic behavior and regularity of solutions.

This paper is organized as follows. We investigate the existence and uniqueness of weak solutions of problems (1.1)–(1.3) in Sect. 3. Using energy techniques, we also proved the asymptotic behavior and regularity of solutions subsequently.

2 Existence of solutions

In this section, we are going to prove the existence of weak solutions.

Theorem 2.1 *Assume* $u_0 \in H_0^1(\Omega)$, *problems* (1.1)–(1.3) *admit a unique weak solution satisfying Definition* 1.1.

To prove Theorem 2.1, we first consider the following elliptic problem

$$\begin{cases} \dfrac{u-u_0}{h} + \Delta\left(\varepsilon\Delta u + \ln\left(\Delta u + \sqrt{1+(\Delta u)^2}\right)\right) + \lambda|u|^{p-2}u = 0, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $h = T/n$, $\varepsilon > 0$ and u_0 is the initial value.

Theorem 2.2 *Assume* $u_0 \in H_0^1(\Omega)$, *there exists a unique weak solution* $u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ *with* $\Delta u_1 \in H_0^1(\Omega)$ *for initial-boundary value problem* (2.1).

Proof We will prove the existence of weak solutions by variation methods.

Let us consider the following functional on the space $V = H_0^1(\Omega) \cap H^2(\Omega)$,

$$\begin{aligned} J(v) = &\int_\Omega \frac{|v-u_0|^2}{2h}\mathrm{d}x + \int_\Omega \frac{\varepsilon|\Delta v|^2}{2}\mathrm{d}x \\ &+ \int_\Omega \int_0^{\Delta v} \ln\left(s + \sqrt{1+s^2}\right)\mathrm{d}s\mathrm{d}x + \lambda\int_\Omega \frac{|v|^p}{p}\mathrm{d}x. \end{aligned} \tag{2.2}$$

In addition, letting $f(t) = \int_0^t \ln(s + \sqrt{1+s^2})\mathrm{d}s$, we know that $f'(t) = \ln(t + \sqrt{1+t^2})$, $f''(t) = \frac{1}{\sqrt{1+t^2}} > 0$ and $f(0) = f'(0) = 0$. Hence, $f(t) \geq f(0) = 0$. It is obvious that

$$\int_\Omega \int_0^{\Delta v} \ln\left(s + \sqrt{1+s^2}\right)\mathrm{d}s\mathrm{d}x \geq 0.$$

Therefore, we see that

$$J(v) \geq \int_\Omega \frac{\varepsilon|\Delta v|^2}{2}\mathrm{d}x + \lambda\int_\Omega \frac{|v|^p}{p}\mathrm{d}x.$$

By $\lambda > 0$, $p > 2$ and the Poincaré inequality, we know that $J(v) \to +\infty$, as $\|v\|_{H^2} \to +\infty$. Hence, $J(v)$ satisfies the coercive condition. On the other hand, since

$\int_0^t \ln(s+\sqrt{1+s^2})\mathrm{d}s$ is a convex function, $J(v)$ is weakly lower semi-continuous on V. So, it follows from the theory in [2] that there exists $u_1 \in V$ such that

$$J(u_1) = \inf_{v\in V} J(v),$$

which implies that $u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ is a minimizer of the functional $J(v)$ in V.

Now for every $\varphi \in C_0^\infty$ and every $\varepsilon \in \mathbb{R}$, since $u_1 + \varepsilon\varphi \in V$, $F(0) \le F(\varepsilon)$, where

$$F(\varepsilon) = J(u_1 + \varepsilon\varphi).$$

Thus, we get $F'(0) = 0$, which is

$$\int_\Omega \frac{u_1 - u_0}{h}\varphi \mathrm{d}x + \int_\Omega \Delta\left(\varepsilon\Delta u_1 + \ln\left(\Delta u_1 + \sqrt{1+(\Delta u_1)^2}\right)\right)\varphi \mathrm{d}x \\ + \lambda \int_\Omega |u_1|^{p-2}u_1\varphi \mathrm{d}x = 0. \tag{2.3}$$

Therefore, the function u_1 is a weak solution of the corresponding Euler–Lagrange equation of $J(v)$, which is problem (2.1). For every $\eta \in C_0^\infty(\Omega)$, there exists a unique $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $-\Delta\varphi = \eta$. Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the unique solution for equation

$$-\Delta w = \frac{u_1 - u_0}{h} + \lambda|u_1|^{p-2}u_1.$$

By $u_1 - u_0, |u_1|^{p-2}u_1 \in H_0^1(\Omega)$, we know that $w \in H_0^1(\Omega) \cap H^3(\Omega)$. Hence by (2.3), we have

$$\int_\Omega (-\Delta w)\varphi \mathrm{d}x = \int_\Omega \left(\varepsilon\Delta u_1 + \ln\left(\Delta u_1 + \sqrt{1+(\Delta u_1)^2}\right)\right)\eta \mathrm{d}x.$$

On the other hand, we know that

$$\int_\Omega (-\Delta w)\varphi \mathrm{d}x = \int_\Omega w(-\Delta\varphi)\mathrm{d}x = \int_\Omega w\eta \mathrm{d}x.$$

Therefore, we derive

$$f_\varepsilon(\Delta u_1) = \varepsilon\Delta u_1 + \ln(\Delta u_1 + \sqrt{1+(\Delta u_1)^2}) = w.$$

For function $f_\varepsilon(t) = \varepsilon t + \ln(t + \sqrt{1+t^2})$, we know that

$$\varepsilon < f_\varepsilon'(t) = \varepsilon + \frac{1}{\sqrt{1+t^2}} \le \varepsilon + 1.$$

So its inverse function $g_\varepsilon(t) = f_\varepsilon^{-1}(t)$ exists and satisfies

$$\frac{1}{1+\varepsilon} \le g_\varepsilon'(t) < \frac{1}{\varepsilon}.$$

Hence, we obtain

$$\Delta u_1 = g_\varepsilon(w) \in H_0^1(\Omega).$$

So we complete the proof of the existence.

Now we prove the uniqueness. Suppose that there exists another weak solution $\widetilde{u}_1$ of problem (2.1). Then, it follows from (2.3) that, for every $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\int_\Omega \frac{\widetilde{u}_1 - u_0}{h}\varphi \mathrm{d}x + \int_\Omega \Delta\left(\varepsilon\Delta\widetilde{u}_1 + \ln\left(\Delta\widetilde{u}_1 + \sqrt{1+(\Delta\widetilde{u}_1)^2}\right)\right)\varphi \mathrm{d}x \\ + \lambda\int_\Omega |\widetilde{u}_1|^{p-2}\widetilde{u}_1\varphi \mathrm{d}x = 0.$$

So that,

$$\int_\Omega \frac{\widetilde{u}_1 - u_1}{h}\varphi \mathrm{d}x + \int_\Omega \varepsilon\Delta(\widetilde{u}_1 - u_1)\Delta\varphi \mathrm{d}x + \lambda\int_\Omega (|\widetilde{u}_1|^{p-2}\widetilde{u}_1 - |u_1|^{p-2}u_1)\varphi \mathrm{d}x \\ + \int_\Omega \left(\ln\left(\Delta\widetilde{u}_1 + \sqrt{1+(\Delta\widetilde{u}_1)^2}\right) - \ln\left(\Delta u_1 + \sqrt{1+(\Delta u_1)^2}\right)\right)\Delta\varphi \mathrm{d}x = 0.$$

Choosing $\varphi = \widetilde{u}_1 - u_1$, we have

$$\int_\Omega \frac{(\widetilde{u}_1 - u_1)^2}{h}\mathrm{d}x + \int_\Omega \varepsilon\Delta^2(\widetilde{u}_1 - u_1)(\Delta\widetilde{u}_1 - \Delta u_1)\mathrm{d}x \\ + \int_\Omega \left(\ln\left(\Delta\widetilde{u}_1 + \sqrt{1+(\Delta\widetilde{u}_1)^2}\right) - \ln\left(\Delta u_1 + \sqrt{1+(\Delta u_1)^2}\right)\right)(\Delta\widetilde{u}_1 - \Delta u_1)\mathrm{d}x \\ + \lambda\int_\Omega (|\widetilde{u}_1|^{p-2}\widetilde{u}_1 - |u_1|^{p-2}u_1)(\widetilde{u}_1 - u_1)\mathrm{d}x = 0.$$

Since function $\ln(t + \sqrt{1+t^2})$ is increasing, we know that every term on the left-hand side is nonnegative. Therefore, we conclude that $u_1 = \widetilde{u}_1$ a.e. in Ω and complete the proof of the uniqueness. □

Next, we discuss the parabolic problem

$$\begin{cases} u_t + \Delta(\varepsilon\Delta u + \ln(\Delta u + \sqrt{1+(\Delta u)^2})) + \lambda|u|^{p-2}u = 0, & \text{in } \Omega_T, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \end{cases} \tag{2.4}$$

where $\varepsilon > 0$.

Theorem 2.3 *Assume $u_0 \in H_0^1(\Omega)$, problem (2.4) admits a unique weak solution $u_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)) \cap L^2(0,T; H^3(\Omega))$ with $\Delta u_\varepsilon \in L^2(0,T; H_0^1(\Omega))$, which satisfies the following estimates*

$$\frac{1}{2}\int_\Omega u_\varepsilon^2 \mathrm{d}x + \int_0^t \int_\Omega \left(\varepsilon \Delta u_\varepsilon + \ln\left(\Delta u_\varepsilon + \sqrt{1+(\Delta u_\varepsilon)^2}\right)\right) \Delta u_\varepsilon \mathrm{d}x\mathrm{d}\tau + \lambda \int_0^t \int_\Omega |u_\varepsilon|^p \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x, \tag{2.5}$$

and

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x + \int_0^t \int_\Omega \left(\varepsilon + \frac{1}{\sqrt{1+|\Delta u_\varepsilon|^2}}\right) |\nabla \Delta u_\varepsilon|^2 \mathrm{d}x\mathrm{d}\tau + \lambda \int_\Omega (p-1)|u_\varepsilon|^{p-2}|\nabla u_\varepsilon|^2 \mathrm{d}x \le \frac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x. \tag{2.6}$$

Proof By Theorem 2.2, we define $u_k \in H_0^1(\Omega) \cap H^2(\Omega)$, $k = 1, 2, \ldots, n$ to be the weak solution of the following elliptic problems

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} + \Delta\left(\varepsilon \Delta u_k + \ln\left(\Delta u_k + \sqrt{1+(\Delta u_k)^2}\right)\right) + \lambda |u_k|^{p-2} u_k = 0, \\ u_k|_{\partial\Omega} = 0, \quad \Delta u_k|_{\partial\Omega} = 0. \end{cases} \tag{2.7}$$

Therefore, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \frac{u_k - u_{k-1}}{h}\varphi \mathrm{d}x + \int_\Omega \Delta\left(\varepsilon \Delta u_k + \ln(\Delta u_k + \sqrt{1+(\Delta u_k)^2})\right)\varphi \mathrm{d}x + \lambda \int_\Omega |u_k|^{p-2} u_k \varphi \mathrm{d}x = 0.$$

Choosing $\varphi = \Delta u_k$, we have

$$\frac{1}{h}\int_\Omega (\nabla u_k - \nabla u_{k-1})\nabla u_k \mathrm{d}x + \int_\Omega \left(\varepsilon + \frac{1}{\sqrt{1+|\Delta u_k|^2}}\right)|\nabla \Delta u_k|^2 \mathrm{d}x + \lambda \int_\Omega (p-1)|u_k|^{p-2}|\nabla u_k|^2 \mathrm{d}x = 0,$$

that is

$$\frac{1}{2h}\int_\Omega |\nabla u_k|^2 \mathrm{d}x + \int_\Omega \left(\varepsilon + \frac{1}{\sqrt{1+|\Delta u_k|^2}}\right)|\nabla \Delta u_k|^2 \mathrm{d}x + \lambda \int_\Omega (p-1)|u_k|^{p-2}|\nabla u_k|^2 \mathrm{d}x \le \frac{1}{2h}\int_\Omega |\nabla u_{k-1}|^2 \mathrm{d}x. \tag{2.8}$$

Next, we construct an approximate solution u_h of problem (2.4) by defining

$$u_h(x,t)=\begin{cases} u_0(x), & t=0,\\ u_1(x), & 0<t\le h,\\ \cdots,\cdots, & \\ u_j(x), & (j-1)h<t\le jh,\\ \cdots,\cdots, & \\ u_n(x), & (n-1)h<t\le nh=T. \end{cases} \tag{2.9}$$

For every $t\in[0,T]$, (2.8) implies

$$\|\nabla u_h(x,t)\|_{L^2(\Omega)}^2\le\|\nabla u_0\|_{L^2(\Omega)}^2.$$

From the above inequality, we see that

$$\|\nabla u_h(x,t)\|_{L^\infty(0,T;L^2(\Omega))}\le\|\nabla u_0\|_{L^2(\Omega)}^2. \tag{2.10}$$

Summing up the inequalities in (2.8), we derive that

$$\int_0^T\int_\Omega\left(\varepsilon+\frac{1}{\sqrt{1+|\Delta u_h|^2}}\right)|\nabla\Delta u_h|^2\mathrm{d}x\mathrm{d}\tau\le\|\nabla u_0\|_{L^2(\Omega)}^2, \tag{2.11}$$

$$\lambda\int_0^T\int_\Omega(p-1)|u_h|^{p-2}|\nabla u_h|^2\mathrm{d}x\mathrm{d}\tau\le\|\nabla u_0\|_2^2. \tag{2.12}$$

Thus,

$$\|\Delta u_h\|_{L^2(0,T;H_0^1(\Omega))}+\|\ln\left(|\Delta u_h|+\sqrt{1+|\Delta u_h|^2}\right)\|_{L^2(0,T;H_0^1(\Omega))}\le C.$$

(2.11) implies that

$$\begin{aligned}\int_0^T\int_\Omega|\nabla\left(\ln\left(\Delta u_h+\sqrt{1+(\Delta u_h)^2}\right)\right)|^2\mathrm{d}x\mathrm{d}\tau&=\int_0^T\int_\Omega\frac{|\nabla\Delta u_h|^2}{1+|\Delta u_h|^2}\mathrm{d}x\mathrm{d}\tau\\&\le\int_0^T\int_\Omega\frac{|\nabla\Delta u_h|^2}{\sqrt{1+|\Delta u_h|^2}}\mathrm{d}x\mathrm{d}\tau\le C.\end{aligned}$$

By $\Delta u_h|_{\partial\Omega}=0$, we know that

$$\begin{aligned}&\|u_h\|_{L^\infty(0,T;H_0^1(\Omega))}+\|u_h\|_{L^2(0,T;H^3(\Omega))}\\&\quad+\|\ln\left(\Delta u_h+\sqrt{1+(\Delta u_h)^2}\right)\|_{L^2(0,T;H_0^1(\Omega))}\le C.\end{aligned}$$

Therefore, we may choose a subsequence (we also denote it by the original sequence for simplicity) such that

$$\begin{aligned} u_h &\rightharpoonup u_\varepsilon, && \text{weakly-*in} \quad L^\infty(0,T;H_0^1(\Omega)),\\ u_h &\rightharpoonup u_\varepsilon, && \text{weakly in} \quad L^2(0,T;H^3(\Omega)),\\ \Delta u_h &\rightharpoonup \Delta u_\varepsilon, && \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)),\\ \ln\left(\Delta u_h + \sqrt{1+(\Delta u_h)^2}\right) &\rightharpoonup \xi_\varepsilon, && \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \end{aligned} \tag{2.13}$$

which follows that ([5], Chapter 2)

$$\|u_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u_\varepsilon\|_{L^2(0,T;H^3(\Omega))} + \|\xi_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le C. \tag{2.14}$$

For each $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot,T) = 0$ and for every $k \in \{1,2,\ldots,n\}$, taking $\varphi(x,kh)$ as a test function in (2.7), we know that

$$\begin{aligned} &\frac{1}{h}\int_\Omega u_k\varphi(x,kh)\mathrm{d}x - \frac{1}{h}\int_\Omega u_{k-1}\varphi(x,kh)\mathrm{d}x + \lambda\int_\Omega |u_k|^{p-2}u_k\varphi(x,kh)\mathrm{d}x\\ &\quad - \int_\Omega \nabla\left(\varepsilon\Delta u_k + \ln\left(\Delta u_k + \sqrt{1+(\Delta u_k)^2}\right)\right)\nabla\varphi(x,kh)\mathrm{d}x = 0. \end{aligned} \tag{2.15}$$

Summing up all the equalities and using $\varphi(\cdot,T) = \varphi(\cdot,nh) = 0$, we deduce

$$\begin{aligned} &h\sum_{k=1}^{n-1}\int_\Omega \frac{u_h(x,kh)[\varphi(x,kh) - \varphi(x,(k+1)h)]}{h}\mathrm{d}x - \int_\Omega u_0\varphi(x,h)\mathrm{d}x\\ &\quad - h\sum_{k=1}^{n}\int_\Omega \nabla\left(\varepsilon\Delta u_h + \ln\left(\Delta u_h + \sqrt{1+(\Delta u_h)^2}\right)\right)\nabla\varphi(x,kh)\mathrm{d}x\\ &\quad + h\lambda\sum_{k=1}^{n}\int_\Omega |u_h|^{p-2}u_h\varphi(x,kh)\mathrm{d}x = 0. \end{aligned} \tag{2.16}$$

Passing to the limits as $h \to 0$, we obtain from (2.13), (2.14), (2.16) that

$$\begin{aligned} &-\int_0^T\int_\Omega u_\varepsilon\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}\tau - \int_\Omega u_0\varphi(x,0)\mathrm{d}x\\ &\quad - \int_0^T\int_\Omega \nabla(\varepsilon\Delta u_\varepsilon + \xi_\varepsilon)\nabla\varphi\mathrm{d}x\mathrm{d}\tau + \lambda\int_0^T\int_\Omega |u_\varepsilon|^{p-2}u_\varepsilon\varphi\mathrm{d}x\mathrm{d}\tau = 0. \end{aligned} \tag{2.17}$$

In addition, if $\varphi \in C_0^\infty(\Omega_T)$, we obtain

$$
\begin{aligned}
&-\int_0^T \int_\Omega u_\varepsilon \frac{\partial \varphi}{\partial t} \mathrm{d}x\mathrm{d}\tau \\
&\quad = \int_0^T \int_\Omega \nabla(\varepsilon \Delta u_\varepsilon + \xi_\varepsilon)\nabla\varphi \mathrm{d}x\mathrm{d}\tau - \lambda \int_0^T \int_\Omega |u_\varepsilon|^{p-2} u_\varepsilon \varphi \mathrm{d}x\mathrm{d}\tau. \qquad (2.18)
\end{aligned}
$$

Noticing that $u_\varepsilon \in L^2(0, T; H^3(\Omega))$ and $\xi_\varepsilon \in L^2(0, T; H_0^1(\Omega))$, we see that

$$
\frac{\partial u_\varepsilon}{\partial t} \in L^2(0, T; H^{-1}(\Omega)).
$$

As $u_\varepsilon \in L^2(0, T; H_0^1(\Omega))$, it follows from the compact imbedding relation

$$
H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),
$$

that

$$
u_\varepsilon \in C([0, T]; L^2(\Omega)).
$$

As the function u_ε satisfies (2.17), we only need to show that $\xi_\varepsilon = \ln(\Delta u_\varepsilon + \sqrt{1 + (\Delta u_\varepsilon)^2})$ a.e. in Ω_T to prove the existence of weak solutions. Taking u_ε as a test function in (2.4), we have

$$
\begin{aligned}
&\int_\Omega \frac{1}{2} |u_\varepsilon(T)|^2 \mathrm{d}x - \int_\Omega \frac{1}{2} |u_0|^2 \mathrm{d}x + \int_0^T \int_\Omega (\varepsilon \Delta u_\varepsilon + \xi_\varepsilon) \Delta u_\varepsilon \mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda \int_0^T \int_\Omega |u_\varepsilon|^p \mathrm{d}x\mathrm{d}\tau = 0. \qquad (2.19)
\end{aligned}
$$

Choosing u_k as a test function in (2.7), we have

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega u_k^2 \mathrm{d}x + \int_\Omega h\Delta \left(\varepsilon \Delta u_k + \ln \left(\Delta u_k + \sqrt{1 + (\Delta u_k)^2} \right) \right) u_k \mathrm{d}x \\
&\quad + h\lambda \int_\Omega |u_k|^p \mathrm{d}x \le \frac{1}{2} \int_\Omega u_{k-1}^2 \mathrm{d}x.
\end{aligned}
$$

Summing up the above equalities for $k = 1, 2, \dots, n$, we derive that

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega u_h^2(T) \mathrm{d}x + \int_0^T \int_\Omega \left(\varepsilon \Delta u_h + \ln \left(\Delta u_h + \sqrt{1 + (\Delta u_h)^2} \right) \right) \Delta u_h \mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda \int_0^T \int_\Omega |u_h|^p \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2} \int_\Omega u_0^2 \mathrm{d}x. \qquad (2.20)
\end{aligned}
$$

Using the fact

$$\left(\ln\left(\xi+\sqrt{1+\xi^2}\right)-\ln\left(\eta+\sqrt{1+\eta^2}\right)\right)(\xi-\eta)\ge 0,$$

for all $\xi,\eta\in\mathbb{R}$, we easily know that

$$\int_0^T\int_\Omega\left(\ln\left(\Delta u_h+\sqrt{1+(\Delta u_h)^2}\right)-\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)(\Delta u_h-\Delta v)\mathrm{d}x\mathrm{d}\tau\ge 0,$$

for every $v\in L^2(0,T;H^2(\Omega))$. Thus, from (2.20) we have that

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_h^2(T)\mathrm{d}x-\frac{1}{2}\int_\Omega u_0^2\mathrm{d}x+\int_0^T\int_\Omega\varepsilon|\Delta u_h|^2\mathrm{d}x\mathrm{d}\tau+\lambda\int_0^T\int_\Omega|u_h|^p\mathrm{d}x\mathrm{d}\tau\\&+\int_0^T\int_\Omega\left(\left(\ln\left(\Delta u_h+\sqrt{1+(\Delta u_h)^2}\right)\right)\Delta v+\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta u_h\right)\mathrm{d}x\mathrm{d}\tau\\&-\int_0^T\int_\Omega\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta v\mathrm{d}x\mathrm{d}\tau\le 0.\end{aligned}$$

Passing to limits as $h\to 0$ and noticing

$$\begin{aligned}\|u_\varepsilon(T)\|_{L^2(\Omega)}&\le\liminf_{h\to 0}\|u_h(T)\|_{L^2(\Omega)},\\ \|\Delta u_\varepsilon\|_{L^2(\Omega_T)}&\le\liminf_{h\to 0}\|\Delta u_h\|_{L^2(\Omega_T)},\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_\varepsilon^2(T)\mathrm{d}x-\frac{1}{2}\int_\Omega u_0^2\mathrm{d}x+\int_0^T\int_\Omega\varepsilon|\Delta u_\varepsilon|^2\mathrm{d}x\mathrm{d}\tau+\lambda\int_0^T\int_\Omega|u_\varepsilon|^p\mathrm{d}x\mathrm{d}\tau\\&+\int_0^T\int_\Omega\xi_\varepsilon\Delta v\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_\Omega\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta u_\varepsilon\mathrm{d}x\mathrm{d}\tau\\&-\int_0^T\int_\Omega\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta v\mathrm{d}x\mathrm{d}\tau\le 0.\end{aligned}\tag{2.21}$$

Combining (2.19) with (2.21), we have, for every $v\in L^2(0,T;H^2(\Omega))$,

$$\begin{aligned}&\int_0^T\int_\Omega\xi_\varepsilon\Delta u_\varepsilon\mathrm{d}x\mathrm{d}\tau-\int_0^T\int_\Omega\xi_\varepsilon\Delta v\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_\Omega\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta u_\varepsilon\mathrm{d}x\mathrm{d}\tau\\&-\int_0^T\int_\Omega\left(\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)\Delta v\mathrm{d}x\mathrm{d}\tau\ge 0,\end{aligned}$$

which is

$$\int_0^T\int_\Omega\left(\xi_\varepsilon-\ln\left(\Delta v+\sqrt{1+(\Delta v)^2}\right)\right)(\Delta u_\varepsilon-\Delta v)\mathrm{d}x\mathrm{d}\tau\ge 0.$$

For each $\gamma > 0$, $\omega \in C_0^\infty(\Omega_T)$, we choose $v \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ to be the solution of $\Delta v = \Delta u_\varepsilon - \lambda\omega$ in the above inequality to have

$$\int_0^T \int_\Omega \omega(\xi_\varepsilon - \ln((\Delta u_\varepsilon - \gamma\omega) + \sqrt{1 + (\Delta u_\varepsilon - \gamma\omega)^2}))\mathrm{d}x\mathrm{d}\tau \geq 0.$$

Passing to limits as $\gamma \to 0$ and using Lebesgue's dominated convergence theorem, we get

$$\int_0^T \int_\Omega \omega(\xi_\varepsilon - \ln(\Delta u_\varepsilon + \sqrt{1 + (\Delta u_\varepsilon)^2}))\mathrm{d}x\mathrm{d}\tau \geq 0,$$

for every $\omega \in C_0^\infty(\Omega_T)$ and conclude that $\xi_\varepsilon = \ln(\Delta u_\varepsilon + \sqrt{1 + (\Delta u_\varepsilon)^2})$ a. e. in Ω_T. By approximation, we use (2.20) to obtain (2.5) and use (2.8) to obtain (2.6). Therefore, we finish the proof of the existence of weak solutions.

The proof of the uniqueness of weak solutions is similar to the proof of uniqueness of problem (2.1), so we omit the details. Thus, we complete the proof of Theorem 2.3. □

Proof of Theorem 2.1 First, by Theorem 2.3, we know that

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \varepsilon\|u_\varepsilon\|_{L^2(0,T;H^3(\Omega))} + \|\ln(\Delta u_\varepsilon \\ &+\sqrt{1 + (\Delta u_\varepsilon)^2})\|_{L^2(0,T;H_0^1(\Omega))} \leq C. \end{aligned}$$

Therefore, we can extract a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u, && \text{weakly-}^*\text{in} \quad L^\infty(0, T; H_0^1(\Omega)), \\ \sqrt{\varepsilon}\nabla\Delta u_\varepsilon &\rightharpoonup \zeta, && \text{weakly in} \quad L^2(\Omega_T), \\ \xi_\varepsilon = \ln(\Delta u_\varepsilon + \sqrt{1 + (\Delta u_\varepsilon)^2}) &\rightharpoonup \xi, && \text{weakly in} \quad L^2(0, T; H_0^1(\Omega)), \end{aligned}$$

which follows that ([5], Chapter 2)

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\zeta\|_{L^2(\Omega_T)} + \|\xi\|_{L^2(0,T;H_0^1(\Omega))} \leq C. \tag{2.22}$$

Using (2.17), we deduce that

$$\begin{aligned} &-\int_0^T \int_\Omega u_\varepsilon \frac{\partial \varphi}{\partial t}\mathrm{d}x\mathrm{d}\tau - \int_\Omega u_0\varphi(x, 0)\mathrm{d}x \\ &- \int_0^T \int_\Omega \nabla(\varepsilon\Delta u_\varepsilon + \xi_\varepsilon)\nabla\varphi\mathrm{d}x\mathrm{d}\tau + \lambda\int_0^T \int_\Omega |u_\varepsilon|^{p-2}u_\varepsilon\varphi(x, 0)\mathrm{d}x\mathrm{d}\tau = 0. \end{aligned}$$

Letting $\varepsilon \to 0$, we see that

$$
\begin{aligned}
&-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \mathrm{d}x\mathrm{d}\tau - \int_\Omega u_0 \varphi(x,0)\mathrm{d}x \\
&\quad - \int_0^T \int_\Omega \nabla \xi \nabla \varphi \mathrm{d}x\mathrm{d}\tau + \lambda \int_0^T \int_\Omega |u|^{p-2} u \varphi(x,0)\mathrm{d}x\mathrm{d}\tau = 0. \qquad (2.23)
\end{aligned}
$$

Choosing $\varphi \in C_0^\infty(\Omega_T)$, we get

$$
-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \mathrm{d}x\mathrm{d}\tau - \int_0^T \int_\Omega \nabla \xi \nabla \varphi \mathrm{d}x\mathrm{d}\tau = 0,
$$

which implies that

$$
\frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(\Omega)).
$$

As $u \in L^2(0,T;H_0^1(\Omega))$, it follows from the compact imbedding relation

$$
H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),
$$

that

$$
u \in C([0,T];L^2(\Omega)).
$$

On the other hand, (2.6) implies that

$$
\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x \le \int_\Omega |\nabla u_0|^2 \mathrm{d}x, \qquad (2.24)
$$

$$
\int_0^t \int_\Omega \frac{1}{\sqrt{1+|\Delta u_\varepsilon|^2}} |\nabla \Delta u_\varepsilon|^2 \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2} \int_\Omega |\nabla u_0|^2 \mathrm{d}x. \qquad (2.25)
$$

Denote

$$
\omega_\varepsilon = \Delta u_\varepsilon.
$$

Noticing that $|\nabla|\omega_\varepsilon|| \le |\nabla \omega_\varepsilon|$, we conclude that

$$
\begin{aligned}
\int_0^t \int_\Omega |\nabla \ln\left(|\omega_\varepsilon| + \sqrt{1+|\omega_\varepsilon|^2}\right)|^2 \mathrm{d}x\mathrm{d}\tau &= \int_0^t \int_\Omega \left| \frac{\nabla |\omega_\varepsilon|}{\sqrt[4]{1+|\omega_\varepsilon|^2}} \right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\le \int_0^t \int_\Omega \frac{|\nabla \omega_\varepsilon|^2}{\sqrt{1+|\omega_\varepsilon|^2}} \mathrm{d}x\mathrm{d}\tau \\
&\le \frac{1}{2} \int_\Omega |\nabla u_0|^2 \mathrm{d}x \le C.
\end{aligned}
$$

Setting $v_\varepsilon = \ln(|\omega_\varepsilon| + \sqrt{1+|\omega_\varepsilon|^2})$ and using $v_\varepsilon|_{\partial\Omega} = 0$, we know that $v_\varepsilon \in L^2(0,T;H_0^1(\Omega))$ and

$$\|v_\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} \le C\int_0^T\int_\Omega |\nabla v_\epsilon|^2 \mathrm{d}x\mathrm{d}\tau \le C. \tag{2.26}$$

By $N = 2$, we see that $H_0^1(\Omega) \hookrightarrow L^\varphi(\Omega)$ with $\varphi = \exp t^2 - 1$. Then, we have $L^2(0,T;H_0^1(\Omega)) \hookrightarrow L^1(0,T;L^\varphi(\Omega))$, and that there exist two positive numbers C_1, C_2 such that

$$\int_0^T\int_\Omega \exp\left(\frac{v_\varepsilon}{C_1||\nabla v_\varepsilon||_{L^2(\Omega_T)}}\right)^2 \mathrm{d}x\mathrm{d}\tau \le C_2|\Omega_T|.$$

In addition, for every $\delta > 0$,

$$e^{2t} \le e^{(\frac{t}{\delta})^2} + e^{4\delta^2}.$$

Choosing $t = v_\varepsilon$ and $\delta = C_1\|\nabla v_\varepsilon\|_{L^2(\Omega_T)}$ in the above inequality, we derive

$$\begin{aligned}\int_0^T\int_\Omega e^{2v_\varepsilon}\mathrm{d}x\mathrm{d}\tau &\le \int_{\Omega_T}\exp\left(\frac{v_\varepsilon}{C_1\|\nabla v_\varepsilon\|_{L^2(\Omega_T)}}\right)^2\mathrm{d}x\mathrm{d}\tau + e^{4C_1^2\|\nabla v_\varepsilon\|^2_{L^2(\Omega_T)}}|\Omega_T| \\ &\le C|\Omega_T|,\end{aligned}$$

which further implies that

$$\int_0^T\int_\Omega |\omega_\varepsilon|^2\mathrm{d}x\mathrm{d}\tau = \int_{\Omega_T}|\Delta u_\varepsilon|^2\mathrm{d}x\mathrm{d}\tau \le C|\Omega_T| \le C. \tag{2.27}$$

It follows from $u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega))$ that

$$\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{2.28}$$

Therefore, we can extract a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \rightharpoonup u, \qquad \text{weakly in} \quad L^2(0,T;H^2(\Omega)), \tag{2.29}$$

which follows that ([5], Chapter 2)

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{2.30}$$

We only need to show that $\xi = \ln(\Delta u + \sqrt{1+(\Delta u)^2})$ a. e. in Ω_T to prove the existence of weak solutions of problems (1.1)–(1.3).

Taking u as a test function in (2.23), we know that

$$\int_\Omega \frac{1}{2}u^2(T)\mathrm{d}x - \int_\Omega \frac{1}{2}u_0^2\mathrm{d}x + \int_0^T\int_\Omega \xi\Delta u\mathrm{d}x\mathrm{d}\tau + \lambda\int_0^T\int_\Omega |u|^p\mathrm{d}x\mathrm{d}\tau = 0. \quad (2.31)$$

Passing to limits as $\varepsilon \to 0$ and noticing

$$\|u(T)\|_{L^2(\Omega)} \le \liminf_{\varepsilon\to 0} \|u_\varepsilon(T)\|_{L^2(\Omega)},$$
$$\|\Delta u\|_{L^2(\Omega_T)} \le \liminf_{\varepsilon\to 0} \|\Delta u_\varepsilon\|_{L^2(\Omega_T)},$$

we obtain

$$\begin{aligned}&\int_\Omega \left[\frac{1}{2}u^2(T) - \frac{1}{2}u_0^2\right]\mathrm{d}x + \lambda\int_0^T\int_\Omega |u|^p\mathrm{d}x\mathrm{d}\tau\\ &+\int_0^T\int_\Omega \Big[\xi\Delta v + \ln\left(\Delta v + \sqrt{1+(\Delta v)^2}\right)\Delta u\\ &-\ln\left(\Delta v + \sqrt{1+(\Delta v)^2}\right)\Delta v\Big]\mathrm{d}x\mathrm{d}\tau \le 0.\end{aligned}$$

Using (2.31), we have

$$\begin{aligned}&\int_0^T\int_\Omega \xi\Delta u\mathrm{d}x\mathrm{d}\tau - \int_0^T\int_\Omega \xi\Delta v\mathrm{d}x\mathrm{d}\tau + \int_0^T\int_\Omega \left(\ln\left(\Delta v + \sqrt{1+(\Delta v)^2}\right)\right)\Delta u\mathrm{d}x\mathrm{d}\tau\\ &-\int_0^T\int_\Omega \left(\ln\left(\Delta v + \sqrt{1+(\Delta v)^2}\right)\right)\Delta v\mathrm{d}x\mathrm{d}\tau \ge 0,\end{aligned}$$

which is

$$\int_0^T\int_\Omega \left(\xi - \ln\left(\Delta v + \sqrt{1+(\Delta v)^2}\right)\right)(\Delta u - \Delta v)\mathrm{d}x\mathrm{d}\tau \ge 0.$$

For each $\gamma > 0$, $\omega \in C_0^\infty(\Omega_T)$, we choose $v \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ to be the solution of $\Delta v = \Delta u - \gamma\omega$ in the above inequality to have

$$\int_0^T\int_\Omega \omega\left(\xi - \ln\left(\Delta u - \gamma\omega + \sqrt{1+(\Delta u - \gamma\omega)^2}\right)\right)\mathrm{d}x\mathrm{d}\tau \ge 0.$$

Passing to limits as $\gamma \to 0$ and using Lebesgue's dominated convergence theorem, we get

$$\int_0^T\int_\Omega \omega\left(\xi - \ln\left(\Delta u + \sqrt{1+(\Delta u)^2}\right)\right)\mathrm{d}x\mathrm{d}\tau \ge 0,$$

for every $\omega \in C_0^\infty(\Omega_T)$ and conclude that $\xi = \ln(\Delta u + \sqrt{1+(\Delta u)^2})$ a. e. in Ω_T.

It follows from (2.23) that u is a weak solution of problems (1.1)–(1.3). Therefore, we finish the proof of the existence of weak solutions.

The proof of uniqueness of weak solutions is obvious, so we omit the details. The proof of Theorem 2.1 is complete. □

3 Asymptotic behavior

This section is devoted to the asymptotic behavior of solutions. To this purpose, we first show that

Theorem 3.1 *The weak solution u satisfies that for any $0 \le \rho \in C^2(\overline{\Omega})$,*

$$\frac{1}{2}\int_\Omega \rho(x)|u(x,t)|^2dx - \frac{1}{2}\int_\Omega \rho(x)|u_0(x)|^2dx + \lambda \iint_{Q_t} \rho(x)|u(x,\tau)|^p dxd\tau$$
$$= -\iint_{Q_t} \ln\left(\Delta u + \sqrt{1+(\Delta u)^2}\right) \Delta(\rho(x)u(x,\tau))dxd\tau, \tag{3.1}$$

where $Q_t = \Omega \times (0,t)$.

Proof In the proof of Theorem 2.3, we have

$$f(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 \mathrm{d}x \in C([0,T]).$$

Similarly, we can also easily prove that for any $0 \le \rho(x) \in C^2(\overline{\Omega})$,

$$f_\rho(t) = \frac{1}{2}\int_\Omega \rho(x)|u(x,t)|^2 \mathrm{d}x \in C([0,T]).$$

Consider the functional

$$\Phi_\rho[v] = \frac{1}{2}\int_\Omega \rho(x)|v(x)|^2 \mathrm{d}x.$$

It is easy to see that $\Phi_\rho[v]$ is a convex functional on $L^2(\Omega)$.

For any $\tau \in (0,T)$ and $h > 0$, we have

$$\Phi_\rho[u(\tau+h)] - \Phi_\rho[u(\tau)] \ge \langle u(\tau+h) - u(\tau), \rho(x)u(x,\tau)\rangle.$$

By $\frac{\delta\Phi_\rho[v]}{\delta v} = \rho(x)v$, for any fixed $t_1, t_2 \in [0,T], t_1 < t_2$, integrating the above inequality with respect to τ over (t_1,t_2) , we have

$$\int_{t_2}^{t_2+h} \Phi_\rho[u(\tau)]\mathrm{d}\tau - \int_{t_1}^{t_1+h} \Phi_\rho[u(\tau)]\mathrm{d}\tau \ge \int_{t_1}^{t_2} \langle u(\tau+h) - u(\tau), \rho(x)u\rangle \mathrm{d}\tau.$$

Multiplying the both sides of the above inequality by $\frac{1}{h}$, and letting $h \to 0$, we obtain

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \geq \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, \rho(x)u \right\rangle \mathrm{d}\tau.$$

Similarly, we have

$$\Phi_\rho[u(\tau)] - \Phi_\rho[u(\tau - h)] \leq \langle (u(\tau) - u(\tau - h)), \rho(x)u \rangle.$$

Thus,

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \leq \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, \rho(x)u \right\rangle \mathrm{d}\tau,$$

and hence,

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] = \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, \rho(x)u \right\rangle \mathrm{d}\tau.$$

Taking $t_1 = 0$, $t_2 = t$, we get from the definition of solutions that

$$\begin{aligned} &\Phi_\rho[u(t)] - \Phi_\rho[u(0)] \\ &= \int_0^t \left\langle -\Delta \left(\ln \left(\Delta u + \sqrt{1 + (\Delta u)^2} \right) \right) - \lambda |u|^{p-2} u, \rho(x) u(\tau) \right\rangle \mathrm{d}\tau \\ &= -\int_0^t \left\langle \ln \left(\Delta u + \sqrt{1 + (\Delta u)^2} \right), \Delta[\rho(x) u(\tau)] \right\rangle \mathrm{d}\tau - \int_0^t \langle \lambda |u|^{p-2} u, \rho(x) u(\tau) \rangle \mathrm{d}\tau. \end{aligned}$$

The proof is complete. □

Theorem 3.2 *Let u be the weak solution of problems* (1.1)–(1.3)*, then*

$$\int_\Omega |u(x,t)|^2 dx \leq \frac{1}{(\frac{p-2}{2}|\Omega|^{\frac{2-p}{2}} \lambda t + C_1)^\alpha}, \quad \alpha = \frac{2}{p-2}, C_1 > 0.$$

Proof Taking $\rho(x) = 1$ in equality (3.1), we have

$$\begin{aligned} &\frac{1}{2} \int_\Omega |u(x,t)|^2 \mathrm{d}x - \frac{1}{2} \int_\Omega |u_0(x)|^2 \mathrm{d}x \\ &= -\int_0^t \int_\Omega \ln \left(\Delta u + \sqrt{1 + (\Delta u)^2} \right) \Delta u \mathrm{d}x \mathrm{d}t - \lambda \iint_{Q_t} |u|^p \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{3.2}$$

Let $f(t) = \frac{1}{2} \int_\Omega |u(x,t)|^2 \mathrm{d}x$. By (3.2), we have

$$f'(t) = -\int_\Omega \ln \left(\Delta u + \sqrt{1 + (\Delta u)^2} \right) \Delta u \mathrm{d}x - \lambda \int_\Omega |u|^p \mathrm{d}x \leq 0.$$

Noticing that $\ln(\Delta u + \sqrt{1+(\Delta u)^2})\Delta u \ge 0$ and using the Hölder inequality, we conclude that

$$\int_\Omega |u(x,t)|^2 \mathrm{d}x \le |\Omega|^{\frac{p-2}{p}} \left(\int_\Omega |u|^p \mathrm{d}x\right)^{2/p},$$

that is $f(t) \le |\Omega|^{\frac{p-2}{p}} \lambda^{-\frac{2}{p}} |f'(t)|^{2/p}$.

Again by $f'(t) \le 0$, we know that $f'(t) \le -|\Omega|^{\frac{2-p}{2}} \lambda f(t)^{p/2}$, and hence,

$$\int_\Omega |u(x,t)|^2 \mathrm{d}x \le \frac{1}{(\frac{p-2}{2}|\Omega|^{\frac{2-p}{2}}\lambda t + C_1)^\alpha}, \quad \alpha = \frac{2}{p-2}, C_1 > 0.$$

The proof is complete. □

4 Regularity of solutions

In this section, we consider the regularity of solutions for problems (1.1)–(1.3).

Theorem 4.1 *If u is weak solution of problems* (1.1)–(1.3), *for any* (x_1,t_1), $(x_2,t_2) \in Q_T$, *we have*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{1/2}),$$

where C is a constant depending only on p.

Proof Let

$$u_\varepsilon(x,t) = J_\varepsilon u(x,t) = \int_0^T \int_{|x-y|<\varepsilon} j_\varepsilon(x-y,t-s)u(y,s)\mathrm{d}y\mathrm{d}s,$$

where $j_\varepsilon(x-y,t-s)$ is the mollifier.

For any $x_1, x_2 \in \Omega$, we have

$$\begin{aligned}
&u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)\\
&= \int_0^T \int_{R^2} j_\varepsilon(x_1-y,t-s)u(y,s)\mathrm{d}y\mathrm{d}s - \int_0^T \int_{R^2} j_\varepsilon(x_2-y,t-s)u(y,s)\mathrm{d}y\mathrm{d}s\\
&= \int_0^T \int_{R^2} \frac{\partial j_\varepsilon(zx_1 + (1-z)x_2 - y, t-s)}{\partial z} u(y,s)\mathrm{d}z\mathrm{d}y\mathrm{d}s\\
&= \int_0^T \int_{R^2} \int_0^1 \nabla_x j_\varepsilon(zx_1 + (1-z)x_2 - y, t-s)(x_1 - x_2)u(y,s)\mathrm{d}z\mathrm{d}y\mathrm{d}s\\
&= -\int_0^T \int_{R^2} \int_0^1 \nabla_y j_\varepsilon(zx_1 + (1-z)x_2 - y, t-s)(x_1 - x_2)u(y,s)\mathrm{d}z\mathrm{d}y\mathrm{d}s\\
&= \int_0^T \int_{R^2} \int_0^1 j_\varepsilon(zx_1 + (1-z)x_2 - y, t-s)\nabla_y u(y,s)\mathrm{d}z\mathrm{d}y\mathrm{d}s(x_1 - x_2).
\end{aligned}$$

Therefore,

$$|u_\varepsilon(x_1,t)-u_\varepsilon(x_2,t)| \leq \int_0^T\int_{R^2}\int_0^1 |j_\varepsilon(zx_1+(1-z)x_2-y,t-s)||\nabla_y u(y,s)|\mathrm{d}z\mathrm{d}y\mathrm{d}s|x_1-x_2|,$$

and by $u \in L^2(0,T;H^2(\Omega))$, we obtain

$$|u_\varepsilon(x_1,t)-u_\varepsilon(x_2,t)| \leq C|x_1-x_2|. \tag{4.1}$$

Set $0<\varepsilon<t_1<t_2<T$. Let $\Delta t = t_2-t_1$, $B_\rho = B_{(\Delta t)^{1/2}}(x_0)$, $x_0\in\Omega$, choose ρ sufficiently small, such that $B_\rho \subset \Omega$, $\varphi \in C_0^2(B_\rho)$. Therefore, we can obtain

$$\begin{aligned}
&\int_{B_\rho}\varphi(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\mathrm{d}x\\
&\quad=\int_{B_\rho}\varphi(x)\int_0^1\frac{\partial u_\varepsilon(x,st_2+(1-s)t_1)}{\partial s}\mathrm{d}s\mathrm{d}x\\
&\quad=\Delta t\int_{B_\rho}\varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon}u(y,\tau)\\
&\qquad\cdot j_{\varepsilon t}(x-y,st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x\\
&\quad=-\Delta t\int_{B_\rho}\varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon}u(y,\tau)\\
&\qquad\cdot j_{\varepsilon\tau}(x-y,st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x.
\end{aligned}\tag{4.2}$$

Fixed $(x,t)\in Q_T,\ 0<\varepsilon<t<T-\varepsilon$, we have $j_\varepsilon(x-y,t-\tau)\in C_0^2(Q_T)$, from definition of weak solution

$$\begin{aligned}
&\int_0^T\int_{|x-y|<\varepsilon} j_{\varepsilon\tau}(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)\mathrm{d}y\mathrm{d}\tau\\
&\quad=\int_0^T\int_{|x-y|<\varepsilon}\ln\left(\Delta_y u+\sqrt{1+(\Delta_y u)^2}\right)\Delta_y j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\\
&\qquad+\lambda\int_0^T\int_{|x-y|<\varepsilon}|u|^{p-2}uj_\varepsilon(x-y,st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau,
\end{aligned}$$

and hence, (4.2) is converted into

$$\begin{aligned}
&\int_{B_\rho}\varphi(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\mathrm{d}x\\
&\quad=-\Delta t\int_{B_\rho}\varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon}\ln\left(\Delta_y u+\sqrt{1+(\Delta_y u)^2}\right)\\
&\qquad\cdot\Delta_y j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x
\end{aligned}$$

$$- \lambda\Delta t \int_{B_\rho} \varphi(x) \int_0^1 \int_0^T \int_{|x-y|<\varepsilon} |u|^{p-2}u$$
$$\cdot\, j_\varepsilon(x-y, st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x$$
$$= -\Delta t \int_0^1 \int_{B_\rho} \Delta_x\varphi(x) \int_0^T \int_{|x-y|<\varepsilon} \ln\left(\Delta_y u + \sqrt{1+(\Delta_y u)^2}\right)$$
$$\cdot\, j_\varepsilon(x-y, st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\mathrm{d}s$$
$$- \lambda\Delta t \int_{B_\rho} \varphi(x) \int_0^1 \int_0^T \int_{|x-y|<\varepsilon} |u|^{p-2}u$$
$$\cdot\, j_\varepsilon(x-y, st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x.$$

Taking

$$\varphi(x) = \varphi_h(x) = \int_{x_0-(\Delta t)^{1/2}e}^{x} \int_{-h}^{(\Delta t)^{1/2}-|y-x_0|-2h} \delta_h(s)\mathrm{d}s\mathrm{d}y,$$

where $e=(1,1)$, $\delta(s)\in C_0^2(R)$; $\delta(s)\geq 0$; $\delta(s)=0$, as $|s|\geq 1$;$\int_R \delta(s)\mathrm{d}s = 1$. For $h>0$, define $\delta_h(s)=\frac{1}{h}\delta(\frac{s}{h})$.

Hence,

$$\int_{B_\rho} \varphi_h(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\mathrm{d}x$$
$$= -\Delta t \int_0^1 \int_{I_\rho} \delta_h((\Delta t)^{1/2}-|x-x_0|-2h)\frac{x_0-x}{|x-x_0|} J_\varepsilon\left(\ln\left(\Delta u+\sqrt{1+(\Delta u)^2}\right)\right)\mathrm{d}x\mathrm{d}s$$
$$- \lambda\Delta t \int_{B_\rho} \varphi(x) \int_0^1 \int_0^T \int_{|x-y|<\varepsilon} |u|^{p-2}u$$
$$\cdot\, j_\varepsilon(x-y, st_2+(1-s)t_1-\tau)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\mathrm{d}x.$$

Noticing that for $x\in B_\rho$, $\lim\limits_{h\to 0}\varphi_h(x)=1$, and if $|x-x_0|<(\Delta t)^{1/2}-h$, then $\delta_h((\Delta t)^{1/2}-|x-x_0|-2h)=0$, $\delta_h\leq\frac{C}{h}$ and

$$m(B_\rho\backslash B_{(\Delta t)^{1/2}-h})\leq Ch(\Delta)^{1/2}.$$

By $J_\varepsilon(\ln(\Delta u+\sqrt{1+(\Delta u)^2}))\leq C$ and $u\in L^\infty(0,T;H_0^1(\Omega))$, therefore

$$\left|\int_{B_\rho} \varphi_h(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\mathrm{d}x\right|\leq C(\Delta t)^{3/2}.$$

Letting $h\to 0$, we obtain

$$\left|\int_{B_\rho} (u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\mathrm{d}x\right|\leq C(\Delta t)^{3/2}.$$

Applying the mean value theorem, we see that for some $x^* \in B_\rho$ such that

$$|u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| \le C(\Delta t)^{1/2}.$$

Taking this into account and using (4.1), it follows that

$$\begin{aligned}
&|u_\varepsilon(x, t_2) - u_\varepsilon(x, t_1)| \\
&\quad \le |u_\varepsilon(x, t_2) - u_\varepsilon(x^*, t_2)| + |u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| + |u_\varepsilon(x^*, t_1) - u_\varepsilon(x, t_1)| \\
&\quad \le C(\Delta t)^{1/2},
\end{aligned}$$

and letting $\varepsilon \to 0$, we known that u is Hölder continuous. The proof is complete. □

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