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Derivations and 2-Local Derivations on Matrix Algebras and Algebras of Locally Measurable Operators

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Abstract

Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. We show that every derivation $D : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{M})$, $n \geq 2$, can be represented as a sum $D = D_m + \overline{\delta}$, where D_m is an inner derivation and $\overline{\delta}$ is a derivation induced by a derivation δ from $\mathcal{A}$ into $\mathcal{M}$. If $\mathcal{A}$ commutes with $\mathcal{M}$, we prove that every 2-local inner derivation $\Delta : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{M}), n \geq 2$, is an inner derivation. In addition, if $\mathcal{A}$ is commutative and commutes with $\mathcal{M}$, then every 2-local derivation $\Delta : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{M}), n \geq 2$, is a derivation. Let $\mathcal{R}$ be a finite von Neumann algebra of type I with center $\mathcal{Z}$ and $LS(\mathcal{R})$ be the algebra of locally measurable operators affiliated with $\mathcal{R}$. We also prove that if the lattice $\mathcal{Z}_{\mathcal{P}}$ of all projections in $\mathcal{Z}$ is atomic, then every derivation $D : \mathcal{R} \rightarrow LS(\mathcal{R})$ is an inner derivation.

Keywords Derivation · 2-Local derivation · Locally measurable operator · Von Neumann algebra

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1 Introduction

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ the field of complex numbers and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear map δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *Jordan derivation* if $\delta(a^2) = \delta(a)a + a\delta(a)$ for each a in $\mathcal{A}$. A linear map δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for each a, b in $\mathcal{A}$. Let m be an element in $\mathcal{M}$, the map $\delta_m : \mathcal{A} \to \mathcal{M},\ a \to \delta_m(a) := ma - am$, is a derivation. A derivation $\delta : \mathcal{A} \to \mathcal{M}$ is said to be an *inner derivation* when it can be written in the form $\delta = \delta_m$ for some m in $\mathcal{M}$. A fundamental result, due to Sakai [18], states that every derivation on a von Neumann algebra is an inner derivation.

An algebra $\mathcal{A}$ is called *regular* (in the sense of von Neumann) if for each a in $\mathcal{A}$ there exists b in $\mathcal{A}$ such that $a = aba$. Let $\mathcal{R}$ be a von Neumann algebra. We denote $S(\mathcal{R})$ and $LS(\mathcal{R})$, respectively, the algebras of all measurable and locally measurable operators affiliated with $\mathcal{R}$. For a faithful normal semi-finite trace τ on $\mathcal{R}$, we denote the algebra of all τ-measurable operators from $S(\mathcal{R})$ by $S(\mathcal{R}, \tau)$ (cf. [1,4,14]). If $\mathcal{R}$ is an abelian von Neumann algebra, then it is $*$-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) essentially bounded measurable complex functions on a measurable space (Ω, Σ, μ), and therefore, $LS(\mathcal{R}) = S(\mathcal{R}) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ is a unital commutative regular algebra of all measurable complex functions on (Ω, Σ, μ). In this case inner derivations on the algebra $S(\mathcal{R})$ are identically zero, i.e., trivial.

Ber et al. [9] obtain necessary and sufficient conditions for existence of non-trivial derivations on commutative regular algebras. In particular, they prove that the algebra $L^0(0, 1)$ of all measurable complex functions on the interval (0, 1) admits non-trivial derivations. Let $\mathcal{R}$ be a properly infinite von Neumann algebra. Ayupov and Kudaybergenov [4] show that every derivation on the algebra $LS(\mathcal{R})$ is an inner derivation.

In 1997, Šemrl [17] introduced 2-local derivations and 2-local automorphisms. A map $\Delta : \mathcal{A} \to \mathcal{M}$ (not necessarily linear) is called a *2-local derivation* if, for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{M}$ such that $D_{x,y}(x) = \Delta(x)$ and $D_{x,y}(y) = \Delta(y)$. In particular, if, for every $x, y \in \mathcal{A}$, $D_{x,y}$ is an inner derivation, then we call Δ is a *2-local inner derivation.* Niazi and Peralta [15] introduce the notion of weak-2-local derivation (respectively, *-derivation) and prove that every weak-2-local *-derivation on M_n is a derivation. 2-local derivations and weak-2-local derivations have been investigated by many authors on different algebras and many results have been obtained in [3–8,11,13,15–17,19].

Let $\mathcal{H}$ be a infinite-dimensional separable Hilbert space. In [17] Šemrl shows that every 2-local derivation on $\mathcal{B}(\mathcal{H})$ is a derivation. Kim and Kim [13] give a short proof of that every 2-local derivation on a finite-dimensional complex matrix algebra is a derivation. Ayupov and Kudaybergenov [3] extend this result to an arbitrary von Neumann algebra. Ayupov et al. [5] prove that if $\mathcal{R}$ is a finite von Neumann algebra of type I without abelian direct summands, then each 2-local derivation on the algebra $LS(\mathcal{R}) = S(\mathcal{R})$ is a derivation. In the same paper, the authors also show that if $\mathcal{R}$ is an abelian von Neumann algebra such that the lattice of all projections in $\mathcal{R}$ is not atomic, then there exists a 2-local derivation on the algebra $S(\mathcal{R})$ which is not a derivation. Zhang and Li [19] construct an example of a 2-local derivation on the algebra of all triangular complex 2×2 matrices which is not a derivation.

Ayupov et al. [5] show that if $\mathcal{A}$ is a unital commutative regular algebra, then every 2-local derivation on the algebra $M_n(\mathcal{A})$, $n \geq 2$, is a derivation. Ayupov and Arzikulov [8] show that if $\mathcal{A}$ is a unital commutative ring, then every 2-local inner derivation on $M_n(\mathcal{A})$, $n \geq 2$, is an inner derivation. Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. He et al. [11] prove that if every Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$ is an inner derivation then every 2-local derivation from $M_n(\mathcal{A})$ $(n \geq 3)$ into $M_n(\mathcal{M})$ is a derivation.

Throughout this paper, $\mathcal{A}$ is an algebra with unit 1 over $\mathbb{C}$ and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule. We say that $\mathcal{A}$ *commutes with* $\mathcal{M}$ if $am = ma$ for every $a \in \mathcal{A}$ and $m \in \mathcal{M}$. From now on, $M_n(\mathcal{A})$, for $n \geq 2$, will denote the algebra of all $n \times n$ matrices over $\mathcal{A}$ with the usual operations. By the way, we denote any element in $M_n(\mathcal{A})$ by $(a_{rs})_{n\times n}$, where $r, s \in \{1, 2, \ldots, n\}$; E_{ij}, $i, j \in \{1, 2, \ldots, n\}$, the matrix units in $M_n(\mathbb{C})$; and $x \otimes E_{ij}$, the matrix whose (i, j)-th entry is x and zero elsewhere. We use A_{ij} for the (i, j)-th entry of $A \in M_n(\mathcal{A})$ and denote $\operatorname{diag}(x_1, \ldots, x_n)$ or $\operatorname{diag}(x_i)$ the diagonal matrix with entries $x_i \in \mathcal{A}$, $i \in \{1, 2, \ldots, n\}$, in the diagonal positions. Particularly, we denote $\operatorname{diag}(x_i)$ by $\operatorname{diag}(x)$, where $x_i = x$ for every $i \in \{1, 2, \ldots, n\}$.

Let $\delta : \mathcal{A} \to \mathcal{M}$ be a derivation. Setting

$$\overline{\delta}((a_{ij})_{n\times n}) = (\delta(a_{ij}))_{n\times n}, \tag{1.1}$$

we obtain a well-defined linear operator from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$, where $M_n(\mathcal{M})$ has a natural structure of $M_n(\mathcal{A})$-bimodule. Moreover, $\overline{\delta}$ is a derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. If $\mathcal{A}$ is a commutative algebra, then the restriction of $\overline{\delta}$ onto the center of the algebra $M_n(\mathcal{A})$ coincides with the given δ.

In this paper we give characterizations of derivations, 2-local inner derivations and 2-local derivations from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. In Sect. 2, we show that a derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, can be decomposed as a sum of an inner derivation and a derivation induced by a derivation from $\mathcal{A}$ to $\mathcal{M}$ as (1.1), as follows:

$$D = D_B + \overline{\delta}.$$

In addition, the representation of the above form is unique if and only if $\mathcal{A}$ commutes with $\mathcal{M}$. Let $\mathcal{R}$ be a finite von Neumann algebra of type I with center $\mathcal{Z}$ and $LS(\mathcal{R})$ be the algebra of locally measurable operators affiliated with $\mathcal{R}$. we prove that if the lattice $\mathcal{Z}_{\mathcal{P}}$ of all projections in $\mathcal{Z}$ is atomic, then every derivation $D : \mathcal{R} \to LS(\mathcal{R})$ is an inner derivation.

In Sect. 3, we consider 2-local inner derivations and 2-local derivations from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. For the case that $\mathcal{A}$ commutes with $\mathcal{M}$, we obtain that every inner 2-local derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ is an inner derivation. In addition, if $\mathcal{A}$ is commutative, we prove that every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is a derivation. Let $\mathcal{R}$ be an arbitrary von Neumann algebra without abelian direct summands. We also show every 2-local derivation $\Delta : \mathcal{R} \to LS(\mathcal{R})$ is a derivation.

2 Derivations

Let $\mathcal{A}$ be an algebra with unit 1 over $\mathbb{C}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Let $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, be a derivation. Firstly, we define a map $D_{rs}^{ij} : \mathcal{A} \to \mathcal{M}$ by

$$D_{rs}^{ij}(a) = [D(a \otimes E_{rs})]_{ij}, \quad a \in \mathcal{A},\ i, j, r, s \in \{1, 2, \ldots, n\}.$$

For any $a, b \in \mathcal{A}$ and some fixed $m \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} D_{rs}^{ij}(ab) &= [D(ab \otimes E_{rs})]_{ij} \\ &= [D((a \otimes E_{rm})(b \otimes E_{ms}))]_{ij} \\ &= [D(a \otimes E_{rm})(b \otimes E_{ms})]_{ij} + [(a \otimes E_{rm})D(b \otimes E_{ms})]_{ij} \\ &= \delta_{js}[D(a \otimes E_{rm})]_{im}b + \delta_{ir}a[D(b \otimes E_{ms})]_{mj}, \end{aligned}$$

where δ is the Kronecker's delta. It follows that

$$D_{rs}^{ij}(ab) = \delta_{js}[D(a \otimes E_{rm})]_{im}b + \delta_{ir}a[D(b \otimes E_{ms})]_{mj}. \tag{2.1}$$

For any $m \in \{1, 2, \ldots, n\}$, we deduce from the equality (2.1) that

$$D_{mm}^{mm}(ab) = D_{mm}^{mm}(a)b + aD_{mm}^{mm}(b),$$

thus $D_{mm}^{mm} : \mathcal{A} \to \mathcal{M}$ is a derivation. We abbreviate the derivation D_{mm}^{mm} by D^m. Particularly, we denote the derivation D_{11}^{11} by D^1.

Theorem 2.1 *Every derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, can be represented as a sum*

$$D = D_B + \overline{\delta}, \tag{2.2}$$

where D_B is an inner derivation implemented by an element $B \in M_n(\mathcal{M})$ and $\overline{\delta}$ is a derivation of the form (1.1) induced by a derivation δ from $\mathcal{A}$ into $\mathcal{M}$. Furthermore, if this representation is unique for every derivation D, then $\mathcal{A}$ commutes with $\mathcal{M}$ (i.e., $am = ma$ for every $a \in \mathcal{A},\ m \in \mathcal{M}$); and if $\mathcal{A}$ commutes with $\mathcal{M}$ then this representation is always unique.

Before the proof of Theorem 2.1, we first present the following lemma.

Lemma 2.2 *For every $i, j, r, s, m \in \{1, 2, \ldots, n\}$ and every $a \in \mathcal{A}$ the following equalities hold:*

(i) $D_{rs}^{ij} = 0$, *$i \neq r$ and $j \neq s$,*
(ii) $D_{rj}^{ij}(a) = D_{rm}^{im}(a) = D_{rm}^{im}(1)a$, *if $i \neq r$,*
(iii) $D_{is}^{ij}(a) = D_{ms}^{mi}(a) = aD_{ms}^{mj}(1)$, *if $j \neq s$,*

(iv) $D^{im}_{jm}(1) = -D^{mj}_{mi}(1)$,
(v) $D^{ij}_{ij}(a) = D^{im}_{im}(1)a - aD^{jm}_{jm}(1) + D^m(a)$.

Proof It obviously follows from (2.1) that statements (i), (ii) and (iii) hold. We only need to prove (iv) and (v).

(iv): In the case $i = j$, we have

$$\begin{aligned}0 &= [D(1 \otimes E_{ii})]_{ii} = [D((1 \otimes E_{im})(1 \otimes E_{mi}))]_{ii}\\ &= [D((1 \otimes E_{im}))(1 \otimes E_{mi})]_{ii} + [(1 \otimes E_{im})D((1 \otimes E_{mi}))]_{ii}\\ &= D^{im}_{im}(1) + D^{mi}_{mi}(1),\end{aligned}$$

i.e.,

$$D^{im}_{im}(1) = -D^{mi}_{mi}(1). \tag{2.3}$$

For the case $i \neq j$, we have

$$\begin{aligned}0 &= D(0) = [D((1 \otimes E_{ii})(1 \otimes E_{jj}))]_{ij}\\ &= [D((1 \otimes E_{ii}))(1 \otimes E_{jj})]_{ij} + [(1 \otimes E_{ii})D((1 \otimes E_{jj}))]_{ij}\\ &= [D(1 \otimes E_{ii})]_{ij} + [D(1 \otimes E_{jj})]_{ij}\\ &= D^{ij}_{ii}(1) + D^{ij}_{jj}(1),\end{aligned}$$

i.e.,

$$D^{ij}_{jj}(1) = -D^{ij}_{ii}(1).$$

By (ii), (iii) and equality (2.3), it follows that

$$D^{im}_{jm}(1) = -D^{mj}_{mi}(1).$$

(v): By equality (2.1), we have

$$D^{ij}_{ij}(a) = D^{im}_{im}(1)a + D^{mj}_{mj}(a), \tag{2.4}$$

and

$$D^{ij}_{ij}(a) = D^{im}_{im}(a) + aD^{mj}_{mj}(1). \tag{2.5}$$

Taking $j = m$ in equality (2.4), we obtain that

$$D^{im}_{im}(a) = D^{im}_{im}(1)a + D^m(a). \tag{2.6}$$

By equalities (2.3), (2.5) and (2.6), it follows that

$$D^{ij}_{ij}(a) = D^{im}_{im}(1)a - aD^{jm}_{jm}(1) + D^m(a).$$

The proof is complete. □

Now we are in position to prove Theorem 2.1.

***Proof of Theorem* 2.1.** Let $(a_{rs})_{n\times n}$ be an arbitrary element in $M_n(\mathcal{A})$ and D be a derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. For any $i, j \in \{1, 2, \ldots, n\}$, it follows from Lemma 2.2 that

$$\begin{aligned}
[D((a_{rs})_{n\times n})]_{ij} &= \sum_{r,s=1}^{n} D_{rs}^{ij}(a_{rs}) \\
&= \sum_{r=1}^{n} D_{rj}^{ij}(a_{rj}) + \sum_{s=1}^{n} D_{is}^{ij}(a_{is}) - D_{ij}^{ij}(a_{ij}) \\
&= \sum_{r\neq i} D_{rj}^{ij}(a_{rj}) + \sum_{s\neq j} D_{is}^{ij}(a_{is}) + D_{ij}^{ij}(a_{ij}) \\
&= \sum_{r\neq i} D_{r1}^{i1}(1)a_{rj} + \sum_{s\neq j} a_{is} D_{1s}^{1j}(1) + D_{i1}^{i1}(1)a_{ij} \\
&\quad - a_{ij} D_{j1}^{j1}(1) + D^1(a_{ij}) \\
&= \sum_{r=1}^{n} D_{r1}^{i1}(1)a_{rj} - \sum_{s=1}^{n} a_{is} D_{j1}^{s1}(1) + D^1(a_{ij}) \\
&= \sum_{k=1}^{n} \left(D_{k1}^{i1}(1)a_{kj} - a_{ik} D_{j1}^{k1}(1) \right) + D^1(a_{ij}) \\
&= \left[(D_{s1}^{r1}(1))_{n\times n}(a_{rs})_{n\times n} - (a_{rs})_{n\times n}(D_{s1}^{r1}(1))_{n\times n} \right]_{ij} \\
&\quad + \left[\overline{D^1}((a_{rs})_{n\times n}) \right]_{ij},
\end{aligned}$$

i.e.,

$$\begin{aligned}
[D((a_{rs})_{n\times n})]_{ij} &= \left[\left(D_{s1}^{r1}(1))_{n\times n}(a_{rs})_{n\times n} - (a_{rs})_{n\times n}(D_{s1}^{r1}(1) \right)_{n\times n} \right]_{ij} \\
&\quad + \left[\overline{D^1}((a_{rs})_{n\times n}) \right]_{ij}, \qquad (2.7)
\end{aligned}$$

where $(D_{s1}^{r1}(1))_{n\times n} \in M_n(\mathcal{M})$ and $[(D_{s1}^{r1}(1))_{n\times n}]_{ij} = D_{j1}^{i1}(1)$. By equality (2.7), we have

$$D((a_{rs})_{n\times n}) = \left[(D_{s1}^{r1}(1))_{n\times n}(a_{rs})_{n\times n} - (a_{rs})_{n\times n}(D_{s1}^{r1}(1))_{n\times n} \right] + \left[\overline{D^1}((a_{rs})_{n\times n}) \right].$$

We denote $B = (D_{s1}^{r1}(1))_{n\times n}$ and $\delta = D^1$. Therefore, every derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, can be represented as a sum

$$D = D_B + \overline{\delta}.$$

Suppose that D_M is an inner derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ implemented by an element $M \in M_n(\mathcal{M})$, and $\overline{\zeta}$ is a derivation of the form (1.1) induced by a derivation ζ from $\mathcal{A}$ into $\mathcal{M}$, such that $D_M = \overline{\zeta}$. The first step is to establish the following. □

Claim 1 If $\mathcal{A}$ commutes with $\mathcal{M}$, then $D_M = \overline{\zeta} = 0$.

Proof of Claim 1 If $i \neq j,\ i, j \in \{1, 2, \ldots, n\}$, we have

$$0 = \overline{\zeta}(E_{ij}) = D_M(E_{ij}) = ME_{ij} - E_{ij}M.$$

It follows that $M_{ji} = 0$. Thus, M has a diagonal form, i.e., $M = \mathrm{diag}(M_{kk})$. Suppose that $\overline{\zeta} \neq 0$, then there exists an element $a \in \mathcal{A}$ such that $\zeta(a) \neq 0$. Take $A = \mathrm{diag}(a)$, then $\overline{\zeta}(A) \neq 0$. On the other hand,

$$\overline{\zeta}(A) = D_M(A) = \mathrm{diag}(M_{kk})\mathrm{diag}(a) - \mathrm{diag}(a)diag(M_{kk}) = 0.$$

This is a contradiction. Thus, $\overline{\zeta} = 0$. □

Claim 2 If $\mathcal{A}$ does not commute with $\mathcal{M}$, then there exist D_M and $\overline{\zeta}$, such that $D_M = \overline{\zeta} \neq 0$.

Proof of Claim 2 By assumption, we can take $a \in \mathcal{A}$ and $m \in \mathcal{M}$ such that $ma \neq am$. We define a derivation $\zeta : \mathcal{A} \to \mathcal{M}$ by $\zeta(x) = mx - xm$ for every x in $\mathcal{A}$. We denote $M = \mathrm{diag}(m) \in M_n(\mathcal{M})$, then D_M is an inner derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. Obviously, $D_M = \overline{\zeta}$ and $\overline{\zeta}(\mathrm{diag}(a)) \neq 0$. Thus, $D_M = \overline{\zeta} \neq 0$.

In the following, we show that the representation of the above form is unique if and only if $\mathcal{A}$ commutes with $\mathcal{M}$.

Case 1 If $\mathcal{A}$ commutes with $\mathcal{M}$, we suppose that there exists a derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, which can be represented as $D = D_{B_1} + \overline{\delta_1} = D_{B_2} + \overline{\delta_2}$. This means that $D_{B_1} - D_{B_2} = \overline{\delta_2} - \overline{\delta_1}$. Since $D_{B_1} - D_{B_2} = D_{B_1 - B_2}$ and $\overline{\delta_2} - \overline{\delta_1} = \overline{\delta_2 - \delta_1}$, we have $D_{B_1 - B_2} = \overline{\delta_2 - \delta_1}$. It follows from Claim 1 that $D_{B_1 - B_2} = \overline{\delta_2 - \delta_1} = 0$. i.e., $D_{B_1} = D_{B_2}$ and $\overline{\delta_1} = \overline{\delta_2}$.

Case 2 If $\mathcal{A}$ does not commute with $\mathcal{M}$, by Claim 2, there exist derivations D_M and $\overline{\zeta}$ from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$, $n \geq 2$, such that $D_M = \overline{\zeta} \neq 0$. Let $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, be an arbitrary derivation. By hypothesis, D can be represented as $D = D_B + \overline{\delta}$. We have $D = D_B + \overline{\delta} = D_B + D_M - \overline{\zeta} + \overline{\delta} = D_{B+M} + \overline{\delta - \zeta}$. This means that the derivation D can be represented as $D = D_B + \overline{\delta}$, and as $D = D_{B+M} + \overline{\delta - \zeta}$ too. Therefore, the representation of (2.2) is not unique for every derivation D. It follows from Cases 1 and 2 that the representation of (2.2) is unique if and only if $\mathcal{A}$ commutes with $\mathcal{M}$. The proof is complete. □

As applications of Theorem 2.1, we obtain the following corollaries.

Corollary 2.3 *The following statements are equivalent.*

(i) *Every derivation $\delta : \mathcal{A} \to \mathcal{M}$ is an inner derivation.*
(ii) *Every derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is an inner derivation.*

Proof If $\delta : \mathcal{A} \to \mathcal{M}$ is an inner derivation, by the equality (1.1), obviously, $\overline{\delta} : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is an inner derivation.

(i) implies (ii): Let $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, be an arbitrary derivation. By Theorem 2.1, D can be represented as a sum $D = D_M + \overline{\delta}$, where D_M is an inner derivation. By hypothesis, δ is an inner derivation from $\mathcal{A}$ into $\mathcal{M}$, and therefore, $\overline{\delta}$ is an inner derivation. We know that the sum of two inner derivations is an inner derivation, this means that $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is an inner derivation.

(ii) implies (i): Suppose that δ is a derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\overline{\delta} : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is a derivation. By hypothesis, $\overline{\delta}$ is an inner derivation. then the restriction of $\overline{\delta}$ onto $E_{11}M_n(\mathcal{A})E_{11}$, the subalgebra of $M_n(\mathcal{A})$, is an inner derivation. This means that $\delta : \mathcal{A} \to \mathcal{M}$ is an inner derivation. □

Corollary 2.4 *Let $\mathcal{A}$ be a commutative unital algebra over $\mathbb{C}$. Then every derivation on the matrix algebra $M_n(\mathcal{A})$ $(n \geq 2)$ is inner if and only if every derivation on $\mathcal{A}$ is identically zero, i.e., trivial.*

Let $\mathcal{R}$ be a von Neumann algebra. Denote by $S(\mathcal{R})$ and $LS(\mathcal{R})$, respectively, the sets of all measurable and locally measurable operators affiliated with $\mathcal{R}$. Then the set $LS(\mathcal{R})$ of all locally measurable operators with respect to $\mathcal{R}$ is a unital $*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator and $S(\mathcal{R})$ is a solid $*$-subalgebra in $LS(\mathcal{R})$. If $\mathcal{R}$ is a finite von Neumann algebra, then $S(\mathcal{R}) = LS(\mathcal{R})$ (see, for example, [1,4,14]). Let $\mathcal{A}$ be a commutative algebra with unit 1 over $\mathbb{C}$. We denote by ∇ the set $\{e \in \mathcal{A} : e^2 = e\}$ of all idempotents in $\mathcal{A}$. For $e, f \in \nabla$ we set $e \leq f$ if $ef = e$. Equipped with this partial order, lattice operations $e \vee f = e + f - ef$, $e \wedge f = ef$ and the complement $e^{\perp} = 1 - e$, the set ∇ forms a Boolean algebra. A nonzero element q from the Boolean algebra ∇ is called an *atom* if $0 \neq e \leq q$, $e \in \nabla$, imply that $e = q$. If given any nonzero $e \in \nabla$ there exists an atom q such that $q \leq e$, then the Boolean algebra ∇ is said to be *atomic*.

Let $\mathcal{R}$ be an abelian von Neumann algebra. Theorem 3.4 of [9] implies that every derivation on the algebra $S(\mathcal{R})$ is inner if and only if the lattice $\mathcal{R}_{\mathcal{P}}$ of all projections in $\mathcal{R}$ is atomic. If $\mathcal{R}$ is a properly infinite von Neumann algebra, in [4] the authors show that every derivation on the algebra $LS(\mathcal{R})$ is inner (see [4], Theorem 4.6). In the case of $\mathcal{R}$ is a finite von Neumann algebra of type I, Theorem 3.5 of [4] shows that a derivation on the algebra $LS(\mathcal{R})$ is an inner derivation if and only if it is identically zero on the center of $\mathcal{R}$.

As a direct application of Corollary 2.3, we obtain the following result.

Corollary 2.5 *Let $\mathcal{R}$ be a finite von Neumann algebra of type I with center $\mathcal{Z}$. Then every derivation D on the algebra $LS(\mathcal{R})$ is inner if and only if the lattice $\mathcal{Z}_{\mathcal{P}}$ of all projections in $\mathcal{Z}$ is atomic.*

Proof Let $\mathcal{R}$ be a finite von Neumann algebra of type I with center $\mathcal{Z}$. There exists a family $\{e_n\}_{n\in\mathcal{F}}$, $\mathcal{F} \subseteq \mathbb{N}$, of central projections from $\mathcal{R}$ with $\bigvee_{n\in\mathcal{F}} e_n = 1$ such that the algebra $\mathcal{R}$ is $*$-isomorphic with the C^*-product of von Neumann algebras $e_n\mathcal{R}$ of type I_n, respectively, $n \in \mathcal{F}$, i.e., $\mathcal{R} \cong \bigoplus_{n\in\mathcal{F}} e_n\mathcal{R}$. By Proposition 1.1 of [1], we have that $LS(\mathcal{R}) \cong \prod_{n\in\mathcal{F}} LS(e_n\mathcal{R})$.

Suppose that D is a derivation on $LS(\mathcal{R})$ and δ its restriction onto the center $S(\mathcal{Z})$. Since δ maps each $e_n S(\mathcal{Z})$ into itself, δ generates a derivation δ_n on $e_n S(\mathcal{Z})$ for each $n \in \mathcal{F}$. By Proposition 1.5 of [1], $LS(e_n\mathcal{R}) \cong M_n(e_n S(\mathcal{Z}))$. Let $\overline{\delta}_n$ be the derivation on the matrix algebra $M_n(e_n S(\mathcal{Z}))$ defined as in (1.1). Put

$$\overline{\delta}(\{x_n\}_{n\in\mathcal{F}}) = \{\overline{\delta}_n(x_n)\}, \quad \{x_n\}_{n\in\mathcal{F}} \in LS(\mathcal{R}). \tag{2.8}$$

Then the map $\overline{\delta}$ is a derivation on $LS(\mathcal{R})$. Lemma 2.3 of [1] implies that each derivation D on $LS(\mathcal{R})$ can be uniquely represented in the form $D = D_B + \overline{\delta}$, where D_B is an inner derivation and $\overline{\delta}$ is a derivation given as (2.8).

If D is an arbitrary derivation on $LS(\mathcal{R})$ and δ its restriction onto center $S(\mathcal{Z})$, by Theorem 3.4 of [9], the lattice $\mathcal{Z}_\mathcal{P}$ is atomic if and only if $\delta = 0$. We have $\delta = 0$ if and only if $\delta_n = 0$ for each $n \in \mathcal{F}$. By Corollary 2.3, $\delta_n = 0$ if and only if $\overline{\delta_n} = 0$ for each $n \in \mathcal{F}$. By equality (2.8), $\overline{\delta_n} = 0$ for each $n \in \mathcal{F}$ if and only if $\overline{\delta} = 0$. Therefore, every derivation on the algebra $LS(\mathcal{R})$ is inner derivation if and only if the lattice $\mathcal{Z}_\mathcal{P}$ of all projections in $\mathcal{Z}$ is atomic. The proof is complete. □

Let $\mathcal{R}$ be a properly infinite von Neumann algebra and $\mathcal{M}$ be a $\mathcal{R}$-bimodule of locally measurable operators. In [10], the authors show that every derivation $D : \mathcal{R} \to \mathcal{M}$ is an inner derivation. In the case of $\mathcal{R}$ is a finite von Neumann algebra of type I, we obtain the following result.

Theorem 2.6 *Let $\mathcal{R}$ be a finite von Neumann algebra of type I with center $\mathcal{Z}$. If the lattice $\mathcal{Z}_\mathcal{P}$ of all projections in $\mathcal{Z}$ is atomic, then every derivation $D : \mathcal{R} \to LS(\mathcal{R})$ is an inner derivation.*

Proof Choose a central decomposition $\{e_n\}_{n\in\mathcal{F}}$, $\mathcal{F} \subseteq \mathbb{N}$, of the unity 1 such that $e_n\mathcal{R}$ is a type I_n von Neumann algebra for each $n \in \mathcal{F}$. By hypothesis, it is easy to check that $D(e_n\mathcal{R}) \subseteq e_n LS(\mathcal{R})$ for each $n \in \mathcal{F}$. Thus, we only need to show that the derivation D restricted to $e_n\mathcal{R}$ is an inner derivation for each $n \in \mathcal{F}$.

Let $e_n\mathcal{R}$ be a type I_n ($n \in \mathcal{F}$) von Neumann algebra with center $e_n\mathcal{Z}$. It is well known that $e_n\mathcal{R} \cong M_n(e_n\mathcal{Z})$. We denote the center of $S(e_n\mathcal{R})$ by $\mathcal{Z}(S(e_n\mathcal{R}))$. By Proposition 1.2 of [1], we have $\mathcal{Z}(S(e_n\mathcal{R})) = S(e_n\mathcal{Z})$. By Proposition 1.5 of [1], $LS(e_n\mathcal{R}) = S(e_n\mathcal{R}) \cong M_n(S(e_n\mathcal{Z}))$.

By assumption, the lattice $\mathcal{Z}_\mathcal{P}$ of all projections in $\mathcal{Z}$ is atomic. This implies that the lattice $e_n\mathcal{Z}_\mathcal{P}$ is also atomic for each $n \in \mathcal{F}$. Statements (ii) of Proposition 2.3 and (vi) of Proposition 2.6 of [9] imply that every derivation $\delta : e_n\mathcal{Z} \to S(e_n\mathcal{Z})$ is trivial. By Corollary 2.3, we have that every derivation from $M_n(e_n\mathcal{Z})$ into $M_n(S(e_n\mathcal{Z}))$ is inner. The proof is complete. □

3 2-Local Derivations

This section is devoted to 2-local inner derivations and 2-local derivations from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. Throughout this section, we always assume that $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$ is a 2-local derivation. Firstly, we give the following lemma.

Lemma 3.1 *For every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, there exists a derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$ such that $\Delta(E_{ij}) = D(E_{ij})$ for all $i, j \in \{1, 2, \ldots, n\}$. In particular, if Δ is a 2-local inner derivation, then D is an inner derivation.*

Proof Let $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, be a 2-local derivation. By Theorem 2.1, with the proof similar to the proof of Theorem 3 in [13], it is easy to check that there exists a derivation D such that $\Delta(E_{ij}) = D(E_{ij})$ for all $i, j \in \{1, 2, \ldots, n\}$.

Let Δ be an inner 2-local derivation. We define two matrices S, T in $M_n(\mathcal{A})$ by

$$S = \sum_{i=1}^{n} i1 \otimes E_{ii}, \quad T = \sum_{i=1}^{n-1} E_{ii+1}.$$

By assumption, there exists an inner derivation $D : M_n(\mathcal{A}) \to M_n(\mathcal{M})$ such that

$$\Delta(S) = D(S), \quad \Delta(T) = D(T).$$

Replacing Δ by $\Delta - D$ if necessary, we may assume that $\Delta(S) = \Delta(T) = 0$. Fixed $i, j \in \{1, 2, \ldots, n\}$, by assumption, we can take two elements X, Y in $M_n(\mathcal{M})$ such that

$$\Delta(E_{ij}) = XE_{ij} - E_{ij}X, \quad 0 = \Delta(S) = XS - SX,$$

and

$$\Delta(E_{ij}) = YE_{ij} - E_{ij}Y, \quad 0 = \Delta(T) = YT - TY.$$

It follows from $XS = SX$ that X is a diagonal matrix. We denote X by $\mathrm{diag}(x_k)$. The equality $YT = TY$ implies that Y is of the form

$$Y = \begin{bmatrix} y_1 & y_2 & y_3 & \cdot & \cdot & y_n \\ 0 & y_1 & y_2 & \cdot & \cdot & y_{n-1} \\ 0 & 0 & y_1 & \cdot & \cdot & y_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdot & y_1 & y_2 \\ 0 & 0 & \cdots & \cdot & 0 & y_1 \end{bmatrix}.$$

On the one side, we have

$$\Delta(E_{ij}) = XE_{ij} - E_{ij}X = \mathrm{diag}(x_k)E_{ij} - E_{ij}\mathrm{diag}(x_k) = (x_i - x_j) \otimes E_{ij}.$$

On the other side, we have

$$[\Delta(E_{ij})]_{ij} = [YE_{ij} - E_{ij}Y]_{ij} = 0.$$

Therefore, $\Delta(E_{ij}) = 0$. The proof is complete. □

Theorem 3.2 *Suppose that $\mathcal{A}$ commutes with $\mathcal{M}$. Then every 2-local inner derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is an inner derivation.*

Proof By Lemma 3.1, we may assume that $\Delta(E_{ij}) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$. For any $A \in M_n(\mathcal{A})$, we take a pair (j, i), $j, i \in \{1, 2, \ldots, n\}$, by assumption, there exists an inner derivation D_B, such that $\Delta(A) = D_B(A)$ and $0 = \Delta(E_{ij}) = D_B(E_{ij})$. We have

$$\begin{aligned} E_{ij}\Delta(A)E_{ij} &= E_{ij}D_B(A)E_{ij} \\ &= D_B(E_{ij}AE_{ij}) - D_B(E_{ij})AE_{ij} - E_{ij}AD_B(E_{ij}) = D_B(E_{ij}AE_{ij}) \\ &= D_B(A_{ji} \otimes E_{ij}) = D_B(\text{diag}(A_{ji}, \ldots, A_{ji})E_{ij}) \\ &= D_B(\text{diag}(A_{ji}, \ldots, A_{ji}))E_{ij} + \text{diag}(A_{ji}, \ldots, A_{ji})D_B(E_{ij}) \\ &= (B\text{diag}(A_{ji}, \ldots, A_{ji}) - \text{diag}(A_{ji}, \ldots, A_{ji})B)E_{ij} \\ &= 0, \end{aligned}$$

i.e.,

$$E_{ij}\Delta(A)E_{ij} = 0.$$

Therefore,

$$E_{ji}(E_{ij}\Delta(A)E_{ij})E_{ji} = E_{jj}\Delta(A)E_{ii} = 0,$$

i.e.,

$$[\Delta(A)]_{ji} = 0,$$

for every $j, i \in \{1, 2, \ldots, n\}$. Hence $\Delta(A) = 0$. The proof is complete. □

Corollary 3.3 *Suppose that $\mathcal{A}$ is a unital commutative algebra over $\mathbb{C}$. Then every 2-local inner derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{A})$, $n \geq 2$, is an inner derivation.*

Remark 3.4 The above result is proved in [8]. By comparison, our proof is more simple.

Suppose that $\mathcal{A}$ is an algebra over $\mathbb{C}$ and $\mathcal{B}$ is a unital subalgebra in $\mathcal{A}$. We denote the commutant of $\mathcal{B}$ by $\mathcal{B}' = \{a \in \mathcal{A} : ab = ba, for\ every\ b \in \mathcal{B}\}$. Let $\mathcal{C}$ be a submodule in $\mathcal{B}'$. It follows from Theorem 3.2 that

Corollary 3.5 *Every 2-local inner derivation $\Delta : M_n(\mathcal{B}) \to M_n(\mathcal{C})$, $n \geq 2$, is an inner derivation.*

Theorem 3.6 *Suppose that $\mathcal{A}$ is a commutative algebra which commutes with $\mathcal{M}$. Then every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$, $n \geq 2$, is a derivation.*

Proof The proof is similar to the proof of Theorem 4.3 in [5]. We leave it to the reader. □

Corollary 3.7 *Suppose that $\mathcal{A}$ is a unital commutative algebra over $\mathbb{C}$. Then every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{A})$, $n \geq 2$, is a derivation.*

If $\mathcal{A}$ is a non-commutative algebra, by Theorem 2.1 every derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})(n \geq 2)$ can be represented as a sum $D = D_B + \overline{\delta}$. In [7], the authors apply this representation of derivation to prove the following result.

Theorem 3.8 ([7], Theorem 2.1) *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If every Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$ is a derivation, then every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{A})$, $n \geq 3$, is a derivation.*

Theorem 3.9 *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If $n \geq 6$ is a positive integer but not a prime number, then every 2-local derivation $\Delta : M_n(\mathcal{A}) \to M_n(\mathcal{M})$ is a derivation.*

Proof Suppose that $n = rk$, where $r \geq 3$ and $k \geq 2$. Then $M_n(\mathcal{A}) \cong M_r(M_k(\mathcal{A}))$ and $M_n(\mathcal{M}) \cong M_r(M_k(\mathcal{M}))$. In [2], the author proves that every Jordan derivation from $M_k(\mathcal{A})$ into $M_k(\mathcal{M})(k \geq 2)$ is a derivation ([2], Theorem 3.1). By Theorem 3.8, the proof is complete. □

Let $\mathcal{R}$ be a type I_n $(n \geq 2)$ von Neumann algebra with center $\mathcal{Z}$ and τ be a faithful normal semi-finite trace on $\mathcal{R}$. We denote the centers of $S(\mathcal{R})$ and $S(\mathcal{R}, \tau)$ by $\mathcal{Z}(S(\mathcal{R}))$ and $\mathcal{Z}(S(\mathcal{R}, \tau))$, respectively. By Proposition 1.2 of [1], we have $\mathcal{Z}(S(\mathcal{R})) = S(\mathcal{Z})$ and $\mathcal{Z}(S(\mathcal{R}, \tau)) = S(\mathcal{Z}, \tau_{\mathcal{Z}})$, where $\tau_{\mathcal{Z}}$ is the restriction of the trace τ on $\mathcal{Z}$. By Propositions 1.4 and 1.5 of [1], $S(\mathcal{R}) = LS(\mathcal{R}) \cong M_n(S(\mathcal{Z}))$ and $S(\mathcal{R}, \tau) \cong M_n(S(\mathcal{Z}, \tau_{\mathcal{Z}}))$.

As a direct application of Theorem 3.6, we have the following corollary.

Corollary 3.10 *Suppose that $\mathcal{R}$ is a type I_n, $n \geq 2$, von Neumann algebra and τ is a faithful normal semi-finite trace on $\mathcal{R}$. Then we have*

(1) every 2-local derivation $\Delta : \mathcal{R} \to LS(\mathcal{R})$ is a derivation;
(2) every 2-local derivation $\Delta : \mathcal{R} \to S(\mathcal{R}, \tau)$ is a derivation.

Lemma 3.11 *Let $\Delta : \mathcal{A} \to \mathcal{M}$ be a 2-local derivation. If there exists a central idempotent e in $\mathcal{A}$ which commutates with $\mathcal{M}$, then $\Delta(ea) = e\Delta(a)$, for each a in $\mathcal{A}$.*

Proof For any $a \in \mathcal{A}$, by assumption, there exists a derivation $\delta : \mathcal{A} \to \mathcal{M}$ such that: $\Delta(ea) = \delta(ea)$, and $\Delta(a) = \delta(a)$. By assumption, e is a central idempotent in $\mathcal{A}$ which commutes with $\mathcal{M}$, it follows that $\delta(e) = 0$. Then

$$\Delta(ea) = \delta(ea) = \delta(e)a + e\delta(a) = e\delta(a) = e\Delta(a).$$

The proof is complete. □

Theorem 3.12 *Suppose that $\mathcal{R}$ is a finite von Neumann algebra of type I without abelian direct summands. Then every 2-local derivation $\Delta : \mathcal{R} \to S(\mathcal{R}) = LS(\mathcal{R})$ is a derivation.*

Proof By assumption, $\mathcal{R}$ is a finite von Neumann algebra of type I without abelian direct summands. Then there exists a family $\{P_n\}_{n \in F}$, $F \subseteq \mathbb{N} \backslash 1$, of orthogonal central projections in $\mathcal{R}$ with $\sum_{n\in F} P_n = 1$, such that the algebra $\mathcal{R}$ is $*$-isomorphic with the C^*-product of von Neumann algebras $P_n\mathcal{R}$ of type I_n, respectively, $n \in F$. Then

$$P_n LS(\mathcal{R}) = P_n S(\mathcal{R}) = S(P_n\mathcal{R}) \cong M_n(P_n Z(\mathcal{R})), \quad n \in F.$$

By Lemma 3.11, we have $\Delta(P_n A) = P_n\Delta(A)$, for all $A \in \mathcal{R}$ and each $n \in F$. This implies that Δ maps each $P_n\mathcal{R}$ into $P_n S(\mathcal{R})$. For each $n \in F$, we define $\Delta_n : P_n\mathcal{R} \to P_n S(\mathcal{R})$ by

$$\Delta_n(P_n A) = P_n\Delta(A), \quad A \in \mathcal{R}.$$

By assumption, it follows that Δ_n is a 2-local derivation from $P_n\mathcal{R}$ into $P_n S(\mathcal{R})$ for each $n \in F$. By (1) of Corollary 3.10, we have that Δ_n is a derivation for each $n \in F$. Since $\sum_{n\in F} P_n = 1$, it follows that Δ is a linear mapping. For any $A, B \in \mathcal{R}$, it follows Δ_n is a derivation for each $n \in F$ that

$$\begin{aligned} P_n\Delta(AB) = \Delta_n(P_n AB) &= \Delta_n(P_n A)P_n B + P_n A\Delta_n(P_n B) \\ &= P_n\Delta(A)B + P_n A\Delta(B) \\ &= P_n(\Delta(A)B + A\Delta(B)). \end{aligned}$$

By assumption, $\sum_{n\in F} P_n = 1$, we get

$$\Delta(AB) = \Delta(A)B + A\Delta(B).$$

Therefore, $\Delta : \mathcal{R} \to S(\mathcal{R})$ is a derivation. The proof is complete. □

Ayupov et al. [7] have proved the following result. Now we give a different proof.

Theorem 3.13 ([7], Theorem 3.1) *Let $\mathcal{R}$ be an arbitrary von Neumann algebra without abelian direct summands and $LS(\mathcal{R})$ be the algebra of all locally measurable operators affiliated with $\mathcal{R}$. Then every 2-local derivation $\Delta : \mathcal{R} \to LS(\mathcal{R})$ is a derivation.*

Proof Let $\mathcal{R}$ be an arbitrary von Neumann algebra without abelian direct summands. We know that $\mathcal{R}$ can be decomposed along a central projection into the direct sum of von Neumann algebras of finite type I, type I_∞, type II and type III. By Lemma 3.11, we may consider these cases separately.

If $\mathcal{R}$ is a von Neumann algebra of finite type I, Theorem 3.12 shows that every 2-local derivation from $\mathcal{R}$ into $\mathrm{LS}(\mathcal{R})$ is a derivation.

If $\mathcal{R}$ is a von Neumann algebra of types I_∞, II or III, then the halving Lemma ([12], Lemma 6.3.3) for type I_∞ algebras and ([12], Lemma 6.5.6) for types II or III algebras implies that the unit of $\mathcal{R}$ can be represented as a sum of mutually equivalent orthogonal projections $e_1, e_2, \ldots, e_6$ in $\mathcal{R}$. It is well known that $\mathcal{R}$ is isomorphic to $M_6(\mathcal{A})$, where $\mathcal{A} = e_1\mathcal{R}e_1$. Further, the algebra $LS(\mathcal{R})$ is isomorphic to the algebra $M_6(LS(\mathcal{A}))$. Theorem 3.9 implies that every 2-local derivation from $\mathcal{R}$ into $\mathrm{LS}(\mathcal{R})$ is a derivation. The proof is complete. □

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