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# The Investigation on Two Kinds of Nonlinear Matrix Equations

**Jing Li**[1,2] · **Yuhai Zhang**[1]

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**Abstract**
In this paper, we consider two kinds of nonlinear matrix equations $X+\sum_{i=1}^{m} B_i^* X^{t_i} B_i = I\ (0 < t_i < 1)$ and $X^s - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = I\ (p_i > 1,\ s \geq 1)$. By means of the integral representation of matrix functions, properties of Kronecker product and the monotonic $p$-concave operator fixed point theorem, we derive necessary conditions and sufficient conditions for the existence and uniqueness of the Hermitian positive definite solution for the matrix equations. We also obtain some properties of the Hermitian positive definite solutions, the bounds of the determinant's sum for $A_i^* A_i$ and the spectral radius of $A_i$.

**Keywords** Nonlinear matrix equation · Hermitian positive definite solution · Integral representation · Kronecker product · Spectral radius

**AMS classification** 15A24 · 15A42

## 1 Introduction

In this paper, we consider the Hermitian positive definite (HPD) solutions of the nonlinear matrix equations

$$X+\sum_{i=1}^{m} B_i^* X^{t_i} B_i = I, \quad 0 < t_i < 1, \tag{1.1}$$

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✉ Jing Li
xlijing@sdu.edu.cn

1 School of Mathematics, Shandong University, Jinan 250100, People's Republic of China

2 School of Mathematics and Statistics, Shandong University at Weihai, Weihai 264209, People's Republic of China

and

$$X^s - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = I,\ \ p_i > 1,\ \ s \geq 1,\ \ p_i \neq s, \tag{1.2}$$

where $A_i, B_i$ $(i = 1, 2, \ldots, m)$ are $n \times n$ matrices, $I$ is an $n \times n$ identity matrix and $m$ is a positive integer. Here, $A_i^*$ denotes the conjugate transpose of the matrix $A_i$.

Nonlinear matrix equations with form (1.1) and (1.2) in the case $m = s = 1$ arise from many fields such as nano research, ladder networks, dynamic programming, control theory, stochastic filtering, statistics [1–8] and the references therein.

In the last few years, (1.1) and (1.2) were investigated in some special cases. For (1.1) with $m = 1$, Zhang et al. [9] considered the existence of HPD solutions and the iterative method. Gao and Zhang [10] studied HPD solutions of $X - A^* X^q A = Q$ $(q > 0)$. For (1.2) with $s = 1$, $0 < |p_i| < 1$, there were some contributions in the literature to the solvability, numerical solutions and perturbation analysis [11–15]. Duan et al. [11] obtained the existence of a unique HPD solution by fixed point theorems for monotone and mixed monotone operators in a normal cone. Lim [12] derived the existence of a unique HPD solution by using a strict contraction for the Thompson metric on the open convex cone of positive definite matrices. Duan et al. [13] and Li and Zhang [14,15] discussed perturbation analysis for the HPD solution of this matrix equation.

In addition, the related matrix equations $X^s \pm A^* \mathcal{F}(X) A = Q$ [16–22], $X - \sum_{i=1}^{m} f(\Phi_i(X)) = Q$ [23] and $X^s \pm \sum_{i=1}^{m} A_i^* X^{-t_i} A_i = Q$ $(t_i > 0)$ [24–33] were studied by some scholars. However, (1.1) and (1.2) have not been thoroughly studied either qualitatively or quantitatively. Motivated by this, this paper will focus on the solvability for (1.1) and (1.2) by means of the integral representation of matrix functions, the properties of Kronecker product and the monotonic $p$-concave operator fixed point theorem.

The rest of this paper is organized as follows: In Sect. 2, we give some preliminary lemmas that will be needed to develop this work. In Sect. 3, we discuss the existence of a unique HPD solution of (1.1). Furthermore, in Sect. 4, some properties of HPD solutions to (1.2) are presented. We obtain the trace and the determinant for the HPD solutions, the bounds of eigenvalues and the determinant of $A_i^* A_i$. Finally, in the case $s > p_i > 1$, by the monotonic $p$-concave operator fixed point theorem (which was proposed in [20]), we obtain a sufficient condition for the existence of a unique HPD solution.

The following notations are used throughout this paper. We denote by $\mathcal{C}^{n\times n}$, $\mathcal{H}^{n\times n}$ and $\mathcal{U}^{n\times n}$ the set of all $n \times n$ complex matrices, Hermitian matrices and unitary matrices, respectively. For $A = (a_1, \ldots, a_n) = (a_{ij}) \in \mathcal{C}^{n\times n}$ and a matrix $B$, $A \otimes B = (a_{ij} B)$ is a Kronecker product, and $\mathrm{vec} A$ is a vector defined by $\mathrm{vec} A = (a_1^T, \ldots, a_n^T)^T$. The symbol $\|\cdot\|$ stands for the spectral norm, $\|\cdot\|_F$ is the Frobenius norm. We denote by $\lambda_i(M)$ the eigenvalues of $M$, by $\det(M)$ the determinant of $M$, by $\rho(M)$ the spectral radius of $M$, by $\mathrm{tr}(M)$ the trace of $M$, by $\lambda_1(M)$ and $\lambda_n(M)$ the maximal and minimal eigenvalues of $M$, respectively. For $X, Y \in \mathcal{H}^{n\times n}$, we write $X \geq Y (X > Y)$ if $X - Y$ is a Hermitian positive semi-definite (definite) matrix. For $A, B \in \mathcal{H}^{n\times n}$, the sets $[A, B]$ and $(A, B]$ are defined by $[A, B] = \{X \in \mathcal{H}^{n\times n} | A \le X \le B\}$ and $(A, B] = \{X \in \mathcal{H}^{n\times n} | A < X \le B\}$.

## 2 Preliminaries

In this section, we present some lemmas that will be needed to develop this paper.

**Lemma 2.1** [34]. *Let $A$ and $B$ be positive operators on a Hilbert space $H$ such that $M_1 I \ge A \ge m_1 I > 0$, $M_2 I \ge B \ge m_2 I > 0$ and $B \ge A > 0$. Then $A^t \le (\frac{M_1}{m_1})^{t-1} B^t > 0$, $A^t \le (\frac{M_2}{m_2})^{t-1} B^t$ hold for any $t \ge 1$.*

**Lemma 2.2** [35]. *If $A \ge B > 0$ and $0 \le \gamma \le 1$, then $A^\gamma \ge B^\gamma$.*

**Lemma 2.3** [36]. *For every Hermitian positive definite matrix $X$, it yields that*

$$X^{-p} = \frac{\sin p\pi}{\pi} \int_0^\infty (\lambda I + X)^{-1} \lambda^{-p} \mathrm{d}\lambda, \quad 0 < p < 1. \tag{2.1}$$

**Lemma 2.4** [37, Theorem 1.9.1]. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{p\times q}$, $C \in \mathcal{C}^{n\times k}$, $D \in \mathcal{C}^{q\times r}$. Then*

(i) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
(ii) $(A \otimes B)^* = A^* \otimes B^*$.

**Lemma 2.5** [37, Lemma 1.9.1]. *Let $A \in \mathcal{C}^{l\times m}$, $X \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$. Then*

$$\mathrm{vec}(AXB) = (B^T \otimes A) \cdot \mathrm{vec}X.$$

**Lemma 2.6** [38, Theorem 6.19]. *Let $A \in \mathcal{C}^{m\times m}$ and $B \in \mathcal{C}^{n\times n}$ with eigenvalues $\lambda_i$ and $\mu_j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, respectively. Then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.*

**Lemma 2.7** [39, Theorem 3.2.1]. *For $A \ge 0$, $B \ge 0$, $A, B \in \mathcal{C}^{n\times n}$, then* $\det(A+B) \ge \det(A) + \det(B)$.

## 3 Hermitian Positive Definite Solutions of $X + \sum_{i=1}^m B_i^* X^{t_i} B_i = I$

In this section, some necessary conditions and sufficient conditions for the existence and uniqueness of HPD solutions of (1.1) are derived.

The next theorem proposes a sufficient condition for the existence of HPD solutions of (1.1). Meanwhile, the bounds for HPD solutions of (1.1) are derived.

**Theorem 3.1** *If $\sum_{i=1}^m \lambda_1(B_i^* B_i) < 1$, then the equations*

$$x = 1 - \sum_{i=1}^m \lambda_1(B_i^* B_i) \left(1 - \sum_{i=1}^m \lambda_n(B_i^* B_i) x^{t_i}\right)^{t_i}, \quad x > 0,\ 0 < t_i < 1, \tag{3.1}$$

*and*

$$x = 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\left(1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)x^{t_i}\right)^{t_i}, \quad x > 0,\ 0 < t_i < 1 \tag{3.2}$$

*have real positive solutions. If* $\alpha$ *and* $\beta$ *are the solutions of the above equations, respectively, then Eq.* (1.1) *has a HPD solution in* $[\alpha I, \beta I]$.

*Moreover,*

$$1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i) < \alpha \leq \beta \leq 1. \tag{3.3}$$

***Proof*** Step 1. We will prove (3.1) and (3.2) have positive solutions. Define the sequences $\{\beta_n\}$ and $\{\alpha_n\}$:

$$\beta_0 = 1,\ \ \beta_{n+1} = 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha_n^{t_i} \tag{3.4}$$

and

$$\alpha_n = 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta_n^{t_i},\ \ n = 0, 1, 2, \ldots. \tag{3.5}$$

By the hypothesis of this theorem and the definition of sequences $\{\beta_n\}$ and $\{\alpha_n\}$, we have

$$\begin{aligned} 1 \geq \alpha_0 &= 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i) > 0, \\ \beta_0 = 1 \geq \beta_1 &= 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha_0^{t_i} \geq 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i) = \alpha_0 > 0, \\ \beta_0 = 1 \geq \alpha_1 &= 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta_1^{t_i} \geq 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta_0^{t_i} = \alpha_0. \end{aligned}$$

Suppose $\beta_{k-1} \geq \beta_k \geq \alpha_0 = 1 - \sum\limits_{i=1}^{m} \lambda_1(B_i^* B_i)$ and $1 = \beta_0 \geq \alpha_k \geq \alpha_{k-1}$, then

$$\begin{aligned} \beta_k &= 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha_{k-1}^{t_i} \geq 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha_k^{t_i} = \beta_{k+1} \geq 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i), \\ 1 \geq \alpha_{k+1} &= 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta_{k+1}^{t_i} \geq 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta_k^{t_i} = \alpha_k. \end{aligned}$$

Hence, for each $k$ we have $\beta_k \geq \beta_{k+1} \geq 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)$ and $1 \geq \alpha_{k+1} \geq \alpha_k$, which imply the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are monotonic and bounded. Therefore, they are convergent to certain positive numbers. Let

$$\alpha = \lim_{n\to\infty} \alpha_n, \quad \beta = \lim_{n\to\infty} \beta_n.$$

Taking limits in (3.4) and (3.5) yields

$$\alpha = 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta^{t_i}, \quad \beta = 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha^{t_i},$$

which imply

$$\alpha = 1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\left(1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\alpha^{t_i}\right)^{t_i},$$
$$\beta = 1 - \sum_{i=1}^{m} \lambda_n(B_i^* B_i)\left(1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i)\beta^{t_i}\right)^{t_i}.$$

Therefore, $\alpha$ and $\beta$ satisfy (3.1) and (3.2), respectively. Moreover,

$$1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i) < \alpha \leq \beta \leq 1.$$

Step 2. We will prove that (1.1) has a HPD solution under the assumption $\sum_{i=1}^{m} \lambda_1(B_i^* B_i) < 1$.

Let $\Omega = \left[(1 - \sum_{i=1}^{m} \lambda_1\left(B_i^* B_i\right) I,\ I\right]$. Define

$$F(X) = I - \sum_{i=1}^{m} B_i^* X^{t_i} B_i, \quad X \in \Omega.$$

Obviously, $\Omega$ is a bounded convex closed set and $F$ is continuous on $\Omega$.

For any $X \in \Omega$, we have $X \leq I$. Note that $s \geq 1$. It follows from Lemmas 2.1 and 2.2 that

$$I \geq F(X) = I - \sum_{i=1}^{m} B_i^* X^{t_i} B_i \geq \left(1 - \sum_{i=1}^{m} \lambda_1\left(B_i^* B_i\right)\right) I > 0.$$

Therefore, $F(X) \subseteq \Omega$. By Brouwer's fixed point theorem, the map $F$ has a fixed point $X_0 \in \Omega$, which is a HPD solution of(1.1).

In what follows, we suppose that $X$ is a HPD solution of (1.1).

Step 3. We will prove that $X \in [\alpha I,\ \beta I]$. According to Lemmas 2.1, 2.2 and the sequences defined by (3.4) and (3.5), we have $\alpha_0 I = (1 - \sum_{i=1}^{m} \lambda_1(B_i^* B_i))I \leq X \leq$

$I=\beta_0 I$. It follows from $X=I-\sum_{i=1}^m B_i^* X^{t_i} B_i$ that $X=I-\sum_{i=1}^m B_i^*(I-\sum_{i=1}^m B_i^* X^{t_i} B_i)^{t_i} B_i$. Hence

$$\left(1-\sum_{i=1}^m \lambda_1(B_i^*B_i)\left(1-\sum_{i=1}^m \lambda_n(B_i^*B_i)\lambda_n^{t_i}(X)\right)^{t_i}\right) I \le X$$
$$\le \left(1-\sum_{i=1}^m \lambda_n(B_i^*B_i)\left(1-\sum_{i=1}^m \lambda_1(B_i^*B_i)\lambda_1^{t_i}(X)\right)^{t_i}\right) I. \tag{3.6}$$

Since $\alpha_0 I \le X \le \beta_0 I$, it follows that $\alpha_0 \le \lambda_n(X)$ and $\lambda_1(X) \le \beta_0$. Note that inequality (3.6) implies $\alpha_1 I \le X \le \beta_1 I$. By similar induction, it yields that

$$\alpha_n I \le X \le \beta_n I. \tag{3.7}$$

Taking limits on both sides of inequality (3.7), we have $\alpha I \le X \le \beta I$. □

The next estimates for HPD solutions of (1.1) are more precise than that in Theorem 3.1.

**Corollary 3.1** *If* $\sum_{i=1}^m \lambda_1(B_i^*B_i) < 1$ *, then every HPD solution of* (1.1) *is in* $[I-\sum_{i=1}^m \beta^{t_i} B_i^*B_i,\ I-\sum_{i=1}^m \alpha^{t_i} B_i^*B_i]$, *where* $\alpha$ *and* $\beta$ *are defined as in Theorem* 3.1.

***Proof*** We suppose that $X$ is a HPD solution of (1.1). By Theorem 3.1, it follows that

$$\alpha \le \lambda_n(X), \quad \lambda_1(X) \le \beta. \tag{3.8}$$

Using $X=I-\sum_{i=1}^m B_i^* X^{t_i} B_i$, we obtain $I-\sum_{i=1}^m \lambda_1^{t_i}(X) B_i^*B_i \le X \le I-\sum_{i=1}^m \lambda_n^{t_i}(X) B_i^*B_i$. Applying inequality (3.8) yields $I-\sum_{i=1}^m \beta^{t_i} B_i^*B_i \le X \le I-\sum_{i=1}^m \alpha^{t_i} B_i^*B_i$. □

In what follows, we will discuss the uniqueness of HPD solutions of (1.1) by means of the properties of (1.1).

The following lemma plays an important role for discussing the uniqueness of HPD solution of (1.1).

**Lemma 3.1** *If* $B_1, B_2, \ldots, B_m$ *are* $n\times n$ *complex nonsingular matrices, then* (1.1) *has a HPD solution if and only if there exist* $Q_i \in \mathcal{C}^{n\times n}$, $i=1,2,\ldots,m$, $P\in\mathcal{U}^{n\times n}$, *and diagonal matrices* $\Gamma, \Lambda > 0$ *such that*

$$B_i = P^*\Gamma^{-\frac{t_i}{2}} Q_i \Lambda P,\ i=1,2,\ldots,m,$$

*where* $\Lambda^2+\Gamma=I$ *and* $\sum_{i=1}^m Q_i^*Q_i = I$. *In this case,* $Y=P^*\Gamma P$ *is a HPD solution of* (1.1).

***Proof*** If (1.1) has a HPD solution $Y$, it follows from the spectral decomposition theorem that there exists $P \in \mathcal{U}^{n\times n}$ and a diagonal matrix $\Gamma > 0$ such that $Y = P^*\Gamma P$. Then (1.1) can be rewritten as

$$P^*\Gamma P + \sum_{i=1}^{m} B_i^* P^* \Gamma^{t_i} P B_i = I. \tag{3.9}$$

Multiplying the left side of (3.9) by $P$ and the right side by $P^*$, we have

$$\sum_{i=1}^{m} P B_i^* P^* \Gamma^{t_i} P B_i P^* = I - \Gamma. \tag{3.10}$$

Note that $B_i$ $(i = 1, 2, \ldots, m)$ are nonsingular matrices. Then $0 < Y < I$, which implies

$$0 < \Gamma < I. \tag{3.11}$$

It follows that (3.10) will be turned into the following form

$$\sum_{i=1}^{m}(I-\Gamma)^{-\frac{1}{2}} P B_i^* P^* \Gamma^{t_i} P B_i P^* (I-\Gamma)^{-\frac{1}{2}} = I. \tag{3.12}$$

Let $\Lambda = (I-\Gamma)^{\frac{1}{2}}$, $Q_i = \Gamma^{\frac{t_i}{2}} P B_i P^* \Lambda^{-1}$. It is easy to verify that $\Gamma + \Lambda^2 = I$ and $B_i = P^*\Gamma^{-\frac{t_i}{2}} Q_i \Lambda P$. It follows from (3.12) that $\sum_{i=1}^{m} Q_i^* Q_i = I$.

Conversely, assume there exist $P \in \mathcal{U}^{n\times n}$, $Q_i \in \mathcal{C}^{n\times n}$, $\sum_{i=1}^{m} Q_i^* Q_i = I$ and diagonal matrices $\Gamma, \Lambda > 0$, $\Lambda^2 + \Gamma = I$ such that

$$B_i = P^*\Gamma^{-\frac{t_i}{2}} Q_i \Lambda P, i = 1, 2, \ldots, m.$$

Let $Y = P^*\Gamma P$, then $Y$ is a HPD matrix, and it follows that

$$\begin{aligned} Y + \sum_{i=1}^{m} B_i^* X^{t_i} B_i &= P^*\Gamma P + \sum_{i=1}^{m} P^*\Lambda^* Q^* \Gamma^{-\frac{t_i}{2}} P (P^*\Gamma P)^{t_i} P^* \Gamma^{-\frac{t_i}{2}} Q_i \Lambda P \\ &= P^*\Gamma P + \sum_{i=1}^{m} P^* \Lambda Q_i^* Q_i \Lambda P = P^*(\Gamma + \Lambda^2) P = I, \end{aligned}$$

which implies $Y$ is a HPD solution of (1.1). □

To prove the next theorem, we first verify the following lemmas.

**Lemma 3.2** *Suppose that* $m \geq 1$, $0 < t < 1$ *and* $\frac{mt}{mt+1} < x, y < 1$. *Then*

$$0 < f(x, y, t) = \frac{\sqrt{(1-x)(1-y)}(x^t - y^t)}{(x-y) x^{\frac{t}{2}} y^{\frac{t}{2}}} < \frac{1}{m}.$$

***Proof*** Let $g_1(x) = \frac{(1-x)^{1/2}}{x^{t/2}}$, $\frac{mt}{mt+1} < x < 1$, $0 < t < 1$. It is easy to verify that the function $g_1(x)$ is monotonically decreasing on $(\frac{mt}{mt+1}, 1)$. It follows that

$$g_1(x) < g_1\left(\frac{mt}{mt+1}\right) = \sqrt{\frac{(mt+1)^{t-1}}{(mt)^t}}, \quad \frac{mt}{mt+1} < x < 1. \tag{3.13}$$

Let $g_2(x) = x^t$, $\frac{mt}{mt+1} < x < 1$, $0 < t < 1$. By the mean value theorem, there exists $\xi \in (\frac{mt}{mt+1}, 1)$ such that

$$\frac{g_2(x) - g_2(y)}{x - y} = g_2'(\xi) < t\left(\frac{mt}{mt+1}\right)^{t-1}. \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$0 < f(x, y, t) = g_1(x) \cdot g_1(y) \cdot \frac{g_2(x) - g_2(y)}{x - y} < \frac{1}{m}.$$

□

**Lemma 3.3** *For every Hermitian positive definite matrix $X$ and $0 < t < 1$, it yields that*

$$X^t = \frac{\sin t\pi}{\pi} \int_0^\infty X(\lambda I + X)^{-1} \lambda^{t-1} \mathrm{d}\lambda.$$

***Proof*** Multiplying the left side of (2.1) in Lemma 2.3 by $X$ and letting $t = 1 - p$, we have

$$X^t = \frac{\sin t\pi}{\pi} \int_0^\infty X(\lambda I + X)^{-1} \lambda^{t-1} \mathrm{d}\lambda, \quad 0 < t < 1.$$

□

**Theorem 3.2** *Assume that $B_1, B_2, \ldots, B_m$ are $n \times n$ complex nonsingular matrices and $0 < t_i < 1$. If (1.1) has a HPD solution on $[\frac{mt}{mt+1} I, I]$, then the HPD solution of (1.1) is unique, where $t = \max_{1 \le i \le m}\{t_i\}$.*

***Proof*** If $Y_1$ is a HPD solution of (1.1), according to Lemma 3.1, there exist $P_1 \in \mathcal{U}^{n \times n}$, $Q_i \in \mathcal{C}^{n \times n}$, $i = 1, 2, \ldots, m$ and diagonal matrices $\Gamma_1, \Lambda_1 > 0$ such that

$$B_i = P_1^* \Gamma_1^{-t_i/2} Q_i \Lambda_1 P_1, \quad i = 1, 2, \ldots, m, \tag{3.15}$$

where

$$\sum_{i=1}^m Q_i^* Q_i = I \text{ and } \Lambda_1^2 + \Gamma_1 = I. \tag{3.16}$$

In this case, $Y_1 = P_1^*\Gamma_1 P_1$, where $\Gamma_1 = \mathrm{diag}(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n})$ with $\{\lambda_{1j}\}$ the eigenvalues of $Y_1$.

Similarly, if $Y_2$ is a HPD solution of (1.1), there exist $P_2 \in \mathcal{U}^{n\times n}$, $U_i \in \mathcal{C}^{n\times n}$, $i = 1, 2, \ldots, m$ and diagonal matrices $\Gamma_2, \Lambda_2 > 0$ such that

$$B_i = P_2^*\Gamma_2^{-t_i/2}U_i\Lambda_2 P_2, \quad i = 1, 2, \ldots, m, \tag{3.17}$$

where

$$\sum_{i=1}^{m} U_i^* U_i = I \quad \text{and} \quad \Lambda_2^2 + \Gamma_2 = I. \tag{3.18}$$

In this case, $Y_2 = P_2^*\Gamma_2 P_2$, where $\Gamma_2 = \mathrm{diag}(\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n})$ with $\{\lambda_{2j}\}$ the eigenvalues of $Y_2$.

According to Lemma 3.3, we have

$$\begin{aligned} Y_1 - Y_2 &= \sum_{i=1}^{m} B_i^*\left(Y_2^{t_i} - Y_1^{t_i}\right)B_i \\ &= \sum_{i=1}^{m}\frac{B_i^*\sin t_i\pi}{\pi}\int_0^\infty \left[Y_2(\lambda I + Y_2)^{-1} - Y_1(\lambda I + Y_1)^{-1}\right]\lambda^{t_i-1}B_i\mathrm{d}\lambda \\ &= -\sum_{i=1}^{m}\frac{B_i^*\sin t_i\pi}{\pi}\int_0^\infty (Y_1 - Y_2)(\lambda I + Y_1)^{-1}\lambda^{t_i-1}B_i\mathrm{d}\lambda \\ &\quad + \sum_{i=1}^{m}\frac{B_i^*\sin t_i\pi}{\pi}\int_0^\infty Y_2(\lambda I + Y_2)^{-1}(Y_1 - Y_2)(\lambda I + Y_1)^{-1}\lambda^{t_i-1}B_i\mathrm{d}\lambda. \end{aligned} \tag{3.19}$$

Note that

$$(\lambda I + Y_1)^{-1} = \left(\lambda I + P_1^*\Gamma_1 P_1\right)^{-1} = P_1^{-1}(\lambda I + \Gamma_1)^{-1}P_1 \tag{3.20}$$

and

$$(\lambda I + Y_2)^{-1} = \left(\lambda I + P_2^*\Gamma_2 P_2\right)^{-1} = P_2^{-1}(\lambda I + \Gamma_2)^{-1}P_2. \tag{3.21}$$

Combing (3.15), (3.17) and (3.19)–(3.21), we have

$$\begin{aligned} Y_1 - Y_2 = &-\sum_{i=1}^{m}\frac{\sin t_i\pi}{\pi}\int_0^\infty P_2^*\Lambda_2 U_i^*\Gamma_2^{-\frac{t_i}{2}}P_2(Y_1 - Y_2)P_1^*(\lambda I + \Gamma_1)^{-1}\Gamma_1^{-\frac{t_i}{2}}Q_i\Lambda_1 P_1\lambda^{t_i-1}\mathrm{d}\lambda \\ &+\sum_{i=1}^{m}\frac{\sin t_i\pi}{\pi}\int_0^\infty P_2^*\Lambda_2 U_i^*\Gamma_2^{1-\frac{t_i}{2}}(\lambda I + \Gamma_2)^{-1}P_2(Y_1 - Y_2)P_1^*(\lambda I + \Gamma_1)^{-1} \\ &\times\Gamma_1^{-\frac{t_i}{2}}Q_i\Lambda_1 P_1\lambda^{t_i-1}\mathrm{d}\lambda. \end{aligned} \tag{3.22}$$

Let

$$W = P_2(Y_1 - Y_2)P_1^*. \tag{3.23}$$

Then (3.22) can be rewritten as

$$\begin{aligned} W = &-\sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} \int_0^\infty \Lambda_2 U_i^* \Gamma_2^{-\frac{t_i}{2}} W(\lambda I + \Gamma_1)^{-1}\Gamma_1^{-\frac{t_i}{2}} Q_i \Lambda_1 \lambda^{t_i - 1} \mathrm{d}\lambda \\ &+ \sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} \int_0^\infty \Lambda_2 U_i^* \Gamma_2^{1-\frac{t_i}{2}} (\lambda I + \Gamma_2)^{-1} W(\lambda I + \Gamma_1)^{-1}\Gamma_1^{-\frac{t_i}{2}} Q_i \Lambda_1 \lambda^{t_i - 1} \mathrm{d}\lambda. \end{aligned} \tag{3.24}$$

From (3.24) and Lemmas 2.4 and 2.5, it follows that

$$\begin{aligned} &\operatorname{vec} W \\ &= -\sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} \int_0^\infty \left[(\lambda I + \Gamma_1)^{-1}\Gamma_1^{-\frac{t_i}{2}} Q_i \Lambda_1\right]^T \otimes \left(\Lambda_2 U_i^* \Gamma_2^{-\frac{t_i}{2}}\right) \lambda^{t_i-1}\mathrm{d}\lambda \cdot \operatorname{vec} W \\ &\quad + \sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} \int_0^\infty \left[(\lambda I + \Gamma_1)^{-1}\Gamma_1^{-\frac{t_i}{2}} Q_i \Lambda_1\right]^T \otimes \left[\Lambda_2 U_i^* \Gamma_2^{1-\frac{t_i}{2}} (\lambda I + \Gamma_2)^{-1}\right] \lambda^{t_i-1}\mathrm{d}\lambda \cdot \operatorname{vec} W \\ &= -\sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} (\Lambda_1 \otimes \Lambda_2)(Q_i^T \otimes U_i^*) \left(\Gamma_1^{\frac{-t_i}{2}} \otimes \Gamma_2^{-\frac{t_i}{2}}\right) \int_0^\infty [(\lambda I + \Gamma_1)^{-1} \otimes I]\lambda^{t_i-1}\mathrm{d}\lambda \cdot \operatorname{vec} W \\ &\quad + \sum_{i=1}^{m} \frac{\sin t_i\pi}{\pi} (\Lambda_1 \otimes \Lambda_2)(Q_i^T \otimes U_i^*) \left(\Gamma_1^{-\frac{t_i}{2}} \otimes \Gamma_2^{1-\frac{t_i}{2}}\right) \\ &\quad \int_0^\infty (\lambda I + \Gamma_1)^{-1} \otimes (\lambda I + \Gamma_2)^{-1}\lambda^{t_i-1}\mathrm{d}\lambda \cdot \operatorname{vec} W. \end{aligned} \tag{3.25}$$

Assume that

$$\Lambda_1 = \operatorname{diag}(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}), \quad \Lambda_2 = diag(\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n}).$$

According to (3.11), (3.16) and (3.18) , we have

$$0 < \sigma_{1j} = \sqrt{1 - \lambda_{1j}} < 1, \quad 0 < \sigma_{2j} = \sqrt{1 - \lambda_{2j}} < 1, \quad j = 1, 2, \ldots, n. \tag{3.26}$$

Let

$$\begin{aligned} &B = \Lambda_1 \otimes \Lambda_2, \quad J_i = Q_i^T \otimes U_i^*, \\ &C_i = \left(\Gamma_1^{-t_i/2} \otimes \Gamma_2^{-t_i/2}\right) \cdot \frac{\sin t_i\pi}{\pi} \int_0^\infty \left[(\lambda I + \Gamma_1)^{-1} \otimes I\right] \lambda^{t_i-1}\mathrm{d}\lambda, \\ &D_i = \left(\Gamma_1^{-t_i/2} \otimes \Gamma_2^{1-t_i/2}\right) \cdot \frac{\sin t_i\pi}{\pi} \int_0^\infty (\lambda I + \Gamma_1)^{-1} \otimes (\lambda I + \Gamma_2)^{-1}\lambda^{t_i-1}\mathrm{d}\lambda, \\ &i = 1, 2, \ldots, m. \end{aligned} \tag{3.27}$$

Then (3.25) can be rewritten as

$$\mathrm{vec}W + B\sum_{i=1}^{m} J_i(C_i - D_i)\cdot \mathrm{vec}W = 0. \tag{3.28}$$

By Lemma 2.6, we have

$$\begin{aligned}
B &= \Lambda_1 \otimes \Lambda_2 = \mathrm{diag}(\sigma_{1l}\cdot\sigma_{2j})_{n^2\times n^2},\\
C_i &= \left(\Gamma_1^{-t_i/2}\otimes\Gamma_2^{-t_i/2}\right)\cdot\frac{\sin t_i\pi}{\pi}\int_0^\infty[(\lambda I+\Gamma_1)^{-1}\otimes I]\lambda^{t_i-1}\mathrm{d}\lambda,\\
&= \mathrm{diag}\left(\lambda_{1l}^{-t_i/2}\cdot\lambda_{2j}^{-t_i/2}\cdot\frac{\sin t_i\pi}{\pi}\int_0^\infty(\lambda+\lambda_{1l})^{-1}\lambda^{t_i-1}\mathrm{d}\lambda\right)_{n^2\times n^2}\\
&= \mathrm{diag}\left(\lambda_{1l}^{-\frac{t_i}{2}}\cdot\lambda_{2j}^{-\frac{t_i}{2}}\cdot\lambda_{1l}^{t_i-1}\right)_{n^2\times n^2}\\
&= \mathrm{diag}\left(\lambda_{1l}^{\frac{t_i}{2}-1}\cdot\lambda_{2j}^{-\frac{t_i}{2}}\right)_{n^2\times n^2},\\
D_i &= \left(\Gamma_1^{-t_i/2}\otimes\Gamma_2^{1-t_i/2}\right)\cdot\frac{\sin t_i\pi}{\pi}\int_0^\infty(\lambda I+\Gamma_1)^{-1}\otimes(\lambda I+\Gamma_2)^{-1}\lambda^{t_i-1}\mathrm{d}\lambda\\
&= \mathrm{diag}\left(\lambda_{1l}^{-t_i/2}\cdot\lambda_{2j}^{1-t_i/2}\cdot\frac{\sin t_i\pi}{\pi}\int_0^\infty(\lambda+\lambda_{1l})^{-1}(\lambda+\lambda_{2j})^{-1}\lambda^{t_i-1}\mathrm{d}\lambda\right)_{n^2\times n^2}\\
&= \mathrm{diag}\left(\frac{\lambda_{1l}^{-t_i/2}\cdot\lambda_{2j}^{1-t_i/2}(\lambda_{1l}^{t_i-1}-\lambda_{2j}^{t_i-1})}{\lambda_{2j}-\lambda_{1l}}\right)_{n^2\times n^2},\\
&\quad i=1,2,\ldots,m,\ l,j=1,2,\ldots,n.
\end{aligned}\tag{3.29}$$

It follows that

$$\begin{aligned}
C_i - D_i &= \mathrm{diag}\left(\lambda_{1l}^{\frac{t_i}{2}-1}\cdot\lambda_{2j}^{-\frac{t_i}{2}} - \frac{\lambda_{1l}^{-t_i/2}\cdot\lambda_{2j}^{1-t_i/2}(\lambda_{1l}^{t_i-1}-\lambda_{2j}^{t_i-1})}{\lambda_{2j}-\lambda_{1l}}\right)_{n^2\times n^2}\\
&= \mathrm{diag}\left(\frac{\lambda_{1l}^{t_i}-\lambda_{2j}^{t_i}}{(\lambda_{1l}-\lambda_{2j})\lambda_{1l}^{\frac{t_i}{2}}\lambda_{2j}^{\frac{t_i}{2}}}\right)_{n^2\times n^2},\\
&\quad i=1,2,\ldots,m,\ l,j=1,2,\ldots,n.
\end{aligned}\tag{3.30}$$

Note that $B$ is nonsingular. Multiplying the left side of Eq. (3.28) by $B^{-1}$, we have

$$\begin{aligned}
&B^{-1}\mathrm{vec}W + \sum_{i=1}^{m} J_i(C_i + D_i)\cdot\mathrm{vec}W\\
&= \left[I + \sum_{i=1}^{m} J_i(C_i - D_i)B\right]B^{-1}\cdot\mathrm{vec}W = 0.
\end{aligned}\tag{3.31}$$

A combination of (3.27) and Lemma 2.4 gives

$$\begin{aligned} J_i^* J_i &= (Q_i^T \otimes U_i^*)^* (Q_i^T \otimes U_i^*) = (\overline{Q}_i \otimes U_i)(Q_i^T \otimes U_i^*) \\ &= (\overline{Q}_i Q_i^T) \otimes (U_i U_i^*) = (\overline{Q_i Q_i^*}) \otimes (U_i U_i^*). \end{aligned}$$

It follows (3.16), (3.18) and Lemma 2.6 that $0 < ||J_i|| \leq 1$. Then

$$\left\| \sum_{i=1}^{m} J_i (C_i - D_i) B \right\| \leq \sum_{i=1}^{m} \|(C_i - D_i) B\|. \tag{3.32}$$

By the hypothesis of the theorem, we have $\frac{mt}{mt+1} I < Y_1, Y_2 < I$, which implies that $\frac{mt}{mt+1} < \lambda_{1l}, \lambda_{2j} < 1,\ l, j = 1, 2, \ldots, n$. Note that $\frac{mt_i}{mt_i+1} < \frac{mt}{mt+1},\ i = 1, 2, \ldots, m$. Then $\frac{mt_i}{mt_i+1} < \lambda_{1l}, \lambda_{2j} < 1,\ l, j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m$. Therefore, it follows (3.29) and (3.30) that

$$\begin{aligned} \|(C_i - D_i) B\| &= \max_{l,j} \left\{ \frac{\sqrt{(1-\lambda_{1l})(1-\lambda_{2j})} \left( \lambda_{2j}^{t_i} - \lambda_{1l}^{t_i} \right)}{\left( \lambda_{2j} - \lambda_{1l} \right) \lambda_{2j}^{t_i/2} \lambda_{1l}^{t_i/2}} \right\} \\ &= \max_{l,j} \{ f(\lambda_{1l}, \lambda_{2j}, t_i) \}, \end{aligned} \tag{3.33}$$

where $f(x, y, t)$ is defined in Lemma 3.2.

A combination of Lemma 3.2, (3.32)–(3.33) gives that

$$\left\| \sum_{i=1}^{m} J_i (C_i - D_i) B \right\| < m \cdot \frac{1}{m} = 1,$$

which implies $I + \sum_{i=1}^{m} J_i (C_i - D_i) B$ is nonsingular. It follows (3.31) that $\mathrm{vec} W = 0$. By (3.23), we have $Y_1 = Y_2$. □

## 4 Hermitian Positive Definite Solutions of $X^s - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = I$

In this section, the properties of HPD solutions and coefficient matrices of (1.2) are derived. The sufficient conditions for the existence of a unique HPD solution are given.

We first give the following lemma. This lemma is easy to verify.

**Lemma 4.1** *Let* $g(x) = x^{-p}(x^s - 1),\ x > 1,\ p > s \geq 1$. *Then*

(i) $g$ *is increasing on* $[1, (\frac{p}{p-s})^{\frac{1}{s}}]$ *and decreasing on* $[(\frac{p}{p-s})^{\frac{1}{s}}, +\infty)$;

(ii) *the maximal value of* $g(x)$ *is* $g((\frac{p}{p-s})^{\frac{1}{s}}) = \frac{s(p-s)^{\frac{p}{s}-1}}{p^{\frac{p}{s}}}$.

The spectral radius of coefficient matrices of (1.2) is derived in the next theorem.

**Theorem 4.1** *Assume that* $A_1, A_2, \ldots, A_m$ *are nonsingular matrices. If* (1.2) *with* $p_i > s \geq 1$ *has a HPD solution, then*

$$\rho^2(A_j) \leq \frac{s(p-s)^{\frac{p}{s}-1}}{p^{\frac{p}{s}}},\ \ j = 1, 2, \ldots, m,$$

*where* $p = \min_{1\leq i\leq m}\{p_i\}$.

***Proof*** Suppose that $X$ is a HPD solution of (1.2). Let $Y = X^s$. Then (1.2) can be rewritten as

$$Y - \sum_{i=1}^{m} A_i^* Y^{\frac{p_i}{s}} A_i = I,\ \ p_i > s \geq 1. \tag{4.1}$$

Let $\lambda_j$ be any eigenvalue of $A_j$ $(j = 1, 2, \ldots, m)$ and $e_j$ be the corresponding unit eigenvector of $\lambda_j$. Multiplying the left side of (4.1) by $e_j^*$ and the right side by $e_j$, we have

$$e_j^* Y e_j - e_j^* |\lambda_j|^2 Y^{\frac{p_j}{s}} e_j - e_j^* \sum_{i\neq j}^{m} A_i^* Y^{\frac{p_j}{s}} A_i e_j = e_j^* e_j,$$

which implies

$$e_j^* Y e_j - e_j^* |\lambda_j|^2 Y^{\frac{p_j}{s}} e_j \geq e_j^* e_j. \tag{4.2}$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be eigenvalues of $Y$. Then

$$\lambda_1\left(Y - |\lambda_j|^2 Y^{\frac{p_j}{s}}\right) = \max_{1\leq i\leq n}\left(\sigma_i - |\lambda_j|^2 \sigma_i^{\frac{p_j}{s}}\right).$$

Let $h(x) = x - |\lambda_j|^2 x^{\frac{p}{s}},\ \ x > 0,\ \ p > s \geq 1$. It is easy to verify that $h''(x) = -|\lambda_j|^2 \frac{p}{s}(\frac{p}{s} - 1)x^{\frac{p}{s}-2} < 0$. Therefore, $\max h(x) = h((\frac{s}{p|\lambda_j|^2})^{\frac{s}{p-s}}) = \frac{(p-s)s^{\frac{s}{p-s}}}{p^{\frac{s}{p-s}+1}|\lambda_j|^{\frac{2s}{p-s}}}$. Since $\sigma_i \geq 1$, then $\sigma_i - |\lambda_j|^2 \sigma_i^{\frac{p_j}{s}} \leq \sigma_i - |\lambda_j|^2 \sigma_i^{\frac{p}{s}} = h(\sigma_i) \leq \frac{(p-s)s^{\frac{s}{p-s}}}{p^{\frac{s}{p-s}+1}|\lambda_j|^{\frac{2s}{p-s}}}$. It follows that

$$\lambda_1(Y - |\lambda_j|^2 Y^{\frac{p_j}{s}}) = \max_{1\leq i\leq n}(\sigma_i - |\lambda_j|^2 \sigma_i^{\frac{p_j}{s}}) \leq \frac{(p-s)s^{\frac{s}{p-s}}}{p^{\frac{s}{p-s}+1}|\lambda_j|^{\frac{2s}{p-s}}}.$$

By inequality (4.2), we obtain

$$e_j^* e_j \leq e_j^* \left(Y - |\lambda_j|^2 Y^{\frac{p_j}{s}}\right) e_j \leq \frac{(p-s)s^{\frac{s}{p-s}}}{p^{\frac{s}{p-s}+1}|\lambda_j|^{\frac{2s}{p-s}}},$$

which implies

$$\rho^2(A_j) \leq \frac{s(p-s)^{\frac{p}{s}-1}}{p^{\frac{p}{s}}}.$$

□

In the next theorem, we obtain the bounds of $\sum_{i=1}^m \lambda_n(A_i^* A_i)$, $\sum_{i=1}^m \det(A_i A_i^*)$ and $\det(X)$.

**Theorem 4.2** *Let* $t = \min_{1\leq i\leq m}\{p_i\}$. *If* (1.2) *with* $p_i > s \geq 1$ *has a HPD solution* $X$*, then*

(1) $\sum_{i=1}^m \lambda_n(A_i^* A_i) \leq \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}$;

(2) $\sum_{i=1}^m \det(A_i A_i^*) \leq \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}$ *and* $\delta_1 \leq \det(X) \leq \delta_2$, *where* $\delta_1, \delta_2$ $(\delta_1 \leq \delta_2)$ *are the positive solutions of the equation* $x^{-t}(x^s - 1) - \sum_{i=1}^m \det(A_i A_i^*) = 0,\ x > 1,\ t > s \geq 1$.

***Proof*** (1) Since $X$ is a HPD solution of (1.2), then $X \geq I$. It follows that

$$\lambda_n(X^s) = \lambda_n\left(I + \sum_{i=1}^m A_i^* X^{p_i} A_i\right) \geq 1 + \lambda_n\left(\sum_{i=1}^m A_i^* X^{p_i} A_i\right)$$
$$\geq 1 + \sum_{i=1}^m \lambda_n(A_i^* A_i) \cdot \lambda_n^t(X),$$

which implies $\sum_{i=1}^m \lambda_n(A_i^* A_i) \leq \lambda_n^{-t}(X)(\lambda_n^s(X) - 1)$. According to Lemma 4.1, we have

$$\sum_{i=1}^m \lambda_n(A_i^* A_i) \leq \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}.$$

(2) Since $X$ is a HPD solution of (1.2), then $X^s = I + \sum_{i=1}^m A_i^* X^{p_i} A_i$. Note that $X \geq I$. Then $\det(X) \geq 1$, which implies $(\det(X))^{p_i} \geq (\det(X))^t$. It follows from Lemma 2.7 that

$$\det(X^s) = \det\left(I + \sum_{i=1}^m A_i^* X^{p_i} A_i\right) \geq 1 + \sum_{i=1}^m \det(A_i A_i^*)(\det(X))^t,$$

which implies

$$\sum_{i=1}^m \det(A_i A_i^*) \leq (\det(X))^{-t}((\det(X))^s - 1). \tag{4.3}$$

It follows from Lemma 4.1 that $\sum_{i=1}^{m}\det(A_iA_i^*) \leq \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}$. On the other hand, it is easy to verify that if $\sum_{i=1}^{m}\det(A_iA_i^*) \leq \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}$, the equation $x^{-t}(x^s-1)-\sum_{i=1}^{m}\det(A_iA_i^*)=0$ has two solutions $\delta_1, \delta_2$ $(1<\delta_1\leq\delta_2)$. By inequality (4.3), we have

$$\delta_1 \leq \det(X) \leq \delta_2.$$

□

To derive the bounds of the trace for HPD solutions of (1.2), we need some properties of the trace in the following lemma.

**Lemma 4.2** *Let $A \geq 0$ and $B \geq 0$ be $n\times n$ matrices. Then for $q \geq 1$, it yields that*

(1) $\lambda_n(A)\mathrm{tr}(B) \leq \mathrm{tr}(AB) \leq \lambda_1(A)\mathrm{tr}(B)$,
(2) $\frac{(\mathrm{tr}(A))^q}{n^{q-1}} \leq \mathrm{tr}(A^q) \leq (\mathrm{tr}(A))^q$.

***Proof*** (1) Since $A \geq 0$, so we get $\lambda_n(A)I \leq A \leq \lambda_1(A)I$, then

$$0 \leq \mathrm{tr}((A-\lambda_n(A))B) = \mathrm{tr}(AB-\lambda_n(A)B) = \mathrm{tr}(AB)-\lambda_n(A)\mathrm{tr}(B),$$

which implies $\lambda_n(A)\mathrm{tr}(B) \leq \mathrm{tr}(AB)$.

Similarly, it is easy to verify that $\mathrm{tr}(AB) \leq \lambda_1(A)\mathrm{tr}(B)$.

(2) Since $A \geq 0$, so $\lambda_i(A) \geq 0$, $i=1,2,\ldots,n$. By Hölder's inequality (see Lemma 1.1.2 on Page 1 in [40]), we have

$$\mathrm{tr}(A)=\sum_{i=1}^{n}\lambda_i(A) \leq n^{1-\frac{1}{q}}\left(\sum_{i=1}^{n}\lambda_i^q(A)\right)^{\frac{1}{q}} = n^{1-\frac{1}{q}}(\mathrm{tr}(A^q))^{\frac{1}{q}},$$

which implies $\frac{(\mathrm{tr}(A))^q}{n^{q-1}} \leq \mathrm{tr}(A^q)$.

When $\lambda_1(A)=\lambda_2(A)=\ldots=\lambda_n(A)=0$, obviously $\mathrm{tr}(A^q)=(\mathrm{tr}(A))^q$. If $\lambda_1(A)+\lambda_2(A)+\ldots+\lambda_n(A)\neq 0$, then

$$\begin{aligned}
&\frac{\lambda_1^q(A)}{(\mathrm{tr}(A))^q}+\frac{\lambda_2^q(A)}{(\mathrm{tr}(A))^q}+\cdots+\frac{\lambda_n^q(A)}{(\mathrm{tr}(A))^q}\\
&=\left(\frac{\lambda_1(A)}{\lambda_1(A)+\lambda_2(A)+\cdots+\lambda_n(A)}\right)^q+\cdots+\left(\frac{\lambda_n(A)}{\lambda_1(A)+\lambda_2(A)+\cdots+\lambda_n(A)}\right)^q\\
&\leq \frac{\lambda_1(A)}{\lambda_1(A)+\lambda_2(A)+\cdots+\lambda_n(A)}+\cdots+\frac{\lambda_n(A)}{\lambda_1(A)+\lambda_2(A)+\cdots+\lambda_n(A)}=1,
\end{aligned}$$

which implies $\mathrm{tr}(A^q)=\sum_{i=1}^{n}\lambda_i^q(A) \leq (\mathrm{tr}(A))^q$.

□

**Theorem 4.3** *Let* $t = \min_{1 \le i \le m}\{p_i\}$. *If* (1.2) *with* $p_i > s \ge 1$ *has a HPD solution* $X$, *then*

$$\gamma_1 \le \operatorname{tr}(X) \le \gamma_2,$$

*where* $\gamma_1,\ \gamma_2\ (\gamma_1 < \gamma_2)$ *are the positive solutions of the equation* $n^{t-1}x^{-t}(x^s - n) - \sum_{i=1}^m \lambda_n(A_i^* A_i) = 0$.

***Proof*** Since $X$ is a HPD soluiton of (1.2),

$$X^s = I + \sum_{i=1}^m A_i^* X^{p_i} A_i. \tag{4.4}$$

Note that $s \ge 1,\ \ p_i > 1$ and $\operatorname{tr}(X) \ge n$. Taking the trace on both sides of (4.4), by Lemma 4.2, we obtain

$$\begin{aligned}(\operatorname{tr}(x))^s &\ge \operatorname{tr}(X^s) = \operatorname{tr}(I + \sum_{i=1}^m A_i^* X^{p_i} A_i) = n + \sum_{i=1}^m \operatorname{tr}(A_i A_i^* X^{p_i})\\ &\ge n + \sum_{i=1}^m \lambda_n(A_i A_i^*)\operatorname{tr}(X^{p_i}) \ge n + \sum_{i=1}^m \lambda_n(A_i^* A_i)\cdot n\left(\frac{\operatorname{tr}(X)}{n}\right)^{p_i}\\ &\ge n + n^{1-t}\sum_{i=1}^m \lambda_n(A_i^* A_i)\operatorname{tr}(X)^t,\end{aligned} \tag{4.5}$$

which implies

$$n^{t-1}(\operatorname{tr}(X))^{-t}((\operatorname{tr}(X))^s - n) - \sum_{i=1}^m \lambda_n(A_i^* A_i) \ge 0. \tag{4.6}$$

Let $h_1(x) = n^{t-1}x^{-t}(x^s - n),\ \ x \ge n^{1/s},\ \ t > s \ge 1$. A calculation gives that the maximal value of $h_1(x)$ is $\max h_1(x) = h_1((\frac{nt}{t-s})^{1/s}) = \frac{n^{t(1-\frac{t}{s})}s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}}$. Note that $n^{t(1-\frac{t}{s})} > 1$. It follows Theorem 4.2 (1) that

$$\sum_{i=1}^m \lambda_n(A_i^* A_i) \le \frac{s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}} < \frac{n^{t(1-\frac{t}{s})}s(t-s)^{\frac{t}{s}-1}}{t^{\frac{t}{s}}},$$

which implies the equation $n^{t-1}x^{-t}(x^s - n) - \sum_{i=1}^m \lambda_n(A_i^* A_i) = 0$ has two positive solution $\gamma_1,\ \gamma_2\ (\gamma_1 < \gamma_2)$. By inequality (4.6), we obtain $\gamma_1 \le \operatorname{tr}(X) \le \gamma_2$. □

To prove the uniqueness of the HPD solution of (1.2) with $1 < p_i < s$, we will use the following definition and lemmas which can be found in [41].

Let $\mathcal{X}$ be a real Banach space, and let $K$ be a closed cone in $\mathcal{X}$, $K^+ = K \setminus \{0\}$.

**Definition 1** [41, Definition 3.1]. Let $T : K \to K$, and let $p \geq 0$. We say that

(a) $T$ is increasing if $0 \leq x \leq y$ implies $Tx \leq Ty$,
(b) $T$ is $p$ -concave if $T(\lambda x) \geq \lambda^p Tx$ for all $x \in K$ and $0 < \lambda < 1$.

**Lemma 4.3** [41, Theorem 3.2]. *Let the norm in $\mathcal{X}$ be monotonic. Suppose that $T : K \to K$ is an increasing p-concave mapping with $0 < p < 1$, and that $Tf \in K_f$ for some $f \in K^+$ with $\|f\| = 1$. Suppose in addition that $T : K_f \to K_f$ is continuous in the norm topology. Then there exists a unique $z \in K_f$ such that $Tz = z$.*

Let $\mathcal{X}$ be $M(n)$, which denotes the set of $n \times n$ real matrices. Then a closed cone in $\mathcal{X}$ is given by $\overline{P}(n)$, the set of $n \times n$ real positive semi-definite matrices. The interior of this cone is the set of $n \times n$ real positive definite matrices, which we will denote by $P(n)$.

**Theorem 4.4** *If $A_1, A_2, \ldots, A_m$ are $n \times n$ real nonsingular matrices and $1 < p_i < s$, then (1.2) has a unique positive definite solution $X_0 \in P(n)$.*

***Proof*** Let

$$F(Y) = I + \sum_{i=1}^{m} A_i^* Y^{\frac{p_i}{s}} A_i, \;\; Y \in P(n).$$

It follows from $1 < p_i < s$ that $0 < \frac{p_i}{s} < 1$. Then $F$ is increasing and continuous in the norm topology. Let $q = \max_{1 \leq i \leq m}\{p_i\}$. For any $Y \in P(n)$ and $0 < \lambda < 1$, we have

$$F(\lambda Y) = I + \sum_{i=1}^{m} A_i^* (\lambda Y)^{\frac{p_i}{s}} A_i \geq \lambda^{\frac{q}{s}} \left( I + \sum_{i=1}^{m} A_i^* Y^{\frac{p_i}{s}} A_i \right) = \lambda^{\frac{q}{s}} F(Y),$$

which implies $F$ is $\frac{q}{s}$-concave. Since $I \in P(n)$, $F(I) \in P(n)$ and $\|I\| = 1$, then by Lemma 4.3, there exists a unique $Y_0 \in P(n)$ such that $F(Y_0) = Y_0$. Let $X_0 = Y_0^{\frac{1}{s}}$. Then $X_0 \in P(n)$ is a unique positive definite solution of (1.2). □

***Remark 4.1*** In this section, our method is not valid when $p_i = s$ for some $i$ in equation (1.2). The case of $p_i = s$ for some $i$ in equation (1.2) is worth investigating further.

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