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On Characterizing the Exponential q-Distribution

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Abstract
In this paper, we attempted to characterize the exponential q-distribution through the q-memorylessness property using the q-addition operator and Jackson integral. Moreover, an extended version of k-gamma q-distribution is introduced and the q-moments of this family is computed. Finally, we suggested a new q-inversion method to simulate data from a q-distribution.

Keywords q-Calculus · q-Gamma function · k-Gamma q-distribution · Exponential q-distribution

Mathematics Subject Classification 44A20 · 33E20 · 33E50

1 Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. Recently, many researchers have focused on the q-calculus [1,2,8,12,16], which corresponds to the link between mathematics and physics. The quantum calculus began with Jackson [13,14] in the early twentieth century. The book of Quantum Calculus [7] published by Kac and Cheung covers many of the fundamental aspects of quantum calculus. Chung et al. [6] defined the q-addition operator and discussed its properties. They used it in the properties of the q-logarithmic function and q-exponential.

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The quantum calculus has a lot of applications in different mathematical areas such as number theory, difference equation (see [11]), orthogonal polynomials, probability theory.

In mathematical physics and probability, the q-distribution is more general than classical distribution. It was introduced by Díaz et al. [9,10] in the continuous case and by Charalambos [4] and Cheung and Kac [5] in the discrete case. The construction of a q-distribution is the construction of a q-analogue of ordinary distribution. Mathai in [15] introduced the q-analogue of the gamma distribution with respect to Lebesgue measure. In this paper, gamma q-distribution is introduced with respect to Jackson q-measure. If q goes to 1, we obtain the ordinary calculus. This condition is the necessary condition in the theory of q-calculus.

The aim of this work is not only to generalize the k-gamma q-distribution, $\gamma_{q,k}(\lambda, a)$ with parameters $\lambda > 0$ and $a > 0$, but also to characterize the exponential q-distribution, $\xi_q(\lambda)$, by the q-memorylessness property in the following way:

A random variable X is exponential q-distributed if and only if

$$\mathbb{P}_q\left(X > s \oplus_q t | X > s\right) = \mathbb{P}_q(X > t), \qquad \forall s, t \geq 0.$$

Next, the link between the quantum distribution and the classical distribution is portrayed as shown in the following diagram

$$\begin{tikzcd}
f(x) = x^{a-1}\mathbf{1}_{[0,1]} \arrow[d, leftrightarrow] & \gamma_{q,k}(\lambda, a) \text{ on } [0, \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}] \arrow[l, "q\to 0"'] \arrow[r, "q\to 1"] \arrow[d, "k=1"] & \gamma_k(\lambda, a) \text{ on } \mathbb{R}_+ \arrow[d, "k=1"] \\
f(x) = x^{a-1}\mathbf{1}_{[0,1]} \arrow[d, "a=1"] & \gamma_q(\lambda, a) \text{ on } [0, \frac{1}{1-q}] \arrow[l, "q\to 0"'] \arrow[r, "q\to 1"] \arrow[d, "a=1"] & \gamma(\lambda, a) \text{ on } \mathbb{R}_+ \arrow[d, "a=1"] \\
U_{[0,1]} & \xi_q(\lambda) \text{ on } [0, \frac{1}{1-q}] \arrow[l, "q\to 0"'] \arrow[r, "q\to 1"] & \xi(\lambda) \text{ on } \mathbb{R}_+
\end{tikzcd}$$

The third objective of this work is to simulate data from the exponential q-distribution with parameters $\lambda,\ a > 0$.

This paper is structured as follows: in Sect. 2, some preliminary concepts related to q-derivative, q-integral, q-operators and some essential results are presented to build our work. In Sect. 3, the q-gamma and the q-beta functions are recalled. Some properties and relationships between them are presented. Besides, the new q-gamma function is introduced and its properties are proved. In Sect. 4, the k-gamma q-distribution is generalized with parameters $\lambda,\ a > 0$ and its q-cumulative function is specified. Then, the exponential q-distribution is deduced from the k-gamma q-distribution and its characterization is proved. In Sect. 5, the definition of the q-moments established by Díaz and Pariguan in [9] and the properties of the q-integral are used to define the q-mean and the q-variance. The q-mean of the k-gamma q-distribution is computed.

Finally, in the closing section number 6, we introduced a new method called the q-inversion which is identified in order to simulate the data from a q-distribution; then, it is applied on the exponential q-distribution.

2 Preliminaries

In this section, some useful basic definitions [7,13,14,17] are introduced. We shall start with the q-derivative and the Jackson q-integral. Fixing a real number $0 < q < 1$, the q-derivative of $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R} \setminus \{0\}$ is given by:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

It is also known as the Jackson derivative.

It is manifestly linear,

$$D_q(f(x) + g(x)) = D_q f(x) + D_q g(x)\ .$$

It has a product rule analogous to the ordinary ones, with two equivalent forms

$$D_q(f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) = g(qx)D_q f(x) + f(x)D_q g(x).$$

Similarly, it satisfies a quotient rule,

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}, \quad g \neq 0.$$

In the case $q = 0$, we have

$$D_0 f(x) = \frac{f(x) - f(0)}{x}.$$

For an integer $n \geq 1$, we have that $D_q x^n = [n]_q x^{n-1}$, where

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

We also denote, for all $n \in \mathbb{N}$,

$$[n]_q! = \begin{cases} 1 & \text{if} \quad n = 0, \\ [n]_q[n-1]_q \ldots [1]_q & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}$,

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

If x goes to ∞, we obtain $[\infty]_q = \dfrac{1}{1-q}$ is called a q-analogue of ∞.

Note that $[\infty]_q$ approaches 1 when q goes to 0 and goes to $+\infty$ when q approaches 1.

We recall some usual notations used in the q-theory.

$$
\begin{aligned}
(a+b)_q^n &= \prod_{i=0}^{n-1}(a+q^i b), \quad \forall\, n \in \mathbb{N},\\
(1+a)_q^\infty &= \prod_{i=0}^{\infty}(1+q^i a),\\
(1+a)_q^t &= \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty}, \quad \forall\, t \in \mathbb{R}.
\end{aligned}
$$

A right inverse of the q-derivative is obtained via the Jackson integral.

For $a,\ b \in \mathbb{R}$ the Jackson integral or q-integral of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$ is defined by:

$$
\int_a^b f(x)\mathrm{d}_q x = (1-q)\sum_{n=0}^{\infty} q^n \left(bf(q^n b) - af(q^n a)\right).
$$

It is clear if one lets q approaches 1, then the q-derivative approaches the Newton derivative and the Jackson integral approaches the Riemann integral.

The q-analogue of the integration theorem by a variable change is given by

$$
\int_{u(a)}^{u(b)} f(u)\mathrm{d}_q u = \int_a^b f(u(x))\mathrm{d}_{q^{1/\beta}} u(x), \quad \text{where } \ u(x) = \alpha x^\beta. \tag{1}
$$

The q-analogue of the rule of integration by parts is

$$
\int_a^b g(x) D_q f(x)\mathrm{d}_q x = [f(x)g(x)]_a^b - \int_a^b f(qx) D_q g(x)\mathrm{d}_q x. \tag{2}
$$

For any function $f(x)$ continuous at $x = 0$, we have

$$
\int_0^a D_q f(x)\mathrm{d}_q x = f(a) - f(0) \ \text{ and } \ D_q \int_0^x f(t)\mathrm{d}_q t = f(x). \tag{3}
$$

Notice that for $q = 0$, we get

$$
\int_a^b f(x)\mathrm{d}_0 x = bf(b) - af(a). \tag{4}
$$

Jackson in [13] proposed the q-analogue of the exponential function e^x given by

$$
e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}.
$$

It is clear that $e_q^0 = 1$ and $D_q e_q^x = e_q^x$.

The q-analogue of the identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where the function E_q^x defined by $e_{1/q}^x$ is given also by

$$e_{1/q}^x = E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

The q-logarithm function $\log_q(x)$ is the inverse of the q-exponential function e_q^x, and the function $\mathrm{Log}_q(x)$ is the inverse function of E_q^x.

Many researchers have focused on the operator theory [3,6,17]. In 1994, Chung et al. [6] proposed the q-addition operator and discussed its properties. The q-addition operator is defined by

$$\begin{cases} (a \oplus_q b)^n = \sum_{k=0}^{n} {}_qC_k^n a^k b^{n-k}, & \forall\, n \in \mathbb{N},\ (a \neq b,) \\ (a \oplus_q a)^n = (a+a)^n = 2^n a^n, \end{cases}$$

where

$${}_qC_k^n = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Equivalently $\oplus_q$ is defined as: $\oplus_q : \mathbb{R}^2 \to \mathbb{R}$

$$(a, b) \mapsto a \oplus_q b\ ,$$

such that $a \oplus_q b$ is the unique real verifying $e_q^{a \oplus_q b} = e_q^a e_q^b$.

From the above definition, we have the following property

$k(a \oplus_q b) = ka \oplus_q kb,\ \forall\, k \in \mathbb{R}$.

It is easy to see that this operator is commutative, i.e. $a \oplus_q b = b \oplus_q a$. Also if we take $b = a$, then we have $a \oplus_q a = a + a = 2a$. Finally, if we take $b = 0$, we obtain $a \oplus_q 0 = 0 \oplus_q a = a$.

This operator permits to express the properties of the q-logarithm and q-exponential functions in a more compact form.

(i) $e_q^a e_q^b = e_q^{a \oplus_q b}$,
(ii) $e_q^{na} = (e_q^a)^n, \quad \forall n \in \mathbb{N}$,
(iii) $\log_q(ab) = \log_q(a) \oplus_q \log_q(b)$,
(ir) $\log_q(a^n) = n \log_q(a), \quad \forall n \in \mathbb{N}$.

Thomas in [17] defined the power function, and he used the q-operator in the properties of this function.

A power function based on q-addition is defined by $a_q^x = E_q^{x Log_q(a)},\ \forall\, a > 0$.

This function satisfies $a_q^x a_q^y = a_q^{x \oplus_q y}$, $(a^x)_q^y = a_q^{xy}$ and $(ab)_q^x = a_q^x b_q^x,\ \forall\, a,\, b > 0$.

Chung et al. [6] defined the new q-derivative also called the x-derivative which is given by

$$D_x f(x) = \lim_{\delta x \to 0} \frac{f(x \oplus_q \delta x) - f(x)}{\delta x}. \tag{5}$$

The new q-derivative of $E_q^{\alpha x}$ is $\alpha E_q^{\alpha x}$ with α as a constant indeed,

Example 1

$$\begin{aligned} D_x E_q^{\alpha x} &= \lim_{\alpha\delta x \to 0} \frac{E_q^{(\alpha x \oplus_q \alpha\delta x)} - E_q^{\alpha x}}{\alpha\delta x} \\ &= \lim_{\alpha\delta x \to 0} \frac{E_q^{\alpha x} E_q^{\alpha\delta x} - E_q^{\alpha x}}{\alpha\delta x} \\ &= E_q^{\alpha x} \lim_{\alpha\delta x \to 0} \frac{(1 + \alpha^2\delta x + O(\alpha^2\delta x^2)) - 1}{\alpha\delta x} \\ &= \alpha E_q^{\alpha x}. \end{aligned}$$

With the same manner, we can prove $D_x e_q^{\alpha x} = \alpha e_q^{\alpha x}$.

3 The New q-Gamma Function

Jackson in [13] has shown that the q-beta function has the q-integral representation, which is a q-analogue of Euler's formula:

$$\beta_q(t, s) = \int_0^1 x^{t-1}(1 - qx)_q^{s-1} \mathrm{d}_q x, \quad \forall\, t,\ s > 0. \tag{6}$$

The q-gamma function expressed as Γ_q is defined in [13] by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-qx} \mathrm{d}_q x, \quad \forall\, t > 0. \tag{7}$$

Jackson [13] proved the properties of the q-gamma function

$$\begin{aligned} \Gamma_q(t+1) &= [t]_q \Gamma_q(t), \quad \forall\, t > 0 \\ \Gamma_q(n+1) &= [n]_q!, \quad \forall n \in \mathbb{N} \end{aligned}$$

The relationship between the q-gamma and the q-beta functions is given by

$$\beta_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \quad \forall\, t,\ s > 0. \tag{8}$$

Díaz et al. in [10] defined the q, k-gamma function by

$$\Gamma_{q,k}(t) = \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{t-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} \mathrm{d}_q x,$$

with

$$E_{q^k}^{-\frac{q^k x^k}{[k]_q}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}}}{[k]_q^n [n]_{q^k}!}.$$

The q, k-gamma function is defined on the interval $\left[0, \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}\right]$.

Now we shall define the new q, k-gamma function on the interval $\left[0, \frac{B[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}\right]$ for $B > 0$ in order to generalize the q, k-gamma function.

Definition 1 The new q, k-gamma function $\Gamma_{q,k}^B$ is given by

$$\Gamma_{q,k}^B(t) = \int_0^{\frac{B[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{t-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} \mathrm{d}_q x.$$

If we take $k = 1$, we can deduce the new q-gamma function defined by

$$\Gamma_{q,1}^B(t) = \Gamma_q^B(t) = \int_0^{\frac{B}{1-q}} x^{t-1} E_q^{-qx} \mathrm{d}_q x.$$

Note that the q-gamma function is a special case of the new q-gamma function.

The following proposition gives some properties of the new q-gamma function.

Proposition 1 *For $B > 0$,*

(i) $\Gamma_q^B(1) = 1$.
(ii) $\Gamma_q^B(t+1) = [t]_q \Gamma_q^B(t), \quad \forall t > 0$.
(iii) $\Gamma_q^B(n+1) = [n]_q!, \quad \forall n \in \mathbb{N}$.

Proof For $B > 0$,

(i) $\Gamma_q^B(1) = \int_0^{\frac{B}{1-q}} E_q^{-qx} \mathrm{d}_q x = E_q^0 - \left(E_q^{-\frac{1}{1-q}}\right)^B = 1$. In fact, we have

$$e_q^x E_q^{-x} = 1,$$

we know also

$$e_q^{\frac{x}{1-q}} = \frac{1}{(1-x)_q^{\infty}}, \text{ with } (1-x)_q^{\infty} = \prod_{i=0}^{\infty}(1-q^i x).$$

If x goes to 1, then we obtain

$$e_q^{\frac{1}{1-q}} \to +\infty.$$

Therefore,

$$E_q^{-\frac{1}{1-q}} = \frac{1}{e_q^{\frac{1}{1-q}}} = 0.$$

(ii) Using the q-integration by parts (2), we obtain for $t > 0$,

$$\begin{aligned}
\Gamma_q^B(t+1) &= \int_0^{\frac{B}{1-q}} x^t E_q^{-qx} \mathrm{d}_q x \\
&= -\int_0^{\frac{B}{1-q}} x^t D_q E_q^{-x} \mathrm{d}_q x \\
&= \int_0^{\frac{B}{1-q}} [t]_q x^{t-1} E_q^{-qx} \mathrm{d}_q x \\
&= [t]\Gamma_q^B(t).
\end{aligned}$$

(iii) By induction, for $n = 1$, the formula is true.
By hypothesis of induction, we have $\Gamma_q^B(n+1) = [n]_q!$.
Using the q-integration by parts we obtain,

$$\begin{aligned}
\Gamma_q^B(n+2) &= \int_0^{\frac{B}{1-q}} x^{n+1} E_q^{-qx} \mathrm{d}_q x \\
&= -\int_0^{\frac{B}{1-q}} x^{n+1} D_q E_q^{-x} \mathrm{d}_q x. \\
&= \int_0^{\frac{B}{1-q}} [n+1] x^n E_q^{-qx} \mathrm{d}_q x \\
&= [n+1]_q \Gamma_q^B(n+1) \\
&= [n]_q! [n+1]_q \\
&= [n+1]_q! \,.
\end{aligned}$$

□

4 The k-Gamma q-Distribution

Charalambos was the first who coined the notion of the q-distribution in the discrete case [4,5]. As for the continuous case, Díaz and Pariguan [9] identified the Gaussian q-distribution.

A function $p_q(x)$ is a q-probability density, provided that it satisfies $p_q(x) \geq 0$, $\forall\, x \in \mathbb{R}$, and $\int_{\mathbb{R}} p_q(x) \mathrm{d}_q x = 1$. The q-cumulative distribution function of a real-valued random variable X, is the q-probability that X will take a value less than or equal to x. It gives the area under the probability q-density function from $-[\infty]_q$ to x. It is defined by

$$F_q(x) = \int_{-[\infty]_q}^{x} p_q(s) \mathrm{d}_q s, \quad x \in \mathbb{R}.$$

Note that the q-cumulative function is an increasing function verifying

$$\mathbb{P}_q\left(]a_1, a_2[\right) = \mathbb{P}_q(a_1 < X < a_2) = \int_{a_1}^{a_2} p_q \mathrm{d}_q x = F_q(a_2) - F_q(a_1).$$

Now, we recall the k-gamma density function with parameters $\lambda,\ a > 0$; it is given by

$$\gamma_{k,a}(x) = \frac{\lambda^a}{\Gamma_k(a)} x^{a-1} e^{-(\lambda x)^k} \mathbf{1}_{[0,\infty)}(x),$$

with

$$\Gamma_k(a) = \int_0^{\infty} x^{a-1} e^{-x^k} \mathrm{d}x.$$

The k-gamma q-density function [10] is defined by

$$\gamma_{q,k,a}(x) = \frac{1}{\Gamma_{q,k}(t)} x^{a-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} \mathbf{1}_{\left[0, \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}\right]}(x),$$

with

$$\Gamma_{q,k}(a) = \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} x^{a-1} E_{q^k}^{-\frac{q^k x^k}{[k]_q}} \mathrm{d}_q x,\ a > 0.$$

The construction of the q-analogue of the k-gamma distribution with parameters $\lambda, a > 0$ rests upon determining the q-analogue for the normalization factor.

Therefore, the q-analogue of $\dfrac{\lambda^a}{\Gamma_k(a)}$ is $\dfrac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)}$.

$$\int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} \gamma_{q,k}(a,\lambda)(x)\mathrm{d}_q x = \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} \frac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)} x^{a-1} E_{q^k}^{-\frac{q^k[\lambda]_q^k x^k}{[k]_q}} \mathrm{d}_q x.$$

Take $u = [\lambda]_q x$, then $\frac{u}{[\lambda]_q} = x$ and $\dfrac{1}{[\lambda]_q}\mathrm{d}_q u = \mathrm{d}_q x$.

So, by applying (1), we have,

$$\int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} \gamma_{q,k}(a,\lambda)(x)\mathrm{d}_q x = \frac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)} \int_0^{\frac{[\lambda]_q[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} \left(\frac{u}{[\lambda]_q}\right)^{a-1} E_{q^k}^{-\frac{q^k u^k}{[k]_q}} \frac{1}{[\lambda]_q}\mathrm{d}_q u.$$

Using Definition (1), then we have

$$\int_0^{\frac{[k]^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}} \gamma_{q,k}(a,\lambda)(x)\mathrm{d}_q x = 1.$$

Now, we are ready to generalize the k-gamma q-distribution with parameters $\lambda,\ a > 0$.

Definition 2 The density of the k-gamma q-distribution is defined by

$$\gamma_{q,k}(a,\lambda)(x) = \frac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)} x^{a-1} E_{q^k}^{-\frac{q^k[\lambda]_q^k x^k}{[k]_q}} \mathbf{1}_{\left[0,\ \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}\right]}(x), \text{ with } \lambda > 0,\ a > 0.$$

Note that, if we take $\lambda = 1$, $\gamma_{q,k}(a,1) = \gamma_{q,k,a}$ (Fig. 1).

Theorem 1 *The q-cumulative function of the k-gamma q-distribution, $\gamma_{q,k}(a,\lambda)$, with parameters $\lambda,\ a > 0$ is given by*

$$F_q(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{(1-q)}{\Gamma_{q,k}^{[\lambda]_q}(a)} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}}([\lambda]_q x)^{kn+a}}{[n]_q![k]_q^n(1-q^{kn+a})} & \text{if } 0 \le x \le \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}, \\ 1 & \text{if } x > \frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^{\frac{1}{k}}}. \end{cases}$$

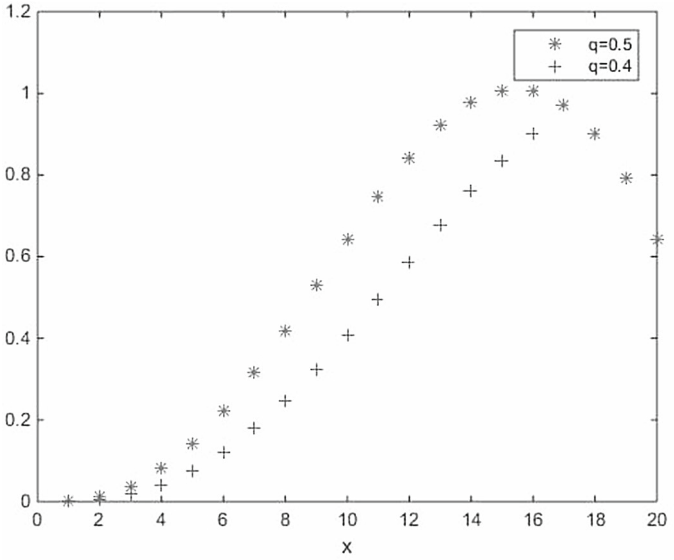

Fig. 1 Curve of $\gamma_{q,k}$ density function for $k = 1,\ a = 3$ and $\lambda = 4$

Proof By using the definition of the q-integral and the expression of E_q^x, we obtain for $x \in [0, \frac{1}{1-q}]$,

$$\begin{aligned}
F_q(x) &= \frac{[\lambda]_q^a}{\Gamma_q^{[\lambda]_q}(a)} \int_0^x s^{a-1} E_{q^k}^{-q^k[\lambda]_q^k s^k} \mathrm{d}_q s \\
&= \frac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} [\lambda]^{kn} s^{kn+a-1}}{[n]_q![k]_q^n} \\
&= \frac{[\lambda]_q^a}{\Gamma_{q,k}^{[\lambda]_q}(a)} (1-q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} [\lambda]_q^{kn} x^{kn+a}}{[n]_q![k]_q^n} \sum_{m=0}^{\infty} q^{m(kn+a)} \\
&= \frac{(1-q)}{\Gamma_{q,k}^{[\lambda]_q}(a)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{kn(n+1)}{2}} ([\lambda]_q x)^{kn+a}}{[n]_q![k]_q^n (1-q^{kn+a})}.
\end{aligned}$$

□

4.1 Characterization of the Exponential q-Distribution

Departing from the k-gamma q-distribution, the gamma and the exponential q-distributions can be deduced, the first by taking $k = 1$, and the second by taking $a = k = 1$.

We note the gamma q-distribution with parameters $\lambda,\ a > 0$ by $\gamma_q(\lambda, a)$, and we write the exponential q-distribution by $\zeta_q(\lambda)$.

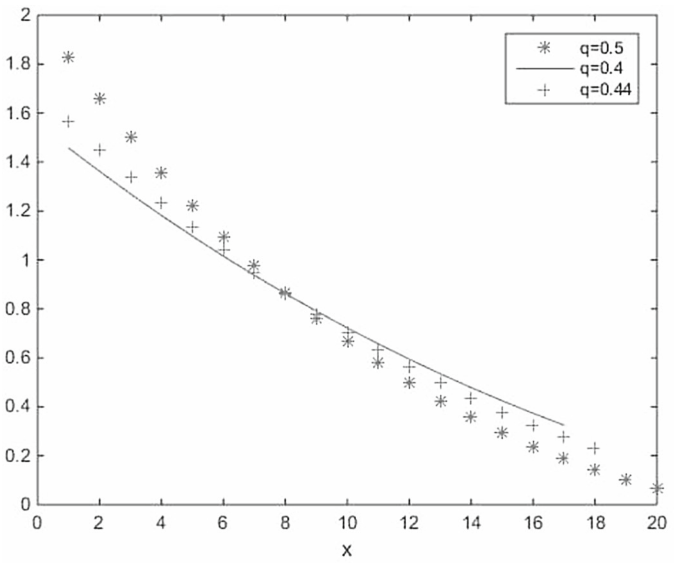

Fig. 2 Curve of the q-exponential density, ξ_q, function for $\lambda = 4$

Definition 3 The density of the exponential q-distribution, $\xi_q(\lambda)$, is defined by

$$\xi_q(x) = [\lambda]_q E_q^{-q[\lambda]x} \mathbf{1}_{[0,\ \frac{1}{1-q}]}(x), \quad for\ \lambda > 0.$$

In Fig. 2, we present the curve of the exponential q-distribution for $\lambda = 4$, with different values of $q = 0, 5$, $q = 0, 4$, and $q = 0, 44$. It is clear that the curve of ξ_q, is decreasing, and the exponential q-distribution as well as the ordinary exponential density has the same curved shape.

Theorem 2 *The q-cumulative function of the exponential q-distribution with parameter $\lambda > 0$ is given by*

$$F_q(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - E_q^{-[\lambda]_q x} & \text{if } 0 \leq x \leq \frac{1}{1-q}, \\ 1 & \text{if } x > \frac{1}{1-q}. \end{cases}$$

Proof For $x \in [0, \frac{1}{1-q}]$, using 3, we have

$$\begin{aligned} F_q(x) &= \int_0^x [\lambda]_q E_q^{-q[\lambda]_q s} \mathrm{d}_q s \\ &= \left[-E_q^{-[\lambda]_q s} \right]_0^x \\ &= 1 - E_q^{-[\lambda]_q x}. \end{aligned}$$

□

In the following result, we prove the q-memorylessness property of the exponential q-distribution. The idea of the characterization of the exponential q-distribution is based on the use of the q-addition operator and its properties (see [6,12]).

Theorem 3 *A random variable X is exponential q-distributed if and only if*

$$\mathbb{P}_q\left(X > s \oplus_q t | X > s\right) = \mathbb{P}_q(X > t), \quad \forall\, s,\ t \geq 0. \tag{9}$$

Proof "⇒" Let X be a random variable with q-exponential distribution.

$$\begin{aligned}\mathbb{P}_q(X \geq t) &= \int_t^{\frac{1}{1-q}} [\lambda]_q E_q^{-q[\lambda]_q x} \mathrm{d}_q x \\ &= E_q^{-[\lambda]_q t}, \quad \forall\, t \geq 0.\end{aligned}$$

Now, we compute the right-hand side of the equality
Note that

$$s \oplus_q t > s \ \Leftrightarrow \ \sum_{k=0}^{n} {}_qC_k^n s^k t^{n-k} = (s \oplus_q t)^n > s^n,\ \forall n \in \mathbb{N}.$$

$$\begin{aligned}\mathbb{P}_q\left(X > s \oplus_q t | X > s\right) &= \frac{\mathbb{P}_q(X > s \oplus_q t) \cap (X > s)}{\mathbb{P}_q(X > s)} \\ &= \frac{\mathbb{P}_q(X > s \oplus_q t)}{\mathbb{P}_q(X > s)} \\ &= \frac{E_q^{-[\lambda]_q (t \oplus_q s)}}{E_q^{-[\lambda]_q s}} \\ &= \frac{E_q^{-[\lambda]_q t \oplus_q (-[\lambda]_q s)}}{E_q^{-[\lambda]_q s}} \\ &= E_q^{-[\lambda]_q t} = \mathbb{P}_q(X \geq t).\end{aligned}$$

Then we obtain the equality.
"⇐"
We suppose that X verify (9) and let f_q be a q-density function of X.

$$\text{Then,}\ \ \mathbb{P}_q(X > t \oplus_q s) = \mathbb{P}_q(X > t)\mathbb{P}_q(X > s); \ \ \forall\, t, s > 0.$$

$$\text{That is} \int_{x > t \oplus_q s} f_q(x) \mathrm{d}_q x = \int_{x > t} f_q(x) \mathrm{d}_q x \int_{x > s} f_q(x) \mathrm{d}_q x.$$

In this step, we take the q-derivative with respect to t and we have:

$$\begin{aligned} -f_q(t \oplus_q s) &= -f_q(t)\left(\int_{x>s} f_q(x)\mathrm{d}_q x\right) \\ f_q(t \oplus_q s) &= f_q(t)\left(1 - \int_0^s f_q(x)\mathrm{d}_q x\right), \\ \text{which imply} \quad \frac{f_q(t \oplus_q s) - f_q(t)}{s} &= -f_q(t)\left(\frac{1}{s}\int_0^s f_q(x)\mathrm{d}_q y\right) \\ &= -f_q(t)\left(\frac{1}{s}(1-q)s\sum_{n=0}^{\infty} q^n f(q^n s)\right) \\ &= -f_q(t)\left((1-q)\sum_{n=0}^{\infty} q^n f(q^n s)\right). \end{aligned}$$

In the second step, we tend s towards 0 and we take $f(0) = q[\lambda]$, then we have,

$$\begin{aligned} \left((1-q)\sum_{n=0}^{\infty} q^n f(q^n s)\right) &\to_{s\to 0} (1-q)\sum_{n=0}^{\infty} q^n f(0) \\ &\to_{s\to 0} f(0) = q[\lambda]. \\ \text{Then,} \quad \lim_{s\to 0} \frac{f_q(t \oplus_q s) - f_q(t)}{s} &= -q[\lambda] f_q(t), \\ \text{hence,} \qquad D_x f_q(t) &= -q[\lambda] f_q(t). \end{aligned}$$

We can assume that $f_q(t) = [\lambda] E_q^{-q[\lambda]t}$ is the solution of the x-differential equation.

Hence, the proof is complete. □

Proposition 2 *The exponential q-distribution interpolates between the uniform distribution on the interval* $[0, 1]$ *and the exponential distribution on* $\mathbb{R}_+$.

Proof If we take $q = 0$, then $[\lambda]_0 = 1$ and $E_0^0 = 1$ as a matter of fact, $f_0(x) = \mathbf{1}_{[0,1]}(x)$.

On the other side, $[\lambda]_q$ converges to λ as q approaches to 1 and $E_q^{-q[\lambda]_q x}$ goes to $e^{-\lambda x}$.

To sum up, the following diagram illustrates these limits:

$$U_{[0,1]} \xleftarrow{q\to 0} \quad \xi_q(\lambda) \quad \xrightarrow{q\to 1} \xi(\lambda)$$

$\square$

5 The q-Moments

Díaz et al. [9,10] introduced the notion of moment in the theory of the q-calculus. It is expressed as follows:

$$_qM_n = \int_{\mathbb{R}} x^n f(x)\mathrm{d}_q x, \quad \text{for } n \in \mathbb{N}.$$

In the following theorem, we compute the q-moment of the k-gamma q-distribution.

Theorem 4 *The q-moment of the k-gamma q-distribution with parameters $\lambda, a > 0$, is given by*

$$_qM_n = \frac{\Gamma_{q,k}^{[\lambda]_q(a+nk)}}{\Gamma_{q,k}^{[\lambda]_q}(a)[\lambda]^{nk}}, \quad \forall\, n \in \mathbb{N}.$$

Proof The idea of this proof is based on the formula of variable change (1),

$$_qM_n = \frac{[\lambda]^a}{\Gamma_{q,k}^{[\lambda](a)}} \int_0^{\frac{[k]_q^{\frac{1}{k}}}{(1-q^k)^k}} x^{a+nk-1} E_{q^k}^{-\frac{q^k[\lambda]^k x^k}{[k]_q}} \mathrm{d}_q x$$

If we make the variable change $u = [\lambda]_q x$, the q-moment becomes

$$\begin{aligned} _qM_n &= \frac{[\lambda]_q^{-nk}}{\Gamma_{q,k}^{[\lambda]_q(a)}} \int_0^{\frac{[\lambda]_q[k]_q^{\frac{1}{k}}}{(1-q^k)^k}} u^{a+nk-1} E_{q^k}^{-\frac{q^k u^k}{[k]_q}} \mathrm{d}_q x \\ &= \frac{\Gamma_{q,k}^{[\lambda]_q}(a+nk)}{\Gamma_{q,k}^{[\lambda]_q}(a)[\lambda]^{nk}}. \end{aligned}$$

$\square$

For $n = 1$, the q-moment is presented as:

$$_qM_1 = \int_{\mathbb{R}} x p_q(x)\mathrm{d}_q x.$$

That is, the first q-moment is called also q-mean of a random variable X with q-probability density function $p_q(x)$.

Definition 4 The q-mean of a random variable X with q-density function $p_q(x)$ is given by

$$\mathbb{E}_q(X) = \int_{\mathbb{R}} x p_q(x) \mathrm{d}_q x = {}_qM_1.$$

Note that the q-expected value operator $\mathbb{E}_q(\cdot)$ is linear in the sense that

$$\mathbb{E}_q(X+Y) = \mathbb{E}_q(X) + \mathbb{E}_q(Y)$$

At this level of analysis, we would set forward certain interesting remarks.

Remark 1
1. From the q-moment of the k-gamma q-distribution, we can deduce the q-mean of the q-gamma distribution in terms of $\mathbb{E}_q(X) = \dfrac{[a]_q}{[\lambda]_q}, \lambda,\ a > 0.$
2. If we take $a = 1$ in the q-mean of the gamma q-distribution, the q-mean of the exponential q-distribution is $\mathbb{E}_q(X) = \dfrac{1}{[\lambda]_q}$.
3. We can check the necessary condition of the q-calculus, *i.e.* if q goes to 1, then the q-mean converges to the ordinary mean.

6 The q-Simulation with the q-Inversion Method

In order to introduce the concept of simulation in the theory of q-calculus, we need to identify the q-analogue of the uniform distribution on $[a, b]$.

Definition 5 A random variable X is called uniform q-distributed, $U_{q[a,b]}$, if its probability density function is given by

$$U_{q[a,b]}(x) = \begin{cases} \frac{1}{[b]_q - [a]_q} & \text{if} [a]_q \leq x \leq [b]_q, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the q-cumulative function F_q of X is a continuous and a strictly increasing on $(0, \frac{1}{1-q})$. Then F_q is a bijective function. We denote by F_q^{-1} its reciprocal function.

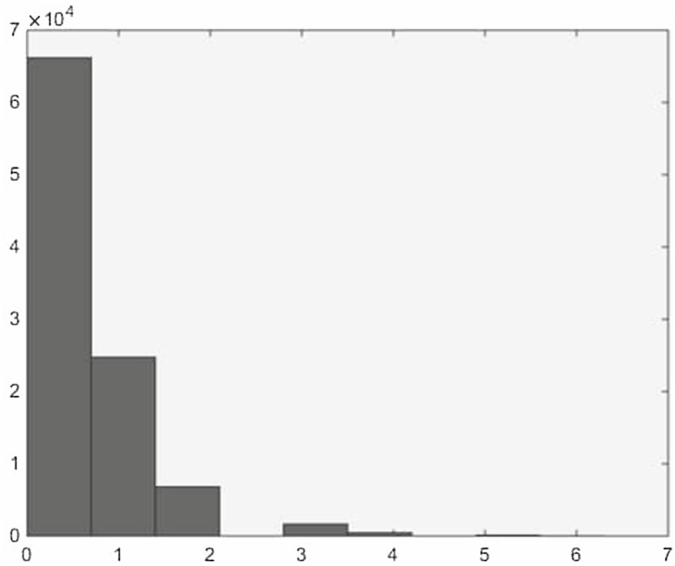

Fig. 3 Histogram of a sample from exponential q-distribution for $\lambda = 8$ and $q = 0, 25$

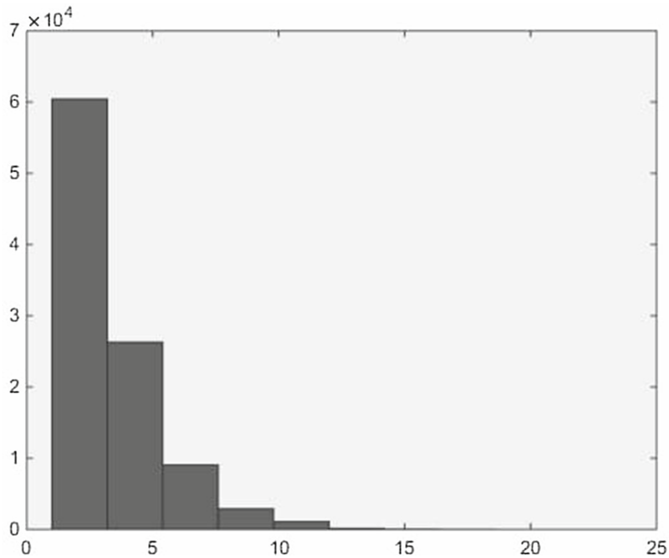

Fig. 4 Histogram of a sample from exponential q-distribution for $\lambda = 8$ and $q = 0, 55$

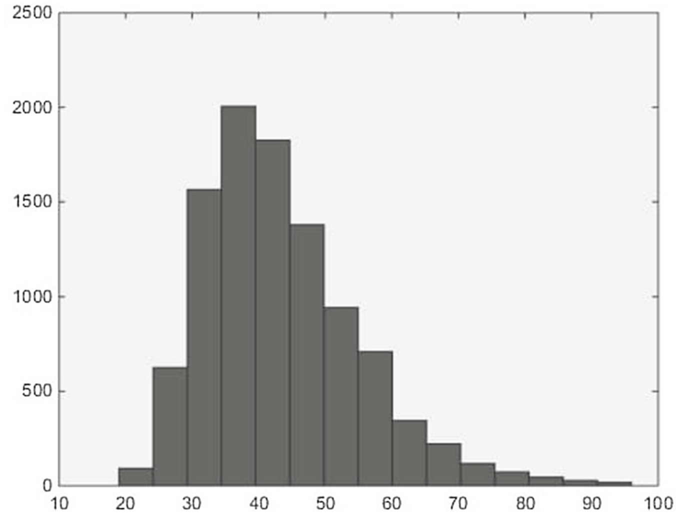

Fig. 5 Histogram of a sample from exponential q-distribution for $\lambda = 8$ and $q = 0, 85$

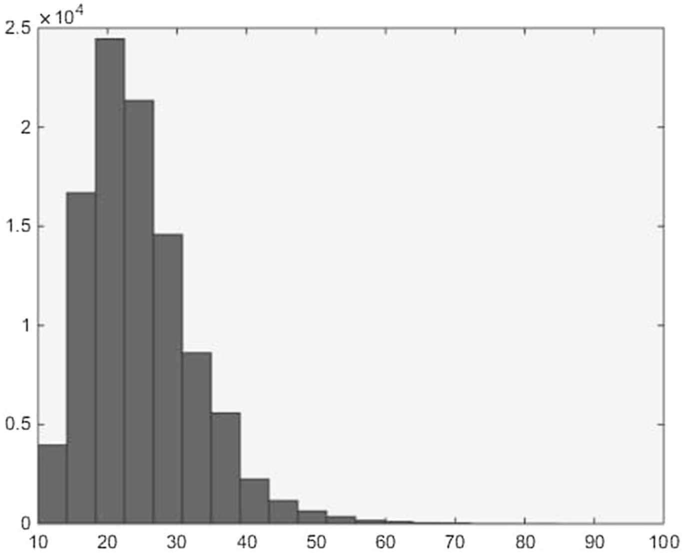

Fig. 6 Histogram of a sample from exponential q-distribution for $\lambda = 8$ and $q = 0, 9$

Theorem 5 *Let X be a random variable with q-cumulative function F_q and let U be a q-uniform random variable on* $[0, 1]$. *Then, X and $F_q^{-1}(U)$ have the same q-distribution.*

Proof Let $t \in [0, 1]$,

$$\begin{aligned}\mathbb{P}_q[F_q^{-1}(U) \leq t] &= \mathbb{P}_q[U \leq F_q(t)] \\ &= F_{U_q}(F_q(t)) \\ &= F_q(t).\end{aligned}$$

Then, X and $F_q^{-1}(u)$ have the same q-distribution. □

According to Theorem 5, the algorithm of q-inversion method is defined as:

1. Simulate u from $U_q[0, 1]$.
2. Compute $F_q^{-1}(u) = x$ which is an observation from the random variable X.

Then, we extend the q-inversion method for generating data to q-distribution.

Now, we apply this algorithm in order to simulate different samples from the exponential q-distribution. We simulated samples from the exponential q-distribution with same size $N = 10{,}000$, and we got the following histograms (see Figs. 3, 4, 5 and 6).

The simulated data were obtained from the q-exponential model for different values of $q \in (0, 1)$. The simulated data were represented by histograms. In order to evaluate the performance of the proposed simulated method, we computed the mean squared error between the estimated density by applying the histogram method and the true-density function. The obtained mean squared errors are around to 10^{-2} for different values of q. Then, the proposed simulated approach is consistent.

Conclusion

Memorylessness refers to the cases when the distribution of a "waiting time" until a certain event does not depend on how much time has already elapsed. Basically the exponential distribution is memoryless. In this paper, we showed that the exponential q-distribution is q-memoryless and corresponds to a link between the uniform distribution on $[0, 1]$ and the classical exponential distribution. This transition may be accounted for in terms of physics as follows: If we take $[0, \frac{1}{1-q}]$ the time scale and try to explore the evolution of a certain phenomenon over time, at time $t = 0$, this phenomenon follows the uniform distribution on $[0, 1]$. However for $q > 0$ it follows the exponential q-distribution. The more q approaches 1, this process approaches the ordinary exponential distribution. From this perspective, we judge that the exponential q-distribution is extremely interesting as it lays the ground for certain constructive and fruitful applications. Having explored the exponential q-distribution, our work is a step that may be taken further. In a future work, we aspire to characterize the gamma q-distribution.

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