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Conservation Laws and τ-Symmetry Algebra of the Gerdjikov–Ivanov Soliton Hierarchy

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Abstract

Two hierarchies of isospectral and nonisospectral Gerdjikov–Ivanov equations are constructed. Conservation laws of the isospectral soliton hierarchy are explicitly derived from a specific Riccati equation that a ratio of two eigenfunctions satisfies. The corresponding K-symmetries and τ-symmetries formulated from the isospectral and nonisospectral hierarchies constitute an infinite-dimensional τ-symmetry algebra for the isospectral hierarchy.

Keywords Matrix spectral problem · Conservation law · τ-Symmetry

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1 Introduction

Many soliton systems possess remarkably rich algebraic characteristics, including the existence of infinitely many symmetries and conservation laws.

For $(1+1)$-dimensional integrable systems, many approaches have been developed to find their conservation laws (CLs), such as the method using the variational identities to formulate generating functions for conserved densities [1,2] in the non-semisimple Lie algebras framework, using adjoint symmetries [3,4] and the expansion technique of ratios of eigenfunctions of spectral problems [5,6]. Among them, generating the CLs from Lax pairs of evolution equations is most popular one [5–9]. The key of this method is to get a conservation density from the spectral problem of Lax pairs. Then by using the obtained conservation density and evolution equation of time, CLs are worked out.

Associated with those CLs are K-symmetries, which do not depend explicitly on space and time variables. In 1987, Li et al. found a general way to construct τ-symmetries [10,11]. These symmetries often constitute a Lie algebra together with K-symmetries. Li and Cheng [12,13] found that there also exist new sets of symmetries for the evolution equations which take τ-symmetries as vector fields. Tu [14] showed that these τ-symmetries may be generated by the generators of the first degree. On the basis of Tu's work, one of the authors (Ma) established a more general skeleton on K-symmetries and τ-symmetries of evolution equations and their Lie algebraic structures [15,16]. In recent years, symmetries of discrete soliton hierarchies were also researched [17–20].

It is well known that the Kaup–Newell equation, the Chen–Lee–Liu equation and the Gerdjikov–Ivanov (GI) equation are three celebrated equations with derivative-type nonlinearities [21–24]. The GI equation

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xx} - 2q^2 r_x - 2q^3 r^2 \\ -r_{xx} - 2q_x r^2 + 2q^2 r^3 \end{pmatrix} \tag{1.1}$$

as an integrable system with fifth-order nonlinear terms, has drawn a great attraction. It has already been proved to be integrable in the Liouville sense by means of trace identity [25,26]. In Refs. [27,28], Fan constructed an N-fold Darboux Transformation (DT) of Eq. (1.1) and derived its soliton solutions. It is well known that some soliton equations can exhibit rogue wave phenomena [29,30]. He et al. researched the rogue waves and breather solutions of the GI equation by using DT [31,32]. The algebro-geometric solutions of the GI equation were given in [33–35]. Recently, Zhang et al. [36] gave its N-soliton solutions by Hirotas bilinear method. But conservation laws and τ-symmetries of the GI soliton hierarchy have not been studied yet.

In this paper, we would like to construct the isospectral and nonisospectral GI hierarchies from a matrix spectral problem associated with the GI equation. The nonisospectral GI hierarchy will be used to present τ-symmetries for the isospetral GI hierarchy. Moreover, a series of CLs of the GI isospectral soliton hierarchy is derived from a Riccati equation which a ratio of two eigenfunctions needs to satisfy. As the application of the obtained hierarchies, the corresponding K-symmetries and

τ-symmetries will be formulated and all those symmetries will be proved to form an infinite-dimensional Lie algebra.

The paper is organized as follows. In Sect. 2, we will discuss basic notions and notations. In Sect. 3, we will obtain the isospectral and nonisospectral GI hierarchies and CLs of the GI isospectral soliton hierarchy. In Sect. 4, two types of symmetries will be constructed and proved to constitute an infinite-dimensional Lie algebra. We conclude the paper in Sect. 5.

2 Basic Notions

Here we mainly follow the notions and notations used in [15] (see also [20]). Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields respectively, and $\mathcal{L}$ be one linear topological space over $\mathbb{C}$. We denote by $\mathscr{L}$ all differentiable vector functions mapping $\mathbb{R}^N \times \mathbb{R} \times \mathcal{L}$ into $\mathcal{L}$.

Definition 1 Let $K = K(u) = K(x, t, u)$, $S = S(u) = S(x, t, u) \in \mathscr{L}$. The Gateaux derivative of $K(u)$ in the direction $S(u)$ with respect to u is defined by

$$K'[S] = K'[u][S(u)] = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon S(u))|_{\varepsilon=0}. \tag{2.1}$$

It is well known that $\mathscr{L}$ forms a Lie algebra with respect to the following product:

$$[\![K, S]\!] = [\![K(u), S(u)]\!] = K'(u)[S(u)] - S'(u)[K(u)] \quad K, S \in \mathscr{L}. \tag{2.2}$$

Assume that $u = u(x, t)$ is a differentiable function or a differential vector function mapping $\mathbb{R}^N \times \mathbb{R}$ into $\mathcal{L}$. We consider an evolution equation

$$u_t = K(x, t, u) \quad K \in \mathscr{L}. \tag{2.3}$$

Definition 2 A function $G = G(x, t, u) \in \mathscr{L}$ is called a symmetry of the equation of (2.3) if G satisfies the linearized equation of (2.3)

$$\frac{\mathrm{d}G}{\mathrm{d}t} = K'(u)[G] \tag{2.4}$$

where d/dt denote the total t-derivative, u satisfies Eq. (2.3) and $K'(u)[G]$ is defined as in (2.1).

Evidently, the linearized equation (2.4) is equivalent to the following equation

$$\frac{\partial G}{\partial t} = [\![K, G]\!] \tag{2.5}$$

where $[\![,]\!]$ is defined as in (2.2). The symmetries defined in Definition 2 are all infinitesimal generators of one-parameter groups of invariant transformations of Eq. (2.3).

Denote by $L(\mathscr{L})$ the linear operators mapping $\mathscr{L}$ into itself. Furthermore, denote by $\mathscr{U}$ the set of differentiable operators mapping $\mathbb{R}^n \times \mathbb{R} \times \mathscr{L}$ into $L(\mathscr{L})$ and suppose that $\Phi K = \Phi(x, t, u)K$ for $\Phi \in \mathscr{U}$, $K \in \mathscr{L}$, $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, $u \in \mathscr{L}$.

Definition 3 Let $\Phi \in \mathscr{U}$, $K \in \mathscr{L}$, the Lie derivative $L_K\Phi \in \mathscr{U}$ of Φ with respect to K is defined by

$$(L_K\Phi) = \Phi'[K] - K'\Phi + \Phi K' \tag{2.6}$$

where the Gateaux derivative $\Phi'[K]$ of the operator $\Phi(u)$ in the direction K with respect to u is defined as (2.1).

Definition 4 An operator $\Phi \in \mathscr{U}$ is called a hereditary symmetry if the following holds:

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K = \Phi(\Phi'[K]S - \Phi'[S]K) \quad K, S \in \mathscr{L}. \tag{2.7}$$

Definition 5 An operator $\Phi \in \mathscr{U}$ is called a strong symmetry if it maps one symmetry of (2.3) into another symmetry of (2.3).

It is easy to see that $\Phi \in \mathscr{U}$ is a strong symmetry of (2.3) if and only if

$$\frac{\partial \Phi}{\partial t} + L_K\Phi = 0. \tag{2.8}$$

3 Isospectral and Nonisospcetral Hierarchies and Conservation Laws

In this section, we first deduce isospectral and nonisospectral GI hierarchies from a matrix spectral problem associated with the GI equation.

Let

$$\varsigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and assume that T denotes the transpose of a matrix. It is well known that the GI hierarchy has the following Lax Pairs [35,36]

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -\frac{1}{2}(\eta^2 - 2qr) & \eta q \\ \eta r & \frac{1}{2}(\eta^2 - 2qr) \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3.1a}$$

and its time evolution

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{3.1b}$$

where $q = q(t, x), r = r(t, x)$ are potential functions and η is a spectral parameter. We assume that $q(x, t)$ and $r(x, t)$ are smooth functions of variables t and x; and their derivatives of any order with respect to x vanish rapidly as $x \to \infty$. The compatibility condition, the zero curvature equation, reads

$$M_t - N_x + [M, N] = 0, \tag{3.2}$$

which yield

$$\begin{aligned}\eta \begin{pmatrix} q \\ r \end{pmatrix}_t &= L_1 L_2 \begin{pmatrix} B \\ -C \end{pmatrix} + \eta^2 \begin{pmatrix} B \\ -C \end{pmatrix} + 2\eta A_0 \begin{pmatrix} q \\ -r \end{pmatrix} \\ &\quad -\eta_t L_1 \begin{pmatrix} q \\ r \end{pmatrix} - 2\eta^2 \eta_t \begin{pmatrix} xq \\ -xr \end{pmatrix},\end{aligned} \tag{3.3}$$

where

$$L_1 = I + 2(q, -r)^{\mathrm{T}} \partial^{-1}(r, q), \quad L_2 = -\varsigma\partial - 2qrI.$$

Setting

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n} (-1)^{n-j} \begin{pmatrix} b_j \\ c_j \end{pmatrix} \eta^{2(n-j)+1}, \tag{3.4}$$

and comparing the coefficients of the same power of η in (3.3), we then see that the related hierarchies of isospectral ($\eta_t = 0$, $A_0 = \frac{1}{2}(-1)^n \eta^{2n}$) and nonisospectral ($\eta_t = \frac{1}{2}(-1)^{n-1}\eta^{2n-1}$, $A_0 = 0$) can be derived respectively, i.e.,

$$u_t = K_n = \Phi^n \begin{pmatrix} q \\ -r \end{pmatrix}, \tag{3.5a}$$

$$u_t = \sigma_n = \Phi^{n-1} \begin{pmatrix} xq_x + \frac{1}{2}q \\ xr_x + \frac{1}{2}r \end{pmatrix}, \tag{3.5b}$$

where n is a positive integer and

$$u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad \Phi = \begin{pmatrix} \partial - 2q\partial^{-1}r_x - 4q\partial^{-1}r^2q & 2q\partial^{-1}q_x - 2q^2 - 4q\partial^{-1}q^2r \\ 2r\partial^{-1}r_x - 2r^2 + 4r\partial^{-1}qr^2 & -\partial - 2r^{-1}q_x + 4r\partial^{-1}q^2r \end{pmatrix}. \tag{3.6}$$

The first nonlinear equation in the GI soliton hierarchy (3.5a) is the GI equation (1.1).

To apply the scheme of generating conservation laws based on Lax pairs, we consider the ratio of the two eigenfunctions

$$\omega = \eta \frac{\phi_2}{\phi_1}. \tag{3.7}$$

Obviously from the spectral problem in (3.1), we see that the ratio ω satisfies the following Riccati equation:

$$q\omega_x = -q^2\omega^2 + (\eta^2 - 2qr)q\omega + \eta^2 qr, \tag{3.8}$$

and the following conservation law relation holds:

$$[qr + q\omega(x,\eta)]_t = \left(A + B\frac{\omega(x,\eta)}{\eta}\right)_x. \tag{3.9}$$

Therefore, for example, letting

$$A = \frac{1}{2}\eta^4 - qr\eta^2 - q^2r^2 + (rq_x - qr_x), \quad B = -q\eta^3 + q_x\eta, \tag{3.10}$$

we have

$$\begin{aligned}&\left(qr + \sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n}}\right)_t\\ &= \left[\frac{1}{2}\eta^4 - qr\eta^2 - q^2r^2 + rq_x - qr_x - \sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n-2}} + \frac{q_x}{q}\sum_{n=0}^{\infty}\frac{\omega_n(x)}{\eta^{2n}}\right]_x,\end{aligned} \tag{3.11}$$

which generates infinitely many conservation laws for the GI equation (1.1). Expand ω into a Laurent series

$$\omega = \sum_{n=0}^{\infty}\omega_n\eta^{-2n}, \tag{3.12}$$

we obtain from the above Riccati equation a recursion relation for defining ω_n:

$$\omega_0 = -qr, \quad \omega_{n+1} = q\left(\frac{\omega_n}{q}\right)_x + \sum_{j=0}^{n}\omega_j\omega_{n-j} + 2qr\omega_n, \quad n \geq 0. \tag{3.13}$$

The first few conservation laws in (3.11) can be computed as follows:

$$\begin{aligned}&(qr)_t = [q_xr - qr_x - q^2r^2]_x,\\ &[qr_x + q^2r^2]_t = [q_xr_x - qr_{xx} - 2q^2rr_x]_x,\\ &\left[-qr_{xx} - qq_xr^2 - 2q^2rr_x\right]_t\\ &\quad = \left[qr_{xxx} + qq_{xx}r^2 + q^2r_x^2 - q_xr_{xx} - q_x^2r^2 + 2qq_xrr_x + 2q^2rr_{xx} - 2q^3r^2r_x - q^4r^4\right]_x.\end{aligned}$$

4 τ-Symmetry Algebra of the GI Soliton Hierarchy

Rewrite the recursion operator Φ of the GI soliton hierarchy in the following form:

$$\Phi = -\varsigma\partial + u^{\mathrm{T}}\delta u e + 2\varsigma u\partial^{-1}u_x^{\mathrm{T}}\varsigma\delta - 2uu^{\mathrm{T}}\delta + 2\varsigma u\partial^{-1}u^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u,$$

and then the Gâteaux derivative of the operator Φ in the direction of $f \in \mathscr{L}$ is

$$\begin{aligned}\Phi'[f] = 2\varsigma\left[f\partial^{-1}(u_x^{\mathrm{T}}\varsigma\delta + u^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u) + u\partial^{-1}(f_x^{\mathrm{T}}\varsigma\delta + f^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u + 2f^{\mathrm{T}}\delta uu^{\mathrm{T}}\delta)\right]\\ + 2(f^{\mathrm{T}}\delta u - fu^{\mathrm{T}}\delta - uf^{\mathrm{T}}\delta).\end{aligned}$$

By the same way, we can obtain the Gâteaux derivative $\Phi'[g], \quad g \in \mathscr{L}$. With the equality

$$\varsigma u\partial^{-1}f_x^{\mathrm{T}}\varsigma\delta g - \varsigma u\partial^{-1}g_x^{\mathrm{T}}\varsigma\delta f = fu^{\mathrm{T}}\delta g - gu^{\mathrm{T}}\delta f,$$

we have

$$\begin{aligned}&\Phi(\Phi'[f]g - \Phi'[g]f)\\ &\quad = \Phi[2\varsigma f\partial^{-1}(u_x^{\mathrm{T}}\varsigma\delta g + u^{\mathrm{T}}\delta g u^{\mathrm{T}}\delta u) - 2\varsigma g\partial^{-1}(u_x^{\mathrm{T}}\varsigma\delta f + u^{\mathrm{T}}\delta f u^{\mathrm{T}}\delta u)\\ &\qquad - 2fu^{\mathrm{T}}\delta g + 2gu^{\mathrm{T}}\delta f].\end{aligned}$$

Similarly, we can arrive at

$$\begin{aligned}&\Phi'[\Phi f]g - \Phi'[\Phi g]f\\ &\quad = 2(\Phi f)^{\mathrm{T}}(\delta u - u^{\mathrm{T}}\delta)g + 2\varsigma(\Phi f)\partial^{-1}(u_x^{\mathrm{T}}\varsigma\delta + u^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u)g + 2\varsigma u\partial^{-1}\Big[(\Phi f)_x^{\mathrm{T}}\varsigma\delta\\ &\qquad + (\Phi f)^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u\Big]g\\ &\qquad - 2(\Phi g)^{\mathrm{T}}(\delta u - u^{\mathrm{T}}\delta)f - 2\varsigma(\Phi g)\partial^{-1}(u_x^{\mathrm{T}}\varsigma\delta + u^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u)f - 2\varsigma u\partial^{-1}\Big[(\Phi g)_x^{\mathrm{T}}\varsigma\delta\\ &\qquad + (\Phi g)^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u\Big]f\\ &\qquad - 2u\Big[(\Phi f)^{\mathrm{T}}\delta g - (\Phi g)^{\mathrm{T}}\delta f\Big] + 4\varsigma u\partial^{-1}u^{\mathrm{T}}\delta\Big[(\Phi f)^{\mathrm{T}}\delta u g - (\Phi g)^{\mathrm{T}}\delta u f\Big].\end{aligned}$$

Lemma 1 *The recursion operator Φ is a hereditary symmetry.*

Proof By using the following equalities

$$\varsigma gu^{\mathrm{T}}\delta f_x = f_x^{\mathrm{T}}\varsigma\delta ug - uf_x^{\mathrm{T}}\varsigma\delta g + \varsigma uf_x^{\mathrm{T}}\delta g, \tag{4.1a}$$

$$\varsigma fu^{\mathrm{T}}\delta g_x = g_x^{\mathrm{T}}\varsigma\delta uf - ug_x^{\mathrm{T}}\varsigma\delta f + \varsigma ug_x^{\mathrm{T}}\delta f, \tag{4.1b}$$

we have

$$
\begin{aligned}
&\Phi(\Phi^{'}[f]g-\Phi^{'}[g]f)-(\Phi^{'}[\Phi f]g-\Phi^{'}[\Phi g]f)\\
&\quad=2\varsigma(fu_x^{\mathrm{T}}\delta g-gu_x^{\mathrm{T}}\delta f)-2(fu_x^{\mathrm{T}}\varsigma\delta g-gu_x^{\mathrm{T}}\varsigma\delta f)+4\varsigma u\partial^{-1}\left[u^{\mathrm{T}}\varsigma\delta(gu^{\mathrm{T}}\delta f-fu^{\mathrm{T}}\delta g)\right]_x\\
&\qquad+4(gu^{\mathrm{T}}\delta f-fu^{\mathrm{T}}\delta g)u^{\mathrm{T}}\delta u-8\varsigma u\partial^{-1}(f^{\mathrm{T}}\varsigma\delta gu^{\mathrm{T}}\delta u_x+u_x^{\mathrm{T}}\varsigma\delta fu^{\mathrm{T}}\delta g-u_x^{\mathrm{T}}\varsigma\delta gu^{\mathrm{T}}\delta f)\\
&\quad=4gu^{\mathrm{T}}\delta fu^{\mathrm{T}}\delta u-4fu^{\mathrm{T}}\delta gu^{\mathrm{T}}\delta u+4\varsigma u\left[(u^{\mathrm{T}}\varsigma\delta gu^{\mathrm{T}}\delta f)-(u^{\mathrm{T}}\varsigma\delta fu^{\mathrm{T}}\delta g)\right]\\
&\quad=0.
\end{aligned}
$$

□

Lemma 2 *The recursion operator Φ is a strong symmetry of the GI hierarchy (3.5a), i.e.,*

$$\Phi'[K_m]=[K_m',\Phi]. \tag{4.2}$$

Proof It is easy to verify

$$
\begin{aligned}
\Phi^{'}[K_0]&=2u(u^{\mathrm{T}}\varsigma\delta-\partial^{-1}u_x^{\mathrm{T}}\varsigma\delta-\partial^{-1}u^{\mathrm{T}}\delta u^{\mathrm{T}}\delta u)\\
&\quad-2\varsigma u(\partial^{-1}u^{\mathrm{T}}\varsigma\delta u^{\mathrm{T}}\delta u+\partial^{-1}u_x^{\mathrm{T}}\delta-u^{\mathrm{T}}\delta)\\
&=(K_0)^{'}\Phi-\Phi(K_0)^{'}.
\end{aligned}
$$

Since Φ is a hereditary and strong symmetry operator for the equation $u_t=K_0$, we have that Φ is a strong symmetry operator for $u_t=\Phi^m K_0=K_m$. Hence, Eq. (4.2) holds. □

Lemma 3

$$\Phi'[\sigma_n]+\Phi\sigma_n'-\sigma_n'\Phi=\Phi^n,\quad n=1,2,\ldots \tag{4.3}$$

Proof When $n=1$, it is easy to verify that

$$\Phi'[\sigma_1]+\Phi\sigma_1'-\sigma_1'\Phi=\Phi. \tag{4.4}$$

Assume that

$$\Phi'[\sigma_m]+\Phi\sigma_m'-\sigma_m'\Phi=\Phi^m, \tag{4.5}$$

then

$$
\begin{aligned}
&(\Phi^{m+1}-\Phi\sigma_{m+1}'+\sigma_{m+1}'\Phi)f\\
&\quad=\Phi^{m+1}f-\Phi(\Phi\sigma_m)'f+(\Phi\sigma_m)'\Phi f\\
&\quad=\Phi^{m+1}f-\Phi(\Phi'[f]\sigma_m+\Phi\sigma_m'[f])+\Phi'[\Phi f]\sigma_m+\Phi\sigma_m'[\Phi f]\\
&\quad=\Phi^{m+1}f-\Phi\Phi'[f]\sigma_m-\Phi(\sigma_m'\Phi+\Phi^m-\Phi'[\sigma_m])f+\Phi'[\Phi f]\sigma_m+\Phi\sigma_m'[\Phi f]\\
&\quad=\Phi\Phi'[\sigma_m]f+\Phi'[\Phi f]\sigma_m-\Phi\Phi'[f]\sigma_m
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi'[\Phi\sigma_m]f \\
&= \Phi'[\sigma_{m+1}]f. \qquad (4.6)
\end{aligned}
$$

where we have used that Φ is a hereditary symmetry operator. Thus Eq. (4.3) holds for any n. □

Theorem 1 *The isospectral flows K_m (3.5a) and nonisospectral flows σ_m (3.5b) of the GI hierarchy constitute an infinitely-dimensional Lie algebra and satisfy the relation,*

$$\llbracket K_m, K_n \rrbracket = 0, \qquad (4.7a)$$

$$\llbracket K_m, \sigma_n \rrbracket = mK_{m+n-1}, \qquad (4.7b)$$

$$\llbracket \sigma_m, \sigma_n \rrbracket = (m-n)\sigma_{m+n-1}, \quad m, n = 1, 2, \ldots. \qquad (4.7c)$$

Proof Here we only prove Eq. (4.7b), since the proof of the other two identities is similar.

We first prove the identity

$$\llbracket K_m, \sigma_1 \rrbracket = mK_m. \qquad (4.8)$$

When $m = 0$, we have

$$
\begin{aligned}
\llbracket K_0, \sigma_1 \rrbracket &= K_0'[\sigma_1] - \sigma_1'[K_0] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} xq_x + \frac{q}{2} \\ xr_x + \frac{r}{2} \end{pmatrix} - \begin{pmatrix} x\partial + \frac{1}{2} & 0 \\ 0 & x\partial + \frac{1}{2} \end{pmatrix} \begin{pmatrix} q \\ -r \end{pmatrix} \qquad (4.9) \\
&= 0
\end{aligned}
$$

Thus Eq. (4.8) is true when $m = 0$.

Assume that Eq. (4.8) holds for $m - 1$, i.e.,

$$\llbracket K_{m-1}, \sigma_1 \rrbracket = (m-1)K_{m-1}, \qquad (4.10)$$

then

$$
\begin{aligned}
\llbracket K_m, \sigma_1 \rrbracket &= \llbracket \Phi K_{m-1}, \sigma_1 \rrbracket \\
&= \Phi'[\sigma_1]K_{m-1} + \Phi K_{m-1}'[\sigma_1] - \sigma_1'[K_m] \\
&= (\Phi + \sigma_1'\Phi - \Phi\sigma_1')K_{m-1} + \Phi K_{m-1}'[\sigma_1] - \sigma_1'[K_m] \qquad (4.11) \\
&= K_m + \Phi\llbracket K_{m-1}, \sigma_1 \rrbracket \\
&= mK_m.
\end{aligned}
$$

So Eq. (4.8) is true for any m.

Now let us consider general identity (4.7b) using the same method. From Eq. (4.8) we know that the identity (4.7b) holds when $n = 1$. Assume that equation (4.7b) hold for $n - 1$, which is

$$[\![K_m, \sigma_{n-1}]\!] = mK_{m+n-2}. \tag{4.12}$$

Then we can obtain

$$\begin{aligned}
[\![K_m, \sigma_n]\!] &= [\![K_m, \Phi\sigma_{n-1}]\!] \\
&= K_m'[\Phi\sigma_{n-1}] - \Phi'[K_m]\sigma_{n-1} - \Phi\sigma_{n-1}'[K_m] \\
&= K_m'[\Phi\sigma_{n-1}] - (K_m'\Phi - \Phi K_m')\sigma_{n-1} - \Phi\sigma_{n-1}'[K_m] \\
&= \Phi[\![K_m, \sigma_{n-1}]\!] \\
&= mK_{m+n-1}.
\end{aligned} \tag{4.13}$$

Thus we complete the proof of Eq. (4.7b). □

Corollary 1 *The vector field $\sigma_2(u)$ is a master symmetry, i.e., it acts as a flow generator via the following relations:*

$$K_{s+1} = \frac{1}{s}[\![K_s, \sigma_2]\!], \tag{4.14a}$$

$$\sigma_{s+1} = \frac{1}{s-2}[\![\sigma_s, \sigma_2]\!] \quad s \neq 2. \tag{4.14b}$$

From the isospectral flows K_m and the nonisospectral flows σ_n, we further define the function τ_0^m and τ_n^m as

$$\tau_0^m = mtK_m + \sigma_1, \tag{4.15a}$$

$$\tau_n^m = \Phi^n\tau_0^m = mtK_{m+n} + \sigma_{n+1}. \tag{4.15b}$$

Theorem 2

$$(\tau_n^m)_t = K_m'[\tau_n^m], \quad m = 0, 1, 2, \ldots; n = 1, 2, \ldots, \tag{4.16}$$

i.e., τ_n^m are sets of symmetries of Eq. (3.5a).

Proof Since Φ maps symmetries of Eq. (3.5a) into symmetries of Eq. (3.5a), it is sufficient to prove

$$(\tau_0^m)_t = K_m'[\tau_0^m]. \tag{4.17}$$

By Eq. (4.15a), we have

$$\begin{aligned}
(\tau_0^m)_t &= mK_m + mtK'[u_t] + \sigma_1'[u_t] \\
&= mK_m + mtK'[K_m] + \sigma_1'[K_m] \\
&= mtK'[K_m] + K_m'[\sigma_1] \\
&= K_m'[\tau_0^m],
\end{aligned} \tag{4.18}$$

and so we conclude that Eq. (4.16) is true. □

Theorem 3 *Every equation in the hierarchy of Eq.* (3.5a) *has two sets of symmetries:* K_m *and* τ_m^n, $m \geq 0$. *They constitute an infinite-dimensional Lie algebra with the commutator relation:*

$$[\![K_m, K_n]\!] = 0, \tag{4.19a}$$
$$[\![K_m, \tau_n^l]\!] = mK_{m+n-1}, \tag{4.19b}$$
$$[\![\tau_l^m, \tau_n^m]\!] = (l-n)\tau_{l+n-1}^m. \tag{4.19c}$$

Proof Here we only prove Eq. (4.19c). Using Eqs. (4.15b) and (4.7), we obtain

$$\begin{aligned}
[\![\tau_l^m, \tau_n^m]\!] &= [\![mtK_{m+l} + \sigma_{l+1}, mtK_{m+n} + \sigma_{n+1}]\!] \\
&= mt\{[\![K_{m+l}, \sigma_{n+1}]\!] + [\![\sigma_{l+1}, K_{m+n}]\!]\} + [\![\sigma_{l+1}, \sigma_{n+1}]\!] \\
&= mt\{(m+l)K_{m+n+l} - (m+n)K_{m+n+l}\} + (l-n)\sigma_{n+l+1} \\
&= (l-n)(mtK_{m+n+l} + \sigma_{n+l+1}) \\
&= (l-n)\tau_{n+l}^m.
\end{aligned}$$

So we accomplish the proof of Theorem 3. □

5 Conclusions

In general, we researched the integrability of the GI soliton hierarchies in this paper. From its Lax pairs, we deduced the isospectral and nonisospectral hierarchies associated with the GI equation. Then we constructed the conservation laws for the obtained isospectral hierarchy. Those conservation laws were generated from taking a Laurent expansion of a ratio of eigenfunctions, which satisfies a Riccati equation, and the first few conservation laws in the series were explicitly presented for the GI equation. From the isospectral and nonisospectral hierarchies, we constructed two different types of symmetries, which are so-called K-symmetries and τ-symmetries, and proved that those two types of symmetries constitute a Lie algebra. The recursion operator Φ of the GI hierarchy was proved to be a hereditary and strong symmetry of the GI hierarchy.

There have been active studies on lumps and their interaction solutions with solitons [37–39]. It would be very interesting to generalize the presented GI equations, both isospectral and nonisospectral, to (2+1)-dimensional equations and consider their lumps and interaction solutions. This will be one of our future projects.

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