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A General Method for the Ulam Stability of Linear Differential Equations

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Abstract
This paper deals with the Ulam stability of linear differential equations by using the method of variation of parameters, which provides a unified method to study the Ulam stability problem of linear differential equations of n-th order with constant and nonconstant coefficients. As an application of the main results, we also obtain the Hyers–Ulam stability of the Cauchy–Euler differential equations of second order, third order and n-th order. Our results make up to some deficiencies in the relevant literature.

Keywords Ulam stability · Linear differential equations · Cauchy–Euler differential equation · Variation of parameters

Mathematics Subject Classification 34D20 · 39B82 · 34D10 · 34G10

1 Introduction

The Ulam stability problem originated from an important talk at the University of Wisconsin organized by Ulam [34]. Such stability problem is first concerned with the stability of group homomorphisms. The essential problem of this type of stability is summarized as follows: "under what conditions can a solution of a perturbed equation close to a solution of the original equation?" Hyers [8] seems to be the first person to study this kind of stability. Specifically, he gave a first affirmative partial answer to the question proposed by Ulam for Banach spaces. Afterward, this work was generalized

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by Rassias [27] for linear mappings by considering an unbounded Cauchy difference. It is worth mentioning that Rassias' work has a great impact on the development of the Ulam stability of functional equations. In the following decades, almost all of the studies associated with the Ulam stability were focused on different types of functional equations and abstract spaces. For more details, the reader can refer to several monographs [6,9,16,29] and the references therein.

Obloza [23] first initiated the study of the Ulam stability of differential equations. A few years later, Alsina and Ger [2] established the Hyers–Ulam stability of the differential equation $y'(x) = y(x)$. That is to say, for a given $\varepsilon > 0$, if f is a differentiable function from an open interval I into $\mathbb{R}$ with $|f'(x) - f(x)| \leq \varepsilon$ for all $x \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ such that $g'(x) = g(x)$ which satisfies $|f(x) - g(x)| \leq 3\varepsilon$ for all $x \in I$. Later, Miura and Takahasi et al. [18, 20,32] further studied the Ulam stability of the differential equation $y'(x) = \lambda y(x)$ in various abstract spaces. By using the same method as in [2], Jung [10] proved the Hyers–Ulam stability of the differential equation $\varphi(x)y'(x) = y(x)$. Furthermore, the Hyers–Ulam stability of the first-order linear differential equation $y'(x) + p(x)y(x) + q(x) = 0$ was extensively studied by Miura et al. [19], Takahasi et al. [33] and Jung [11, 12]. Meantime, Jung [13] also proved the Hyers–Ulam stability of a system of first-order linear differential equations with constant coefficients by the matrix method. Li and Shen [17] established the Hyers–Ulam stability of second order linear differential equations with constant coefficients by the reduced order method. However, this result requires the condition that the associated characteristic equation contains two different positive roots. Using the similar technique, the Ulam stability of linear differential equations with constant coefficients was considered by Cîmpean and Popa [5], Popa and Raşa [25]. Analogously, the Hyers–Ulam stability of third-order linear differential equations with constant coefficients was studied by Abdollahpour and Najati [1]. More generally, Brzdek and Jung [3] investigated the Hyers–Ulam stability for an operator linear equation of second order. Further, Xu et al. [36] proved the Hyers–Ulam stability of linear operator equations of higher order by using the fixed point method. Recently, Brzdek et al. [4] studied the Hyers–Ulam stability of linear operators and hence they also obtained the stability results of linear differential equation of n-th order with constant coefficients. Popa and Raşa [26] investigated the Hyers–Ulam stability of linear differential equations with nonconstant coefficients by the decomposition of the differential operator of n-th order. At the same time, Miura et al. [21] proved the Hyers–Ulam stability of the Chebyshev differential equation $(1 - x^2)y''(x) - xy'(x) + n^2y(x) = 0$ by using the analytical method. Inspired by this work, Shen et al. [31] further considered the Ulam stability of a class of linear differential equations $p(x)y''(x) + q(x)y'(x) + \lambda y(x) = 0$, $\lambda > 0$ when the coefficient functions satisfy certain conditions. This work can be regarded as a generalization of Miura et al. [21]. In recent years, some classical methods for solving differential equations were also applied to study the Ulam stability problem of differential equations, such as the integrating factor method [30,35], the power series method [14] and the Laplace transform method [28] and so on. It should be pointed out that the fixed point technique had been used to study the Ulam stability of the general differential equation $y' = F(x, y)$ [15]. Huang and Li [7] showed the Hyers–Ulam stability of linear functional equations by using various methods, including direct method, iteration method, fixed point method and open mapping theorem. Meanwhile, Mortici et al. [22] solved the inhomogeneous Euler equation of second order and proved its Hyers–Ulam stability on a bounded domain by using the integration method. Soon after, Popa and Pugna [24] obtained the Ulam stability of the inhomogeneous Euler's differential equation of n-th order. This result extends and improves the work of Mortici et al. [22].

Based on the preceding statements, we can find that there is little literature, except for Ref. [26], to uniformly study the Ulam stability of linear differential equations with constant and nonconstant coefficients. The aim of this paper is to provide a unified method, i.e., the variation of parameters, to establish the Ulam stability of linear differential equations.

Let $\mathbb{R}$ denote the set of all real numbers. Unless otherwise stated, the symbol I denotes a subinterval in $\mathbb{R}$. Let $C(I, \mathbb{R})$ and $C^n(I, \mathbb{R})$ denote the set of all real-valued continuous functions and the set of all differentiable functions which have continuous derivatives up to n.

2 The General Solution and Ulam Stability of First-Order Linear Differential Equations

In this section, we shall establish the Ulam stability of first order linear differential equations by the method of variation of parameters.

The nonhomogeneous linear differential equation of first-order is given as follows:

$$y'(x) + p(x)y(x) = f(x), \tag{1}$$

where $x \in I$, $p, f \in C(I, \mathbb{R})$. The associated homogeneous linear differential equation of (1) is

$$y'(x) + p(x)y(x) = 0. \tag{2}$$

Suppose that $y_1(x)$ is a basic solution of (2). It is well known that $Y(x) = \widetilde{c}y_1(x)$ is the general solution of (2), where $\widetilde{c}$ is an arbitrary constant. Using the method of variation of parameters, we can obtain a particular solution of (1) by replacing the constant $\widetilde{c}$ in $Y(x)$ by an undetermined function $u(x)$. Setting $y_p(x) = u(x)y_1(x)$. Now, we assume that $y_p(x)$ is a solution of (1). Substituting y_p into (1), we then obtain

$$u(x)\left(\frac{\mathrm{d}y_1}{\mathrm{d}x} + p(x)y_1(x)\right) + y_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} = f(x). \tag{3}$$

Notice that $y_1(x)$ is a solution of (2), it follows from (3) that

$$y_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} = f(x). \tag{4}$$

By separating the variable and integrating both sides of (4) from x_0 to x with respect to t, we get

$$u(x) = c_1 + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t,$$

where $c_1 \in \mathbb{R}$ is an arbitrary constant. Therefore, we have

$$y_p(x) = u(x)y_1(x) = \left(c_1 + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t\right) y_1(x).$$

Accordingly, we can present the following result.

Theorem 2.1 *Let $p, f \in C(I, \mathbb{R})$ and let $y_1(x)$ be a basic solution of (2). Then, the general solution $y(x)$ of (1) is given by*

$$\begin{aligned} y(x) &= Y(x) + y_p(x) \\ &= \widetilde{c}y_1(x) + \left(c_1 + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t\right) y_1(x) \\ &= cy_1(x) + y_1(x) \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t, \end{aligned}$$

where $c = \widetilde{c} + c_1$ is still an arbitrary constant.

Based on Theorem 2.1, we now prove the Ulam stability of nonhomogeneous linear differential Eq. (1).

Theorem 2.2 *Let $p, f \in C(I, \mathbb{R})$ and let $y_1(x)$ be a basic solution of (2). Assume that $\varphi : I \to [0, +\infty)$ is a continuous function. If $y_\varphi \in C^1(I, \mathbb{R})$ satisfies the differential inequality*

$$\left|y_\varphi'(x) + p(x)y_\varphi(x) - f(x)\right| \le \varphi(x) \tag{5}$$

for all $x \in I$, then there exists $y \in C^1(I, \mathbb{R})$ such that $y(x)$ satisfies (1) and

$$\left|y_\varphi(x) - y(x)\right| \le |y_1(x)| \left|\int_{x_0}^{x} \frac{\varphi(t)}{|y_1(t)|}\mathrm{d}t\right|$$

for all $x \in I$.

Proof For convenience, we write

$$y_\varphi'(x) + p(x)y_\varphi(x) := f_\varphi(x). \tag{6}$$

Then, it follows from (5) that

$$|f_\varphi(x) - f(x)| \le \varphi(x) \tag{7}$$

for all $x \in I$. According to Theorem 2.1 and (6), there exists a constant c such that

$$y_\varphi(x) = cy_1(x) + y_1(x)\int_{x_0}^{x} \frac{f_\varphi(t)}{y_1(t)}\mathrm{d}t, \tag{8}$$

where $x_0 \in I$ is an arbitrary fixed point.

Define

$$y(x) = cy_1(x) + y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t \tag{9}$$

for all $x \in I$. By Theorem 2.1, it is easy to verify that $y(x)$ is a solution of (1). From (7), (8) and (9), we can obtain

$$\begin{aligned} |y_\varphi(x) - y(x)| &= \left| y_1(x)\int_{x_0}^{x} \frac{f_\varphi(t)}{y_1(t)}\mathrm{d}t - y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\mathrm{d}t \right| \\ &= |y_1(x)| \left| \int_{x_0}^{x} \frac{f_\varphi(t) - f(t)}{y_1(t)}\mathrm{d}t \right| \\ &\le |y_1(x)| \left| \int_{x_0}^{x} \frac{\varphi(t)}{|y_1(t)|}\mathrm{d}t \right| \end{aligned} \tag{10}$$

for all $x \in I$. The proof of the theorem is now completed. □

Remark 1 In fact, we all know that $y_1(x) = \mathrm{e}^{-\int_{x_0}^{x} p(t)\mathrm{d}t}$ $(x_0 \in I)$ is a basic solution of the homogeneous linear differential Eq. (2). Then, the general solution of (2) is given by $Y(x) = c\mathrm{e}^{-\int_{x_0}^{x} p(t)\mathrm{d}t}$. Therefore, the inequality (10) can be rewritten as

$$|y_\varphi(x) - y(x)| \le \mathrm{e}^{-\int_{x_0}^{x} p(t)\mathrm{d}t} \left| \int_{x_0}^{x} \varphi(t)\mathrm{e}^{\int_{x_0}^{t} p(\tau)\mathrm{d}\tau}\mathrm{d}t \right|.$$

As a particular case of Theorem 2.2, we can obtain the Hyers–Ulam stability of the first-order nonhomogeneous linear differential Eq. (1) when the interval $I = [a, b]$ is finite.

Corollary 2.3 *Let $p, f \in C([a, b], \mathbb{R})$. For a given $\varepsilon > 0$, if a continuously differentiable function $y_\varepsilon : [a, b] \to \mathbb{R}$ satisfies the following inequality*

$$\left| y'_\varphi(x) + p(x)y_\varphi(x) - f(x) \right| \le \varepsilon$$

for all $x \in [a, b]$, then there exists a solution $y : [a, b] \to \mathbb{R}$ of (1) such that

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$, where

$$K = \mathrm{e}^{\int_a^b |p(x)|\mathrm{d}x}(b - a).$$

3 The General Solution and Ulam Stability of Second-Order Linear Differential Equations

In this section, we shall consider the general solution and Ulam stability of second-order linear differential equations by the method of variation of parameters.

The second-order nonhomogeneous linear differential equation is given as follows:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \tag{11}$$

where $x \in I$, $p, q, f \in C(I, \mathbb{R})$. The corresponding homogeneous linear differential equation of (11) is given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{12}$$

Let y_1 and y_2 be fundamental system of solutions of (12). The general solution of (12) is $Y(x) = \widetilde{c}_1 y_1(x) + \widetilde{c}_2 y_2(x)$. Replacing $\widetilde{c}_1$ and $\widetilde{c}_2$ in $Y(x)$ by two undetermined functions $u_1(x)$ and $u_2(x)$, respectively. Setting $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. Now, we assume that $y_p(x)$ is a solution of (11). Then, we have

$$\begin{aligned} y_p' &= u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2, \\ y_p'' &= u_1 y_1'' + 2u_1' y_1' + u_1'' y_1 + u_2 y_2'' + 2u_2' y_2' + u_2'' y_2. \end{aligned} \tag{13}$$

Substituting (13) into (11), and combining the same items, we can obtain

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= \left(y_1'' + p(x)y_1' + q(x)y_1\right) u_1 + \left(y_2'' + p(x)y_2'\right. \\ &\quad \left. + q(x)y_2\right) u_2 + y_1 u_1'' + y_1' u_1' + y_2 u_2'' + y_2' u_2' \\ &\quad + p(x)\left(y_1 u_1' + y_2 u_2'\right) + y_1' u_1' + y_2' u_2' \\ &= \left(y_1 u_1'\right)' + \left(y_2 u_2'\right)' + p(x)\left(y_1 u_1' + y_2 u_2'\right) + y_1' u_1' + y_2' u_2' \\ &= \left(y_1 u_1' + y_2 u_2'\right)' + p(x)\left(y_1 u_1' + y_2 u_2'\right) + y_1' u_1' + y_2' u_2' \\ &= f(x), \end{aligned} \tag{14}$$

where for the second equal sign we have used the fact y_1 and y_2 are two solutions of (12). Note that y_p is merely a particular solution. Without loss of generality, we can assume that

$$y_1 u_1' + y_2 u_2' = 0, \tag{15}$$

and then we can obtain from (14) that

$$y_1' u_1' + y_2' u_2' = f(x). \tag{16}$$

Since y_1 and y_2 are fundamental system of solutions of (12), it follows that the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

for all $x \in I$. By using Cramer's rule, we can get from (15) and (16) that

$$u_1'(x) = \frac{W_{1,2}(x)}{W(y_1, y_2)(x)}, \quad u_2'(x) = \frac{W_{2,2}(x)}{W(y_1, y_2)(x)}, \tag{17}$$

where

$$W_{1,2}(x) = \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix}, \quad W_{2,2}(x) = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix}.$$

By integrating both sides of (17) from x_0 to x with respect to t, it follows that

$$u_1(x) = \widehat{c}_1 + \int_{x_0}^{x} \frac{W_{1,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t,$$
$$u_2(x) = \widehat{c}_2 + \int_{x_0}^{x} \frac{W_{2,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t,$$

where $\widehat{c}_1$ and $\widehat{c}_2$ are two arbitrary constants. Then, we have

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) = \widehat{c}_1 y_1(x) + \widehat{c}_2 y_2(x) \\ &\quad + y_1(x) \int_{x_0}^{x} \frac{W_{1,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t + y_2(x) \int_{x_0}^{x} \frac{W_{2,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t. \end{aligned}$$

Furthermore, we can obtain the general solution of (11).

Theorem 3.1 *Let $p, q, f \in C(I, \mathbb{R})$ and let $y_1(x)$ and $y_2(x)$ be fundamental system of solutions of (12). Then, the general solution $y(x)$ of (11) can be given by*

$$\begin{aligned} y(x) &= Y(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_1(x) \int_{x_0}^{x} \frac{W_{1,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t \\ &\quad + y_2(x) \int_{x_0}^{x} \frac{W_{2,2}(t)}{W(y_1, y_2)(t)} \mathrm{d}t, \end{aligned}$$

where c_1 and c_2 are two arbitrary constants, $x_0 \in I$ is an arbitrary fixed point.

According to Theorem 3.1, we can obtain the Ulam stability of the second-order nonhomogeneous linear differential Eq. (11).

Theorem 3.2 *Let $p, q, f \in C(I, \mathbb{R})$ and let $y_1(x)$ and $y_2(x)$ be fundamental system of solutions of (12). Assume that $\varphi : I \to [0, +\infty)$ is a continuous function. If $y_\varphi \in C^2(I, \mathbb{R})$ satisfies the differential inequality*

$$\left| y_\varphi''(x) + p(x)y_\varphi'(x) + q(x)y_\varphi(x) - f(x) \right| \le \varphi(x) \tag{18}$$

for all $x \in I$, then there exists $y \in C^2(I, \mathbb{R})$ such that $y(x)$ satisfies (11) and

$$\left| y_\varphi(x) - y(x) \right| \le \left| \int_{x_0}^{x} \left| \frac{W(y_1, y_2)(t, x)}{W(y_1, y_2)(t)} \right| \varphi(t) \mathrm{d}t \right|$$

for all $x \in I$, *where*

$$W(y_1, y_2)(t, x) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}.$$

Proof Here, we set

$$y_\varphi''(x) + p(x)y_\varphi'(x) + q(x)y_\varphi(x) := f_\varphi(x). \tag{19}$$

From (18), it follows that the inequality (11) holds. By Theorem 3.1 and (19), there exist two constants c_1 and c_2 such that

$$y_\varphi(x) = c_1 y_1(x) + c_2 y_2(x) + y_1(x) \int_{x_0}^{x} \frac{-y_2(t) f_\varphi(t)}{W(y_1, y_2)(t)} \mathrm{d}t + y_2(x) \int_{x_0}^{x} \frac{y_1(t) f_\varphi(t)}{W(y_1, y_2)(t)} \mathrm{d}t, \tag{20}$$

where $x_0 \in I$ is an arbitrary fixed point.

Define

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_1(x) \int_{x_0}^{x} \frac{-y_2(t) f(t)}{W(y_1, y_2)(t)} \mathrm{d}t + y_2(x) \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(y_1, y_2)(t)} \mathrm{d}t \tag{21}$$

for all $x \in I$. By Theorem 3.1, we know that $y(x)$ is a solution of (12). In view of (11), (20) and (21), we can infer that

$$\begin{aligned} &\left|y_\varphi(x) - y(x)\right| \\ &= \left| y_1(x) \int_{x_0}^{x} \frac{-y_2(t)\left(f_\varphi(t) - f(t)\right)}{W(y_1, y_2)(t)} \mathrm{d}t + y_2(x) \int_{x_0}^{x} \frac{y_1(t)\left(f_\varphi(t) - f(t)\right)}{W(y_1, y_2)(t)} \mathrm{d}t \right| \\ &= \left| \int_{x_0}^{x} \frac{-y_1(x) y_2(t)\left(f_\varphi(t) - f(t)\right)}{W(y_1, y_2)(t)} \mathrm{d}t + \int_{x_0}^{x} \frac{y_2(x) y_1(t)\left(f_\varphi(t) - f(t)\right)}{W(y_1, y_2)(t)} \mathrm{d}t \right| \\ &\le \left| \int_{x_0}^{x} \left| \frac{W(y_1, y_2)(t, x)}{W(y_1, y_2)(t)} \right| |\varphi(t) \mathrm{d}t| \right|, \end{aligned}$$

for all $x \in I$. This completes the proof. □

Remark 2 Let $I = [a, b]$ be a finite closed interval and let ε be an arbitrary positive number. If we take $\varphi(t) \equiv \varepsilon$, then we can obtain the Hyers–Ulam stability of (11).

Corollary 3.3 *Let* $p, q, f \in C([a, b], \mathbb{R})$. *For a given* $\varepsilon > 0$, *if a twice continuously differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality*

$$\left|y_\varphi''(x) + p(x)y_\varphi'(x) + q(x)y_\varphi(x) - f(x)\right| \le \varepsilon$$

for all $x \in [a, b]$, *then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (11) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$*, where*

$$K = \max_{x \in [a,b]} \int_a^x \left| \frac{W(y_1, y_2)(t, x)}{W(y_1, y_2)(t)} \right| \mathrm{d}t.$$

As a particular case, we can establish the Ulam stability of second order nonhomogeneous linear differential equations with constant coefficients. Specifically, the differential equation is given as follows:

$$y''(x) + \gamma y'(x) + \delta y(x) = f(x), \tag{22}$$

where γ, δ are real constants, $f \in C(I, \mathbb{R})$. The associated characteristic equation is

$$\lambda^2 + \gamma\lambda + \delta = 0. \tag{23}$$

Corollary 3.4 *Let* $\varphi : I \to [0, +\infty)$ *be a continuous function. Assume that the characteristic Eq. (23) has two distinct real roots* λ_1 *and* λ_2*. If* $y_\varphi \in C^2(I, \mathbb{R})$ *satisfies the differential inequality*

$$\left|y''_\varphi(x) + \gamma y'_\varphi(x) + \delta y_\varphi(x) - f(x)\right| \le \varphi(x) \tag{24}$$

for all $x \in I$*, then there exists a solution* $y \in C^2(I, \mathbb{R})$ *of (22) such that*

$$\left|y_\varphi(x) - y(x)\right| \le \frac{1}{|\lambda_1 - \lambda_2|} \left| \int_{x_0}^x \left| \mathrm{e}^{\lambda_2(x-t)} - \mathrm{e}^{\lambda_1(x-t)} \right| \varphi(t) \mathrm{d}t \right|$$

for all $x \in I$*.*

Proof Notice that $y_1(x) = \mathrm{e}^{\lambda_1 x}$ and $y_2(x) = \mathrm{e}^{\lambda_2 x}$ are fundamental system of solutions of the associated homogeneous linear differential equation of (22). The desired result can be obtained from Theorem 3.2. □

Corollary 3.5 *Let* $\varphi : I \to [0, +\infty)$ *be a continuous function. Assume that the characteristic Eq. (23) has two real roots* λ_1 *and* λ_2 *with* $\lambda_1 = \lambda_2$*. If* $y_\varphi \in C^2(I, \mathbb{R})$ *satisfies the differential inequality (24) for all* $x \in I$*, then there exists a solution* $y \in C^2(I, \mathbb{R})$ *of (22) such that*

$$\left|y_\varphi(x) - y(x)\right| \le \left| \int_{x_0}^x \left|(x - t)\mathrm{e}^{\lambda_1(x-t)}\right| \varphi(t) \mathrm{d}t \right|$$

for all $x \in I$*.*

Proof It is well known that $y_1(x) = \mathrm{e}^{\lambda_1 x}$ and $y_2(x) = x\mathrm{e}^{\lambda_1 x}$ are fundamental system of solutions of the associated homogeneous linear differential equation of (22). Similarly, the desired result can be proved by Theorem 3.2. □

Corollary 3.6 *Let $\varphi : I \to [0, +\infty)$ be a continuous function. Assume that the characteristic Eq. (23) has a pair of complex conjugate roots λ_1 and λ_2 with $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, where $\alpha \in \mathbb{R}$, $\beta > 0$. If $y_\varphi \in C^2(I, \mathbb{R})$ satisfies the differential inequality (24) for all $x \in I$, then there exists a solution $y \in C^2(I, \mathbb{R})$ of (22) such that*

$$|y_\varphi(x) - y(x)| \leq \beta \left| \int_{x_0}^{x} \left| e^{\alpha(x-t)} \sin[\beta(x-t)] \right| \varphi(t) dt \right|$$

for all $x \in I$.

Proof In view of the theory of ordinary differential equations, we know that $y_1(x) = e^{\alpha x} \cos \beta x$ and $y_2(x) = e^{\alpha x} \sin \beta x$ are fundamental system of solutions of the associated homogeneous linear differential equation of (22). By Theorem 3.2, the desired result can be derived. □

In addition, as an application of Theorem 3.2, we can show the Hyers–Ulam stability of the Cauchy–Euler differential equation of second order. The general form of the nonhomogeneous Cauchy–Euler differential equation of second order is given by

$$x^2 y''(x) + \mu x y'(x) + \nu y(x) = f(x), \tag{25}$$

where $\mu, \nu \in \mathbb{R}$, $f \in C([a, b], \mathbb{R})$, $0 < a < b$. The auxiliary equation is

$$\lambda^2 - (\mu - 1)\lambda + \nu = 0. \tag{26}$$

The following stability results can be obtained depending on the different cases of the auxiliary Eq. (26).

Corollary 3.7 *Let $f \in C([a, b], \mathbb{R})$ and let $\mu, \nu \in \mathbb{R}$ with $(\mu - 1)^2 - 4\nu > 0$, i.e., the auxiliary Eq. (25) has two distinct real roots λ_1 and λ_2. For a given $\varepsilon > 0$, if a twice continuously differentiable function $y_\varepsilon : [a, b] \to \mathbb{R}$ satisfies the following inequality*

$$\left| x^2 y_\varphi''(x) + \mu x y_\varphi'(x) + \nu y_\varphi(x) - f(x) \right| \leq \varepsilon \tag{27}$$

for all $x \in [a, b]$, then there exists a solution $y : [a, b] \to \mathbb{R}$ of (25) such that

$$|y_\varepsilon(x) - y(x)| \leq K\varepsilon$$

for all $x \in [a, b]$, where

$$K = \frac{1}{|\lambda_1 - \lambda_2|} \max_{x \in [a,b]} \int_a^x \frac{1}{t} \left| \left(\frac{x}{t}\right)^{\lambda_1} - \left(\frac{x}{t}\right)^{\lambda_2} \right| dt.$$

Proof Notice that the inequality (27) is equivalent to

$$\left| y_\varphi''(x) + \frac{\mu}{x} y_\varphi'(x) + \frac{\nu}{x^2} y_\varphi(x) - \frac{f(x)}{x^2} \right| \leq \frac{\varepsilon}{x^2}, \tag{28}$$

since $x \in [a, b]$ and $0 < a < b$. By Theorem 3.2, there exists $y \in C^2([a, b], \mathbb{R})$ such that

$$y''(x) + \frac{\mu}{x} y'(x) + \frac{\nu}{x^2} y(x) = \frac{f(x)}{x^2} \tag{29}$$

for all $x \in [a, b]$. Clearly, the Eqs. (25) and (29) are equivalent. So y is also solution of (25).

Moreover, since $\lambda_1 \neq \lambda_2$, we know that $y_1(x) = x^{\lambda_1}$ and $y_2(x) = x^{\lambda_2}$ are fundamental system of solutions of the associated homogeneous linear differential equation of (25). Then, it follows from Theorem 3.2 and (28) that the desired inequality holds. □

Corollary 3.8 *Let $f \in C([a, b], \mathbb{R})$ and let $\mu, \nu \in \mathbb{R}$ with $(\mu - 1)^2 - 4\nu = 0$, i.e., the auxiliary Eq. (25) has two equal real roots $\lambda_1 = \lambda_2$. For a given $\varepsilon > 0$, if a twice continuously differentiable function $y_\varepsilon : [a, b] \to \mathbb{R}$ satisfies the following inequality (27) for all $x \in [a, b]$, then there exists a solution $y : [a, b] \to \mathbb{R}$ of (25) such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$, where

$$K = \max_{x\in[a,b]} \int_a^x \frac{1}{t} \left| \left(\frac{x}{t}\right)^{\lambda_1} \ln\left(\frac{x}{t}\right) \right| \mathrm{d}t.$$

Proof In this case, $y_1(x) = x^{\lambda_1}$ and $y_2(x) = x^{\lambda_1} \ln x$ are fundamental system of solutions of the associated homogeneous linear differential equation of (25). The proof is similar to Corollary 3.7 and so is omitted. □

Corollary 3.9 *Let $f \in C([a, b], \mathbb{R})$ and let $\mu, \nu \in \mathbb{R}$ with $(\mu - 1)^2 - 4\nu < 0$, i.e., the auxiliary Eq. (25) has a pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where $\alpha \in \mathbb{R}$, $\beta > 0$. For a given $\varepsilon > 0$, if a twice continuously differentiable function $y_\varepsilon : [a, b] \to \mathbb{R}$ satisfies the following inequality (27) for all $x \in [a, b]$, then there exists a solution $y : [a, b] \to \mathbb{R}$ of (25) such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$, where

$$K = \frac{1}{\beta} \max_{x\in[a,b]} \int_a^x \frac{1}{t} \left| \left(\frac{x}{t}\right)^{\alpha} \sin\left[\beta\left(\frac{x}{t}\right)\right] \right| \mathrm{d}t.$$

Proof In this case, $y_1(x) = x^\alpha \cos(\beta \ln x)$ and $y_2(x) = x^\alpha \sin(\beta \ln x)$ are fundamental system of solutions of the associated homogeneous linear differential equation of (25). The proof is similar to Corollary 3.7 and so is omitted. □

Remark 3 In [22], the authors proved the Hyers–Ulam stability of the Eq. (25) when the auxiliary Eq. (26) has two distinct real roots and two equal real roots, respectively. But this case is not considered if the auxiliary Eq. (26) has a pair of complex conjugate roots. So Corollary 3.9 can be regarded as a supplement to [22].

4 The General Solution and Ulam Stability of Third-Order Linear Differential Equations

Using the similar argument as in Sect. 3, this section will deal with the general solution and Ulam stability of three-order linear differential equations.

The general form of the three-order nonhomogeneous linear differential equation is given by

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x), \tag{30}$$

where $p, q, r, f \in C(I, \mathbb{R})$. The associated homogenous linear differential equation of (30) is

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0. \tag{31}$$

Let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be fundamental system of solutions of (31). Then, the general solution of (31) is $Y(x) = \widetilde{c}_1 y_1(x) + \widetilde{c}_2 y_2(x) + \widetilde{c}_3 y_3(x)$. By using the method of variation of parameters, we set $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x)$, where $u_1(x), u_2(x)$ and $u_3(x)$ are three undetermined functions. In order to solve these functions, we assume that $y_p(x)$ is a solution of (30). Then, we get

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2 + u_3 y_3' + u_3' y_3, \tag{32}$$

$$y_p'' = u_1 y_1'' + 2u_1' y_1' + u_1'' y_1 + u_2 y_2'' + 2u_2' y_2' + u_2'' y_2 + u_3 y_3'' + 2u_3' y_3' + u_3'' y_3, \tag{33}$$

$$\begin{aligned} y_p''' &= u_1 y_1''' + 3u_1' y_1'' + 3u_1'' y_1' + u_1''' y_1 + u_2 y_2''' + 3u_2' y_2'' + 3u_2'' y_2' + u_2''' y_2 \\ &\quad + u_3 y_3''' + 3u_3' y_3'' + 3u_3'' y_3' + u_3''' y_3. \end{aligned} \tag{34}$$

Substituting these equalities (32–34) into (30), we can infer that

$$\begin{aligned} & y_p''' + p(x)y_p'' + q(x)y_p' + r(x)y_p \\ &= \left(y_1''' + p(x)y_1'' + q(x)y_1' + r(x)y_1\right) u_1 + \left(y_2''' + p(x)y_2'' + q(x)y_2' + r(x)y_2\right) u_2 \\ &\quad + \left(y_3''' + p(x)y_3'' + q(x)y_3' + r(x)y_3\right) u_3 + p(x) \left(u_1'' y_1 + u_2'' y_2 + u_3'' y_3 \right. \\ &\quad \left. + 2u_1' y_1' + 2u_2' y_2' + 2u_3' y_3'\right) + q(x) \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right) \\ &\quad + u_1''' y_1 + 3u_1'' y_1' + 3u_1' y_1'' + u_2''' y_2 + 3u_2'' y_2' + 3u_2' y_2'' \\ &\quad + u_3''' y_1 + 3u_3'' y_3' + 3u_3' y_3'' \\ &= p(x) \left(u_1'' y_1 + u_2'' y_2 + u_3'' y_3 + 2u_1' y_1' + 2u_2' y_2' + 2u_3' y_3'\right) + q(x) \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right) \\ &\quad + u_1''' y_1 + 3u_1'' y_1' + 3u_1' y_1'' + u_2''' y_2 + 3u_2'' y_2' + 3u_2' y_2'' + u_3''' y_3 + 3u_3'' y_3' + 3u_3' y_3'' \\ &= \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right)'' + \left(u_1' y_1' + u_2' y_2' + u_3' y_3'\right)' + p(x) \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right)' \\ &\quad + p(x) \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right) + q(x) \left(u_1' y_1 + u_2' y_2 + u_3' y_3\right) + \left(u_1' y_1'' + u_2' y_2'' + u_3' y_3''\right) \\ &= f(x), \end{aligned} \tag{35}$$

where for the second equal sign we have used the fact y_1, y_2 and y_3 are three solutions of (31). By the foregoing hypothesis, y_p is only a particular solution of (30). For the sake of convenience, without loss of generality, we can assume that

$$\begin{cases} u_1' y_1 + u_2' y_2 + u_3' y_3 = 0, \\ u_1' y_1' + u_2' y_2' + u_3' y_3 = 0. \end{cases} \tag{36}$$

Then, it follows from (35) that

$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = f(x). \tag{37}$$

Since y_1, y_2 and y_3 are fundamental system of solutions of (31), the Wronskian

$$W(y_1, y_2, y_3)(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix} \neq 0$$

for all $x \in I$. Using Cramer's rule, it follows from (36) and (37) that

$$\begin{aligned} u_1'(x) &= \frac{W_{1,3}(x)}{W(y_1, y_2, y_3)(x)}, \\ u_2'(x) &= \frac{W_{2,3}(x)}{W(y_1, y_2, y_3)(x)}, \\ u_3'(x) &= \frac{W_{3,3}(x)}{W(y_1, y_2, y_3)(x)}, \end{aligned} \tag{38}$$

where

$$W_{1,3}(x) = \begin{vmatrix} 0 & y_2(x) & y_3(x) \\ 0 & y_2'(x) & y_3'(x) \\ f(x) & y_2''(x) & y_3''(x) \end{vmatrix},$$

$$W_{2,3}(x) = \begin{vmatrix} y_1(x) & 0 & y_3(x) \\ y_1'(x) & 0 & y_3'(x) \\ y_1''(x) & f(x) & y_3''(x) \end{vmatrix},$$

$$W_{3,3}(x) = \begin{vmatrix} y_1(x) & y_2(x) & 0 \\ y_1'(x) & y_2'(x) & 0 \\ y_1''(x) & y_2''(x) & f(x) \end{vmatrix}.$$

By integrating both sides of (38) from x_0 to x with respect to t, then we can obtain

$$\begin{aligned} u_1(x) &= \widehat{c}_1 + \int_{x_0}^{x} \frac{W_{1,3}(t)}{W(y_1, y_2, y_3)(t)} \mathrm{d}t, \\ u_2(x) &= \widehat{c}_2 + \int_{x_0}^{x} \frac{W_{2,3}(t)}{W(y_1, y_2, y_3)(t)} \mathrm{d}t, \\ u_3(x) &= \widehat{c}_3 + \int_{x_0}^{x} \frac{W_{3,3}(t)}{W(y_1, y_2, y_3)(t)} \mathrm{d}t, \end{aligned}$$

where $\widehat{c_1}$, $\widehat{c_2}$ and $\widehat{c_3}$ are three arbitrary constants. Thus, we get

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x) \\ &= \widehat{c_1}y_1(x) + \widehat{c_2}y_2(x) + \widehat{c_3}y_3(x) + y_1(x)\int_{x_0}^{x} \frac{W_{1,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t \\ &\quad + y_2(x)\int_{x_0}^{x} \frac{W_{2,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t + +y_3(x)\int_{x_0}^{x} \frac{W_{3,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t. \end{aligned}$$

Based on the above argument, we can obtain the following result.

Theorem 4.1 *Let $p, q, r, f \in C(I, \mathbb{R})$ and let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be fundamental system of solutions of* (31). *Then, the general solution $y(x)$ of* (30) *can be given by*

$$\begin{aligned} y(x) &= Y(x) + y_p(x) \\ &= c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + y_1(x)\int_{x_0}^{x} \frac{W_{1,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t \\ &\quad + y_2(x)\int_{x_0}^{x} \frac{W_{2,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t + y_3(x)\int_{x_0}^{x} \frac{W_{3,3}(t)}{W(y_1, y_2, y_3)(t)}\mathrm{d}t, \end{aligned}$$

where c_1, c_2 and c_3 are three arbitrary constants, $x_0 \in I$ is an arbitrary fixed point.

By Theorem 4.1, we shall discuss the Ulam stability of the three-order nonhomogeneous linear differential Eq. (30).

Theorem 4.2 *Let $p, q, r, f \in C(I, \mathbb{R})$ and let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be fundamental system of solutions of (31). Assume that $\varphi : I \to [0, +\infty)$ is a continuous function. If $y_\varphi \in C^3(I, \mathbb{R})$ satisfies the differential inequality*

$$\left| y_\varphi'''(x) + p(x)y_\varphi''(x) + q(x)y_\varphi'(x) + r(x)y_\varphi(x) - f(x) \right| \le \varphi(x)$$

for all $x \in I$, then there exists $y \in C^3(I, \mathbb{R})$ such that $y(x)$ satisfies (30) *and*

$$|y_\varphi(x) - y(x)| \le \left| \int_{x_0}^{x} \left| \frac{W(y_1, y_2, y_3)(t, t, x)}{W(y_1, y_2, y_3)(t)} \right| \varphi(t)\mathrm{d}t \right|$$

for all $x \in I$, where

$$W(y_1, y_2, y_3)(t, t, x) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}.$$

Proof Using the same argument as in the proof of Theorem 3.2, we can easily carry out the proof of this theorem. □

In particular, we can obtain the Hyers–Ulam stability of (30) when $I = [a, b]$ be a finite closed interval.

Corollary 4.3 *Let $p, q, r, f \in C([a,b],\mathbb{R})$. For a given $\varepsilon > 0$, if a three times continuously differentiable function $y_\varepsilon : [a,b] \to \mathbb{R}$ satisfies the following inequality*

$$\left|y_\varphi'''(x) + p(x)y_\varphi''(x) + q(x)y_\varphi'(x) + r(x)y_\varphi(x) - f(x)\right| \le \varepsilon$$

for all $x \in [a,b]$, then there exists a solution $y : [a,b] \to \mathbb{R}$ of (30) such that

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a,b]$, where

$$K = \max_{x\in[a,b]} \int_a^x \left|\frac{W(y_1, y_2, y_3)(t,t,x)}{W(y_1, y_2, y_3)(t)}\right| \mathrm{d}t.$$

According to Theorem 4.2, we can obtain the Ulam stability of third-order nonhomogeneous linear differential equations with constant coefficients. The general form of the differential equation is given by

$$y'''(x) + \gamma y''(x) + \delta y'(x) + \eta y(x) = f(x), \tag{39}$$

where γ, δ, η are real constants, $f \in C(I,\mathbb{R})$. The corresponding characteristic equation is

$$\lambda^3 + \gamma\lambda^2 + \delta\lambda + \eta = 0. \tag{40}$$

By distinguishing various roots of the characteristic equation, the following results can be obtained.

Corollary 4.4 *Let $\varphi : I \to [0,+\infty)$ be a continuous function. Assume that the characteristic Eq. (40) has three distinct real roots λ_1, λ_2 and λ_3. If $y_\varphi \in C^3(I,\mathbb{R})$ satisfies the differential inequality*

$$\left|y_\varphi'''(x) + \gamma y_\varphi''(x) + \delta y_\varphi'(x) + \eta y_\varphi'(x) - f(x)\right| \le \varphi(x) \tag{41}$$

for all $x \in I$, then there exists a solution $y \in C^3(I,\mathbb{R})$ of (39) such that

$$\begin{aligned}|y_\varphi(x) - y(x)| \le \frac{1}{|(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)|}\Bigg|\int_{x_0}^x \Bigg|&\lambda_1\left(\mathrm{e}^{\lambda_3(x-t)} - \mathrm{e}^{\lambda_2(x-t)}\right)\\ &+ \lambda_2\left(\mathrm{e}^{\lambda_1(x-t)} - \mathrm{e}^{\lambda_3(x-t)}\right) + \lambda_3\left(\mathrm{e}^{\lambda_2(x-t)} - \mathrm{e}^{\lambda_1(x-t)}\right)\Bigg|\varphi(t)\mathrm{d}t\Bigg|\end{aligned}$$

for all $x \in I$.

Proof The proof is quite similar to the proof of Corollary 3.3 and so is omitted. □

Corollary 4.5 *Let $\varphi : I \to [0,+\infty)$ be a continuous function. Assume that the characteristic Eq. (40) has three real roots λ_1, λ_2 and λ_3 with $\lambda_1 = \lambda_2 \neq \lambda_3$. If $y_\varphi \in C^3(I,\mathbb{R})$*

satisfies the differential inequality (41) *for all* $x \in I$, *then there exists a solution* $y \in C^3(I, \mathbb{R})$ *of* (39) *such that*

$$|y_\varphi(x) - y(x)| \leq \frac{1}{(\lambda_1 - \lambda_3)^2} \left| \int_{x_0}^{x} \left| [(\lambda_1 - \lambda_3)(x - t) - 1] \left(e^{\lambda_1 (x-t)} + e^{\lambda_3 (x-t)} \right) \right| \varphi(t) dt \right|$$

for all $x \in I$.

Corollary 4.6 *Let* $\varphi : I \to [0, +\infty)$ *be a continuous function. Assume that the characteristic Eq.* (40) *has three real roots* λ_1, λ_2 *and* λ_3 *with* $\lambda_1 = \lambda_2 = \lambda_3$. *If* $y_\varphi \in C^3(I, \mathbb{R})$ *satisfies the differential inequality* (41) *for all* $x \in I$, *then there exists a solution* $y \in C^3(I, \mathbb{R})$ *of* (39) *such that*

$$|y_\varphi(x) - y(x)| \leq \frac{1}{2} \left| \int_{x_0}^{x} (x - t)^2 e^{\lambda_1 (x-t)} \varphi(t) dt \right|$$

for all $x \in I$.

Corollary 4.7 *Let* $\varphi : I \to [0, +\infty)$ *be a continuous function. Assume that the characteristic Eq.* (40) *has a real root* λ_1 *and a pair of complex conjugate roots* $\lambda_2 = \alpha + i\beta$, $\lambda_3 = \alpha - i\beta$, *where* $\alpha \in \mathbb{R}$, $\beta > 0$. *If* $y_\varphi \in C^3(I, \mathbb{R})$ *satisfies the differential inequality* (41) *for all* $x \in I$, *then there exists a solution* $y \in C^3(I, \mathbb{R})$ *of* (39) *such that*

$$|y_\varphi(x) - y(x)| \leq \frac{1}{\beta[\beta^2 + (\alpha - \lambda_1)^2]} \left| \int_{x_0}^{x} \right| \beta e^{\lambda_1 (x-t)} - e^{\alpha (x-t)} (\beta \cos[\beta(x - t)] + (\alpha - \lambda) \sin[\beta(x - t)]) |\varphi(t) dt|$$

for all $x \in I$.

Moreover, we can infer from Theorem 4.2 that the Cauchy–Euler differential equation of third order is Hyers–Ulam stable. The general form of the nonhomogeneous Cauchy–Euler differential equation of third order is given by

$$x^3 y'''(x) + \mu x^2 y''(x) + \nu x y'(x) + \omega y(x) = f(x), \tag{42}$$

where $\mu, \nu, \omega \in \mathbb{R}$, $f \in C([a, b], \mathbb{R})$, $0 < a < b$. The auxiliary equation is

$$\lambda^3 + (\mu - 3)\lambda^2 + (\nu - \mu + 2)\lambda + \omega = 0. \tag{43}$$

The following stability results can be obtained depending on the different cases of the auxiliary Eq. (43).

Corollary 4.8 *Let* $f \in C([a, b], \mathbb{R})$ *and let* $\mu, \nu, \omega \in \mathbb{R}$. *Assume that the auxiliary Eq.* (43) *has three distinct real roots* λ_1,λ_2 *and* λ_3. *For a given* $\varepsilon > 0$, *if a three times continuously differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality*

$$\left| x^3 y_\varphi'''(x) + \mu x^2 y_\varphi''(x) + \nu x y_\varphi'(x) + \omega y_\varphi(x) - f(x) \right| \leq \varepsilon \tag{44}$$

for all $x \in [a, b]$*, then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (42) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$*, where*

$$K = \frac{1}{|(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)|} \max_{x \in [a,b]} \int_a^x \frac{1}{t} \left|(\lambda_1 - \lambda_2)\left(\frac{x}{t}\right)^{\lambda_3} + (\lambda_2 - \lambda_3)\left(\frac{x}{t}\right)^{\lambda_1} + (\lambda_3 - \lambda_1)\left(\frac{x}{t}\right)^{\lambda_2}\right| \mathrm{d}t.$$

Proof Using the same argument as in Corollary 3.7, the desired result can be derived by Theorem 4.2. □

Corollary 4.9 *Let* $f \in C([a, b], \mathbb{R})$ *and let* $\mu, \nu, \omega \in \mathbb{R}$*. Assume that the auxiliary Eq.* (43) *has a single root* λ_1 *and a double root* λ_2*. For a given* $\varepsilon > 0$*, if a three times continuously differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality* (44) *for all* $x \in [a, b]$*, then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (42) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$*, where*

$$K = \frac{1}{(\lambda_1 - \lambda_2)^2} \max_{x \in [a,b]} \int_a^x \frac{1}{t} \left|\left(\frac{x}{t}\right)^{\lambda_1} - \left(\frac{x}{t}\right)^{\lambda_2} + (\lambda_2 - \lambda_1)\left(\frac{x}{t}\right)^{\lambda_2} \ln\frac{x}{t}\right| \mathrm{d}t.$$

Corollary 4.10 *Let* $f \in C([a, b], \mathbb{R})$ *and let* $\mu, \nu, \omega \in \mathbb{R}$*. Assume that the auxiliary Eq.* (43) *has a triple root* λ_1*. For a given* $\varepsilon > 0$*, if a three times continuously differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality* (44) *for all* $x \in [a, b]$*, then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (42) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$*, where*

$$K = \frac{1}{2} \max_{x \in [a,b]} \int_a^x \frac{1}{t} \left(\frac{x}{t}\right)^{\lambda_1} \ln^2\frac{x}{t} \mathrm{d}t.$$

Corollary 4.11 *Let* $f \in C([a, b], \mathbb{R})$ *and let* $\mu, \nu, \omega \in \mathbb{R}$*. Assume that the auxiliary Eq.* (43) *has a real root* λ_1 *and a pair of complex conjugate roots* $\lambda_2 = \alpha + i\beta$*,* $\lambda_3 = \alpha - i\beta$*, where* $\alpha \in \mathbb{R}$*,* $\beta > 0$*. For a given* $\varepsilon > 0$*, if a three times continuously differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality* (44) *for all* $x \in [a, b]$*, then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (42) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$, *where*

$$K = \frac{1}{\beta[(\alpha-\lambda_1)^2+\beta^2]} \max_{x\in[a,b]} \int_a^x \frac{1}{t} \left| \beta \left(\frac{x}{t}\right)^{\lambda_1} - \left(\frac{x}{t}\right)^{\alpha} \left[\beta \cos\left(\beta \ln \frac{x}{t}\right) \right.\right.$$
$$\left.\left. + (\lambda_1 - \alpha) \sin\left(\beta \ln \frac{x}{t}\right)\right]\right| \mathrm{d}t.$$

5 The General Solution and Ulam Stability of *n*-th Order Linear Differential Equations

More generally, in this section, we will consider the Ulam stability of n-th order linear differential equations.

The general form of the n-th order nonhomogeneous linear differential equation is given by

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = f(x), \tag{45}$$

where $p_k, f \in C(I, \mathbb{R}), k = 0, 1, \ldots, n-1, n \geq 2$. The corresponding homogeneous linear differential equation of (45) is

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0. \tag{46}$$

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be fundamental system of solutions of (46). It is well known that the general solution of (45) is

$$Y(x) = \widetilde{c}_1 y_1(x) + \widetilde{c}_2 y_2(x) + \cdots + \widetilde{c}_n y_n(x),$$

where $\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n$ are arbitrary constants. Using the method of variation of parameters, we can replace $\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n$ by the undetermined functions $u_1(x), u_2(x), \ldots, u_n(x)$, respectively. Then, we set

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x).$$

To determine these functions, we assume that y_p is a solution of (45). Thus, we have

$$y_p' = \sum_{k=1}^{n} \left(u_k' y_k + u_k y_k'\right).$$

In order to conveniently compute the second derivative of y_p, we can assume

$$\sum_{k=1}^{n} u_k' y_k = u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0.$$

Furthermore, we can obtain

$$y_p'' = \sum_{k=1}^{n} \left(u_k' y_k' + u_k y_k''\right).$$

Similarly, if we assume

$$\sum_{k=1}^{n} u_k' y_k' = u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0.$$

Then, we have

$$y_p''' = \sum_{k=1}^{n} \left(u_k' y_k'' + u_k y_k'''\right).$$

Continuing in this manner, in general, we arrive at the following hypothesis and the corresponding derivative.

$$\begin{aligned}\sum_{k=1}^{n} u_k' y_k^{(n-2)} &= u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0,\\ y_p^{(n)} &= \sum_{k=1}^{n} \left(u_k' y_k^{(n-1)} + u_k y_k^{(n)}\right).\end{aligned} \tag{47}$$

Under these assumptions, by substituting y_p and its derivatives up to n into (45), we can infer that

$$\sum_{k=1}^{n} u_k' y_k^{(n-1)} = u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f(x).$$

Combining (47) with the previous assumptions gives the following system of linear equations

$$\begin{cases} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0 \\ \cdots\cdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f(x) \end{cases}.$$

Notice that $y_1, y_2, \ldots, y_n$ are fundamental system of solutions of (46). Then, the Wronskian

$$W(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0$$

for all $x \in I$. Using Cramer's rule, it follows that

$$u_k'(x) = \frac{W_{k,n}(x)}{W(y_1, y_2, \ldots, y_n)(x)}, \quad k = 1, 2, \ldots, n,$$

where $W_{k,n}(x)$ is the determinant of n order obtained by replacing the k-th column of the Wronskian by the n dimensional column vector $(0, 0, \ldots, 0, f(x))^{\mathrm{T}}$.

By integrating both sides of (45) from x_0 to x with respect to t, then we can infer that

$$u_k(x) = \widehat{c}_k + \int_{x_0}^{x} \frac{W_{k,n}(t)}{W(y_1, y_2, \ldots, y_n)(t)} \mathrm{d}t,$$

where $\widehat{c}_k,\ k = 1, 2, \ldots, n$ are arbitrary constants. Therefore, we can obtain

$$\begin{aligned} y_p(x) &= \sum_{k=1}^{n} u_k(x) y_k(x) \\ &= \sum_{k=1}^{n} \widehat{c}_k y_k(x) + \sum_{k=1}^{n} \int_{x_0}^{x} \frac{y_k(x) W_{k,n}(t)}{W(y_1, y_2, \ldots, y_n)(t)} \mathrm{d}t. \end{aligned}$$

Based on the preceding argument, the general solution of the n-th order nonhomogeneous linear differential Eq. (45) can be obtained.

Theorem 5.1 *Let $p_k, f \in C(I, \mathbb{R})$, $k = 1, 2, \ldots, n-1$, and let $y_1(x), y_2(x), \ldots, y_n(x)$ be fundamental system of solutions of (46). Then, the general solution $y(x)$ of (45) can be given by*

$$\begin{aligned} y(x) &= Y(x) + y_p(x) \\ &= \sum_{k=1}^{n} c_k y_k(x) + \sum_{k=1}^{n} \int_{x_0}^{x} \frac{y_k(x) W_{k,n}(t)}{W(y_1, y_2, \ldots, y_n)(t)} \mathrm{d}t, \end{aligned}$$

where $c_k,\ k = 1, 2, \ldots, n$ are arbitrary constants, $x_0 \in I$ is an arbitrary fixed point.

According to Theorem 5.1, the Ulam stability of the n-th order linear differential Eq. (45) can be derived.

Theorem 5.2 *Let $p_k, f \in C(I, \mathbb{R})$, $k = 1, 2, \ldots, n-1$, and let $y_1(x), y_2(x), \ldots, y_n(x)$ be fundamental system of solutions of (46). Assume that $\varphi : I \to [0, +\infty)$ is a continuous function. If $y_\varphi \in C^n(I, \mathbb{R})$ satisfies the differential inequality*

$$\left| y_\varphi^{(n)}(x) + \sum_{k=1}^{n} p_{n-k}(x) y_\varphi^{(n-k)}(x) - f(x) \right| \le \varphi(x)$$

for all $x \in I$, then there exists $y \in C^n(I, \mathbb{R})$ such that $y(x)$ satisfies (45) and

$$|y_\varphi(x) - y(x)| \le \left| \int_{x_0}^{x} \left| \frac{W(y_1, y_2, \cdots, y_n)\left(\overbrace{t, \ldots, t}^{n-1}, x\right)}{W(y_1, y_2, \ldots, y_n)(t)} \right| \varphi(t)\mathrm{d}t \right| \tag{48}$$

for all $x \in I$, where

$$W(y_1, y_2, \ldots, y_n)(\underbrace{t, \ldots, t}_{n-1}, x) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix}.$$

Proof Using the same argument as in Theorem 3.2, we set

$$y_\varphi^{(n)}(x) + \sum_{k=1}^{n} p_{n-k}(x) y_\varphi^{(n-k)}(x) := f_\varphi(x).$$

By Theorem 5.1, there exist $c_1, c_2, \ldots, c_n$ such that

$$y_\varphi(x) = \sum_{k=1}^{n} c_k y_k(x) + \sum_{k=1}^{n} \int_{x_0}^{x} \frac{y_k(x)\widetilde{W}_{k,n}(t)}{W(y_1, y_2, \ldots, y_n)(t)} \mathrm{d}t, \tag{49}$$

where $\widetilde{W}_{k,n}(x)$ is the determinant of n-th order obtained by replacing the k-th column of the Wronskian by the n dimensional column vector $(0, 0, \ldots, 0, f_\varphi(x))^{\mathrm{T}}$.

Define

$$y(x) = \sum_{k=1}^{n} c_k y_k(x) + \sum_{k=1}^{n} \int_{x_0}^{x} \frac{y_k(x) W_{k,n}(t)}{W(y_1, y_2, \ldots, y_n)(t)} \mathrm{d}t. \tag{50}$$

By the property of the determinant, we get

$$\begin{aligned} &\sum_{k=1}^{n} \left(y_k(x)\widetilde{W}_{k,n}(t) - y_k(x) W_{k,n}(t) \right) \\ &\quad = \sum_{k=1}^{n} y_k(x)(-1)^{k+n} f_\varphi(x) M_{k,n}(t) - y_k(x)(-1)^{k+n} f(x) M_{k,n}(t) \\ &\quad = (f_\varphi(x) - f(x)) \sum_{k=1}^{n} (-1)^{k+n} y_k(x) M_{k,n}(t) \end{aligned}$$

$$= (f_\varphi(x) - f(x))W(y_1, y_2, \ldots, y_n)(\underbrace{t, \ldots, t}_{n-1}, x),$$

where $M_{k,n}(t)$ is the cofactor of the element $f_\varphi(x)$ in the determinant $\widetilde{W}_{k,n}(t)$. Obviously, it is the same as the cofactor of the element $f(x)$ in the determinant $W_{k,n}(t)$.

Therefore, the desired inequality (48) can be derived from (49), (50) and the above equality. The proof of the theorem is now completed. □

Similar to Corollary 4.3, the Hyers–Ulam stability of the n-th order nonhomogeneous linear differential Eq. (45) can be derived from Theorem 5.2.

Corollary 5.3 *Let $p_k, f \in C(I, \mathbb{R})$, $k = 1, 2, \ldots, n-1$. For a given $\varepsilon > 0$, if an n times continuously differentiable function $y_\varepsilon : [a, b] \to \mathbb{R}$ satisfies the following inequality*

$$\left| y_\varphi^{(n)}(x) + \sum_{k=1}^{n} p_{n-k}(x) y_\varphi^{(n-k)}(x) - f(x) \right| \leq \varepsilon$$

for all $x \in [a, b]$, then there exists a solution $y : [a, b] \to \mathbb{R}$ of (45) such that

$$|y_\varepsilon(x) - y(x)| \leq K\varepsilon$$

for all $x \in [a, b]$, where

$$K = \max_{x \in [a,b]} \int_a^x \left| \frac{W(y_1, y_2, \ldots, y_n)\left(\overbrace{t, \ldots, t}^{n-1}, x\right)}{W(y_1, y_2, \ldots, y_n)(t)} \right| \mathrm{d}t.$$

Remark 4 In the theory of Ulam stability, the optimal Hyers–Ulam constant K is the minimum positive real number such that the error inequality $|y_\varepsilon(x) - y(x)| \leq K\varepsilon$ holds for all $x \in I$. It can be easily seen from the proofs of the main theorems that each Hyers–Ulam constant K is optimal in Corollaries 3.3, 4.3 and 5.3.

Finally, by Theorem 5.2, we can obtain the Hyers–Ulam stability of the Cauchy–Euler differential equation of n-th order. For simplicity, the general form of the nonhomogeneous Cauchy–Euler differential equation of n-th order is given as follows:

$$x^n y^{(n)}(x) + \sum_{k=1}^{n} \mu_{n-k} x^{n-k} y^{(n-k)}(x) = f(x), \tag{51}$$

where $\mu_k \in \mathbb{R}$, $(k = 1, 2, \ldots, n)$, $f \in C([a, b], \mathbb{R})$ with $0 < a < b$.

Corollary 5.4 *Let $y_1(x), y_2(x), \ldots, y_n(x)$ be fundamental system of solutions of the associated homogeneous equation of (51). For a given $\varepsilon > 0$, if an n times continuously* *differentiable function* $y_\varepsilon : [a, b] \to \mathbb{R}$ *satisfies the following inequality*

$$\left| x^n y_\varphi^{(n)}(x) + \sum_{k=1}^{n} \mu_{n-k} x^{n-k} y_\varphi^{(n-k)}(x) - f(x) \right| \le \varepsilon$$

for all $x \in [a, b]$*, then there exists a solution* $y : [a, b] \to \mathbb{R}$ *of* (51) *such that*

$$|y_\varepsilon(x) - y(x)| \le K\varepsilon$$

for all $x \in [a, b]$*, where*

$$K = \max_{x\in[a,b]} \int_a^x \frac{1}{t^n} \left| \frac{W(y_1, y_2, \cdots, y_n)\left(\overbrace{t, \ldots, t}^{n-1}, x\right)}{W(y_1, y_2, \ldots, y_n)(t)} \right| \mathrm{d}t.$$

Proof By Theorem 5.2, the proof is a direct extension of Corollary 3.7 and so is omitted. □

6 Conclusions

By using the method of variation of parameters, in this paper, we have established the Ulam stability of linear differential equations of first order, second order, third order and n-th order, respectively. This paper presented a unified method to study the Ulam stability problem of linear differential equations of first order and higher order with constant and nonconstant coefficients. Accordingly, the Hyers–Ulam stability of linear differential equations can be directly derived from the main results. Moreover, the stability constant obtained in this paper is optimal. In particular, the Hyers–Ulam stability of the Cauchy–Euler differential equations of second order, third order and n-th order can also be obtained by the relevant conclusions.

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