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Homogenization of the Darcy–Lapwood–Brinkman Flow in a Thin Domain with Highly Oscillating Boundaries

Igor Pažanin[1] · Francisco Javier Suárez-Grau[2]

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Abstract

In this paper, we investigate the flow through a thin corrugated domain filled with fluid-saturated porous medium. The porous medium flow is described by the nonlinear Darcy–Lapwood–Brinkman model acknowledging the viscous shear and the inertial effects. The thickness of the domain is assumed to be of the same small order ε as the period of the oscillating boundaries. Depending on the magnitude of the permeability with respect to ε, we rigorously derive different asymptotic models and compare the results with the non-oscillatory case. We employ a homogenization technique based on the adaption of the unfolding method and deduce the influence of the porous structure and boundary oscillations on the effective flow.

Keywords Darcy–Lapwood–Brinkman equation · Thin domain · Highly oscillating boundary · Unfolding method

Mathematics Subject Classification 35B27 · 35B40 · 76S05

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✉ Francisco Javier Suárez-Grau
fjsgrau@us.es

Igor Pažanin
pazanin@math.hr

1 Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

2 Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain

1 Introduction

Numerous models have been developed in the past 15 decades to describe flows through porous media. When a fluid permeates a porous material, the actual path of an individual fluid particle cannot be followed analytically. The effect at main order, as the fluid slowly percolates the pores of the medium, must be represented by a macroscopic law, which is applicable to fluid with large mass compared with the dimension of the porous structure of a medium, and this is the basis for the Darcy's law [15]. According to this law, the driving force necessary to move a specific volume of fluid at a certain speed through a porous medium is in equilibrium with the resistance generated by internal friction between the fluid and the pore structure. Mathematically, the Darcy law takes the form

$$u = -\frac{K}{\mu}\nabla p, \tag{1}$$

where u and p represent the filter velocity and pressure, K is the permeability of the porous medium and μ is the dynamic coefficient of viscosity. In the presence an external force f , Eq. (1) can be written as

$$u = \frac{K}{\mu}(f - \nabla p), \tag{2}$$

In the case of a homogeneous, isotropic porous medium, the flow governed by this modified Darcy law is of potential type rather than a boundary layer type. This law is valid for a densely packed porous medium whose permeability is very low. The Darcy model takes into account only the frictional force offered by the solid particles to the fluid rather than the usual viscous shear, so this law is only valid if a variety of the conditions are being met, and therefore, it is not applicable in many physically relevant settings. For instance, as a first-order PDE for the velocity, Darcy's equation cannot sustain the no-slip boundary condition imposed on an impermeable boundary. The existence of shear within the porous medium was experimentally demonstrated by Beavers et al. [9], near the boundaries, thus forming a zone of shear influenced by fluid flow. The Darcy equation cannot predict the existence of such a boundary zone because there is not a macroscopic shear term included in the equation. Thus, to take into account the shear, Eq. (2) was generalized by Slattery [27] and Tam [28] taking the form

$$-\frac{\mu}{K}u + \mu_e \Delta u = \nabla p - f, \tag{3}$$

where μ_e is the effective viscosity of the fluid in the porous medium and is a function of the porosity, ϕ. This is the most suitable governing equation for an incompressible creeping flow of a Newtonian fluid within an isotropic, homogeneous porous medium. This equation, originally proposed by Brinkman [10], was justified by Childress [13], Lundgren [19] and Saffman [26]. The Brinkman equation (3) is physically consistent with the previously mentioned experimentally observed boundary shear zone on account of the usual viscous shear force. The Brinkman model is valid for a sparsely packed fluid-saturated porous medium wherein there is more window for a fluid to flow so that the distortion of velocity gives rise to the usual viscous shear force.

Also, if the effects of inertia are important for the process (e.g., due to the curvilinearity of the flow path), the Darcy's law again cannot be applied. As the inertia force increases relative to the viscous force, the streamlines become more distorted and drag increases more rapidly than linearly with velocity. At present, there are several different views as to how the Darcy model should be generalized to include the inertia effect. Lapwood [18] gave a mathematical form incorporating the convective inertial term, $\frac{\rho}{\phi^2}(u\cdot\nabla)u$ in the momentum equation, where ρ is the fluid density (see also Nield and Bejan [24]). Thus, in such situation when viscous shear and macroscopic inertial effects are significant, it has been customary to use the so called Darcy–Lapwood–Brinkman (DLB) equation to model the porous medium flow. This model takes the form:

$$-\mu_e\Delta u+\nabla p+\frac{\mu}{K}u=f-\frac{\rho}{\phi^2}(u\cdot\nabla)u, \tag{4}$$

$$\operatorname{div}u=0. \tag{5}$$

Note that the second-order DLB equation (4) is capable of handling the presence of the solid boundary on which the no-slip condition for the velocity is imposed. Moreover, the effects of flow inertia are also being incorporated making model (4), (5) an important generalization of the Darcy law that has sound physical basis.

We observe that in the absence of viscous shear, Eq. (4) is known as the Darcy–Lapwood equation. If quadratic drag is incorporated in the system, then the above equation becomes

$$-\mu_e\Delta u+\nabla p+\frac{\mu}{K}u+\frac{\rho C_b}{\sqrt{K}}|u|u=f-\frac{\rho}{\phi^2}(u\cdot\nabla)u,$$

where C_b is the is the dimensionless quadratic drag coefficient. This equation is known as Darcy–Lapwood–Brinkman–Forchheimer model. If the Reynolds number Re is very small, then the quadratic drag can be neglected. If $Re\approx O(1)$, then the quadratic drag law holds. For high Reynolds numbers, we will have to use a cubic drag law.

In this paper, we consider that the Reynolds number Re is very small (the quadratic drag can be neglected), and so we study the Darcy–Lapwood–Brinkman model. Due to nonlinearity of Eq. (4), the Darcy–Lapwood–Brinkman model has been mostly treated numerically (see, e.g., Chen et al. [12], Khalilli et al. [17], Umawathi et al. [30]). Analytical treatments are sparse and address only simple 2D fractures with plane-parallel walls under additional assumptions which linearize the momentum equation (4). We refer the reader to the papers by Hamdan et al. [7,21]. In view of that, the goal of this paper is to analyze the 3D fluid flow through a thin layer of porous medium sandwiched between two corrugated walls and governed by (4), (5). First, in Sect. 2, we study the flow in a thin constricted fracture without boundary oscillations, namely:

$$\Omega_\varepsilon=\left\{(x',x_3)\in\mathbb{R}^2\times\mathbb{R}\,:\,x'\in\omega,\ \varepsilon h^-\left(x'\right)<x_3<\varepsilon h^+\left(x'\right)\right\}. \tag{6}$$

Here ω is a smooth bounded open set in $\mathbb{R}^2$, while h^- and h^+ are smooth functions such that $h^+ > h^-$ on $\overline{\omega}$. Assuming that permeability $K = K_\varepsilon$ may depend on the small parameter ε, we employ a homogenization technique with respect to ε and rigorously derive three different effective models (see Theorem 1):

- If $K_\varepsilon \approx \varepsilon^2$, i.e., when the permeability is of order ε^2, we obtain a 2D Darcy law as an effective model including the effects of the domain's geometry and the porous structure.
- If $K_\varepsilon \ll \varepsilon^2$, we obtain a 2D pressure-driven Darcy law as an effective model not accounting the effects of the Brinkman (viscous) term.
- If $K_\varepsilon \gg \varepsilon^2$, we obtain no contribution of the porous structure at the macroscopic model. As a result, we obtain a solution in the form of the Poiseuille flow.

We observe that the convective inertial term does not contribute to in the macroscopic models, and only the Darcy and Brinkman terms appear.

Most recently, Pažanin and Siddheshwar [25] considered the similar problem, but in the case of the 2D flow. It is worth mentioning that the effective expression in the critical case provided in Theorem 1 is consistent with the one formally obtained in [25] via two-scale expansion method.
Section 3 is the central part of the present work. Here, we introduce the oscillations at the top and the bottom of the flow domain; namely, we assume that the period of the oscillations has the same small order as the domain thickness. In view of that, the domain to be considered is the following:

$$\Lambda_\varepsilon = \left\{ (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \, : \, x' \in \omega, \; \varepsilon h^- \left(\frac{x'}{\varepsilon} \right) < x_3 < \varepsilon h^+ \left(\frac{x'}{\varepsilon} \right) \right\}, \qquad (7)$$

for periodic functions h^- and h^+. As above, we aim to determine the asymptotic behavior (as $\varepsilon \to 0$) of the flow governed by (4), (5), now posed in Λ_ε. The proof of our results is based on an adaptation of the unfolding method (see Arbogast et al. [6], and Cioranescu et al. [14]), which is strongly related to the two-scale convergence method (see Allaire [2], Nguetseng [23] and also Marušić-Paloka et al. [20]). The unfolding method has been extensively used to study periodic homogenization problems where the size of the periodic cell tends to zero. We refer the reader to a recent works by Anguiano and Suárez-Grau [3–5]. The basic idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable). In the present setting, it is necessary to combine the unfolding method with a rescaling in the height variable in order to be able to work with a domain of a fixed height. In particular, due to the boundary oscillations, an extension operator needs to be constructed in order to extend the pressure to an ε-independent domain. Consequently, we manage to identify the critical size and later on the effects of the microstructure (boundary oscillations) in the corresponding effective equations. It turns out that the critical size is exactly the same as the one we obtain for the non-oscillatory case. Moreover, depending on the magnitude of the permeability K_ε with respect to ε, we derive three different characteristic cases (see Theorem 2):

– If $K_\varepsilon \approx \varepsilon^2$, we obtain a 2D Darcy law as an effective model which includes both the effects of the porous structure and the boundary oscillations given by the local Darcy–Brinkman problems in 3D.
– If $K_\varepsilon \ll \varepsilon^2$, we obtain a 2D pressure-driven Darcy law as an effective model not accounting the effects of the Brinkman (viscous) term, but including the effects of the boundary oscillations provided by the local Hele-Shaw problems in 2D.
– If $K_\varepsilon \gg \varepsilon^2$, as in the non-oscillatory case, we obtain no contribution of the porous structure at the effective model, but here the effects of the boundary oscillations are present through local Stokes problems in 3D.

We remark that convective inertial term also vanishes in the limit, and so only the Darcy and Brinkman terms contribute to the macroscopic models depending on the relation between ε and K_ε.

To conclude, we believe that the analysis presented in this paper is instrumental for understanding the effective behavior of the porous medium flow in thin domains with highly oscillating boundaries. As emphasized above, by considering the (nonlinear) Darcy–Lapwood–Brinkman equation, the important features have been taken into account that cannot be captured by a classical Darcy's law. Consequently, the considered flow naturally finds applications both in industry (chemical reactors, heat exchangers, filtering equipment, etc.) and in geophysical problems; see [24] and the references therein. By employing a homogenization technique, the averaged effects of the boundary oscillations and the porous structure have been elegantly deduced. It should be mentioned that such homogenized system is of practical interest for developing numerical codes since it allows to filter out the small scales of the boundary, having a high computational cost. In view of that, we hope that our results could have an impact on the known engineering practice.

2 Non-oscillatory Case

Throughout the text, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. Correspondingly, for the functions we use the same notation $U = (U', U_3)$, $U' \in \mathbb{R}^2$.

In this section, we study the flow of a viscous fluid in the domain Ω_ε given by

$$\Omega_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^3 \,:\, x' \in \omega,\ \varepsilon h^-(x') < x_3 < \varepsilon h^+(x')\right\},$$

where $h^-, h^+ \in C^1(\omega) \cap C(\overline{\omega})$ such that $h^+ > h^-$ on $\overline{\omega}$. We suppose that the fracture Ω_ε is filled by a fluid-saturated sparsely packed porous medium. As explained in Introduction, the flow through the porous medium is modeled by the Darcy–Lapwood–Brinkman (DLB) equation. In view of that, let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ satisfying

$$\begin{aligned} &-\mu_e \Delta u_\varepsilon + \nabla p_\varepsilon + \frac{\mu}{K_\varepsilon} u_\varepsilon = f - \frac{\rho}{\phi^2}(u_\varepsilon \cdot \nabla) u_\varepsilon, \\ &\operatorname{div} u_\varepsilon = 0. \end{aligned} \tag{8}$$

To complete the problem, we impose a standard no-slip boundary condition

$$u_\varepsilon = 0 \ \text{ on } \quad \partial\Omega_\varepsilon. \tag{9}$$

The right-hand side f is of the form

$$f(x) = (f'(x'), 0), \quad \text{a.e. } x \in \omega,$$

where f is assumed to be in $L^2(\omega \times (h^-_{\min}, h^+_{\max}))^2$. Such choice of f is usual and justified when we deal with thin domains. Indeed, since the thickness of the domain is small, then the vertical component of the force can be neglected and, moreover, the force can be considered independent of the vertical variable.

Using standard techniques (see, e.g., Galdi [16]), it can be established that (8), (9) have at least one solution $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$. The space $L^2_0(\Omega_\varepsilon)$ is the space of functions of $L^2(\Omega_\varepsilon)$ with null integral. Our aim is to study the asymptotic behavior of u_ε and p_ε when the thickness ε tends to zero, taking into account the magnitude of K_ε with respect to ε. For this purpose, we use the dilatation in the vertical variable x_3:

$$y_3 = \frac{x_3}{\varepsilon}, \tag{10}$$

in order to have the functions defined in an open set independent of ε and with height of order one:

$$\Omega = \left\{(x', y_3) \in \mathbb{R}^3 \ : \ x' \in \omega, \ h^-(x') < y_3 < h^+(x')\right\}.$$

We define $\tilde{u}_\varepsilon \in H^1_0(\Omega)^3$, $\tilde{p}_\varepsilon \in L^2(\Omega)/\mathbb{R}$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \ \ \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e.\, (x', y_3) \in \Omega.$$

In view of (10), system (8) can be rewritten in Ω as

$$\begin{cases} -\mu_e \Delta_{x'} \tilde{u}_\varepsilon - \varepsilon^{-2}\mu_e \partial^2_{y_3} \tilde{u}_\varepsilon + \nabla_{x'} \tilde{p}_\varepsilon + \varepsilon^{-1}\partial_{y_3} \tilde{p}_\varepsilon e_3 + \dfrac{\mu}{K_\varepsilon}\tilde{u}_\varepsilon = f' - \dfrac{\rho}{\phi^2}(\tilde{u}_\varepsilon \cdot \nabla_\varepsilon)\tilde{u}_\varepsilon, \\ \operatorname{div}_{x'} \tilde{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3} \tilde{u}_{\varepsilon,3} = 0 \end{cases} \tag{11}$$

with no-slip boundary condition on $\partial\Omega$, i.e.,

$$\tilde{u}_\varepsilon = 0 \ \text{ on } \quad \partial\Omega. \tag{12}$$

The asymptotic behavior of the sequence $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ is provided in the following result:

Theorem 1 *We distinguish three characteristic cases:*

(i) *if* $K_\varepsilon \approx \varepsilon^2$, *with* $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, *then* $(\tilde{u}_\varepsilon/\varepsilon^2, \tilde{p}_\varepsilon)$ *converges weakly, as* ε *tends to zero, in* $H^1(h^-, h^+; L^2(\omega)^3) \times L^2_0(\omega)$ *to* $(\tilde{u}, \tilde{p})$, *with* $\tilde{u}_3 = 0$ *and* $\tilde{u} = 0$ *on* $y_3 = h^-, h^+$. *Moreover,* $\tilde{p} \in H^1(\omega)$ *and* $(\widetilde{U}'(x'), \tilde{p}(x'))$ *is the solution of the effective problem*

$$\begin{cases} \tilde{U}'(x') = \dfrac{2K A_M(x')}{M\mu}\Big(f'(x') - \nabla_{x'}\tilde{p}(x')\Big) & \text{in } \omega, \\ \operatorname{div}_{x'}\tilde{U}'(x') = 0 & \text{in } \omega, \\ \tilde{U}'(x') \cdot n = 0 & \text{on } \partial\omega, \end{cases} \tag{13}$$

where $\tilde{U}(x') = \displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}(x', y_3)\,\mathrm{d}y_3$, $M = \sqrt{\frac{\mu}{K\mu_e}}$ *and the function* $A_M(x')$ *is given by*

$$\begin{aligned} A_M(x') &= \frac{2 - e^{Mh^+(x') - Mh^-(x')} - e^{Mh^-(x') - Mh^+(x')}}{e^{Mh^+(x') - Mh^-(x')} - e^{Mh^-(x') - Mh^+(x')}} \\ &= \frac{1 - ch(Mh^+(x') - Mh^-(x'))}{sh(Mh^+(x') - Mh^-(x'))}. \end{aligned} \tag{14}$$

(ii) *if* $K_\varepsilon \ll \varepsilon^2$, *then* $(\tilde{u}_\varepsilon/\varepsilon^2, (K_\varepsilon/\varepsilon^2)\tilde{p}_\varepsilon)$ *converges weakly, as* ε *tends to zero, in* $H^1(h^-, h^+; L^2(\omega)^3) \times L^2_0(\omega)$ *to* $(\tilde{u}, \tilde{p})$, *with* $\tilde{u}_3 = 0$ *and* $\tilde{u} = 0$ *on* $y_3 = h^-, h^+$. *Moreover,* $\tilde{p} \in H^1(\omega)$ *and* $(\tilde{U}, \tilde{p})$ *is the unique solution of the effective problem*

$$\begin{cases} \tilde{U}'(x', y_3) = -\dfrac{A_0(x')}{\mu}\nabla_{x'}\tilde{p}(x') & \text{in } \omega, \\ \tilde{U}_3(x') = 0 & \text{in } \omega, \\ \operatorname{div}_{x'}\tilde{U}(x') = 0 & \text{in } \omega, \\ \tilde{U}(x') \cdot n = 0 & \text{on } \partial\omega, \end{cases} \tag{15}$$

where $\tilde{U}(x') = \displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}(x', y_3)\,\mathrm{d}y_3$ *and the function* $A_0(x')$ *is given by*

$$A_0(x') = h^+(x') - h^-(x').$$

(iii) *if* $K_\varepsilon \gg \varepsilon^2$, *then* $(\tilde{u}_\varepsilon/\varepsilon^2, \tilde{p}_\varepsilon)$ *converges weakly, as* ε *tends to zero, in* $H^1(h^-, h^+; L^2(\omega)^3) \times L^2_0(\omega)$ *to* $(\tilde{u}, \tilde{p})$, *with* $\tilde{u}_3 = 0$ *and* $\tilde{u} = 0$ *on* $y_3 = h^-, h^+$.

Moreover, $\tilde{p} \in H^1(\omega)$ *and* $(\tilde{U}, \tilde{p})$ *is the unique solution of the effective problem*

$$\begin{cases} \tilde{U}'(x') = \dfrac{A_\infty(x')}{12\mu_e}\big(f'(x') - \nabla_{x'}\tilde{p}(x')\big) & \text{in } \omega, \\ \tilde{U}_3(x') = 0 & \text{in } \omega, \\ \mathrm{div}_{x'}\,\tilde{U}'(x') = 0 & \text{in } \omega, \\ \tilde{U}'(x')\cdot n = 0 & \text{on } \partial\omega, \end{cases} \tag{16}$$

where $\tilde{U}(x') = \displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}(x', y_3)\,\mathrm{d}y_3$ *and the function* $A_\infty(x')$ *is given by*

$$A_\infty(x') = h^+(x')^3 - 3h^-(x')^3 - 3\Big(h^+(x')^2h^-(x') - h^+(x')h^-(x')^2\Big).$$

2.1 Proof of Theorem 2.1

Let us first fix some notation.

We denote by : the full contraction of two matrices, namely for $A = (a_{i,j})_{1\le i,j\le 2}$ and $B = (b_{i,j})_{1\le i,j\le 2}$, we have $A : B = \sum_{i,j=1}^2 a_{ij}b_{ij}$.

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line.

We denote by C a generic positive constant which can change from line to line.

A priori estimates: First, we need to derive the a priori estimates for u_ε. To accomplish this, we employ some technical results which can be verified straightforwardly by a simple change of variables (for the proof, see [20], Lemmas 8 and 11):

Lemma 1 *The following estimates hold:*

$$\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon\|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}, \tag{17}$$

$$\|\varphi_\varepsilon\|_{L^4(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{1}{2}}\|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}, \tag{18}$$

Now, we prove the sharp a priori estimates for the velocity u_ε.

Proposition 1 *For* u_ε *satisfying system* (8), (9),

(i) if $K_\varepsilon \approx \varepsilon^2$, *with* $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$ *or* $K_\varepsilon \ll \varepsilon^2$, *it holds*

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{5}{2}}. \tag{19}$$

(ii) if $K_\varepsilon \gg \varepsilon^2$, *it holds*

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}}. \tag{20}$$

Moreover, in every case it holds

$$\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{3}{2}}. \tag{21}$$

Proof We consider u_ε as test function in the weak formulation of problem (8), and so we get

$$\mu_e \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 \, \mathrm{d}x + \frac{\mu}{K_\varepsilon} \int_{\Omega_\varepsilon} |u_\varepsilon|^2 \, \mathrm{d}x = \int_{\Omega_\varepsilon} f'(x') \cdot u'_\varepsilon \, \mathrm{d}x. \tag{22}$$

Using the Cauchy–Schwarz inequality, $f' \in L^2(\omega)^2$ and (17), we deduce

$$\left| \int_{\Omega_\varepsilon} f'(x') \cdot u'_\varepsilon \, \mathrm{d}x \right| \le C\varepsilon^{\frac{3}{2}} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}},$$

leading to

$$\mu_e \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 \, \mathrm{d}x + \frac{\mu}{K_\varepsilon} \int_{\Omega_\varepsilon} |u_\varepsilon|^2 \, \mathrm{d}x \le C\varepsilon^{\frac{3}{2}} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}. \tag{23}$$

On the one hand, this implies that (21) holds. Consequently, using (17), we get

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{5}{2}}. \tag{24}$$

On the other hand, using (21) in (23), we also obtain

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}. \tag{25}$$

From (24) and (25), we have that

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\left(\varepsilon^{\frac{5}{2}} + \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}\right).$$

We compare $\varepsilon^{\frac{5}{2}}$ with respect to $\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}$ and observe that the critical case is when $K_\varepsilon \approx \varepsilon^2$ which gives estimate (19). In the subcritical case $K_\varepsilon \ll \varepsilon^2$, we also deduce (19), while in the supercritical case we deduce (20). □

Corollary 1 *For $\tilde{u}_\varepsilon$ satisfying system* (11), (12),

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, or $K_\varepsilon \ll \varepsilon^2$, the following estimate holds

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)^3} \le C\varepsilon^2. \tag{26}$$

(ii) if $K_\varepsilon \gg \varepsilon^2$, the following estimate holds

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon\, K_\varepsilon^{\frac{1}{2}}. \tag{27}$$

Moreover, in every cases it holds

$$\|D_{x'}\tilde{u}_\varepsilon\|_{L^2(\Omega)^{2\times 3}} \leq C\varepsilon, \quad \|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon^2. \tag{28}$$

Proof Estimates (26), (27) and (28) are easily obtained from (19), (20) and (21), respectively, by applying the change of variable (10). □

Now, we prove the a priori estimates for the pressure p_ε. For this, we need one more technical lemma addressing the auxiliary divergence problem (see Lemma 20 from [20]).

Lemma 2 *The problem*

$$\begin{cases} \operatorname{div}\varphi_\varepsilon = f_\varepsilon \in L^2_0(\Omega_\varepsilon) & \text{in } \Omega_\varepsilon, \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{29}$$

has a solution $\varphi_\varepsilon \in H^1_0(\Omega_\varepsilon)^3$ such that

$$\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad \|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \leq \frac{C}{\varepsilon}\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{30}$$

Proposition 2 *For p_ε satisfying system (8), (9),*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, or $K_\varepsilon \gg \varepsilon^2$, we have

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \tag{31}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, we have

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon}. \tag{32}$$

Proof We introduce φ_ε as the solution of the auxiliary problem

$$\operatorname{div}\varphi_\varepsilon = p_\varepsilon \in L^2_0(\Omega_\varepsilon) \quad \text{in } \Omega_\varepsilon, \quad \varphi_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

According to Lemma 2, such problem has at least one solution such that

$$\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad \|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \leq \frac{C}{\varepsilon}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

We multiply system (8) by φ_ε, and integrating over Ω_ε, we obtain

$$\begin{aligned}\|p_\varepsilon\|^2_{L^2_0(\Omega_\varepsilon)} &= \left|\int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}\varphi_\varepsilon \,\mathrm{d}x\right| \\ &\leq \left|\mu_e \int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon \,\mathrm{d}x\right| + \left|\int_{\Omega_\varepsilon} f\cdot\varphi_\varepsilon \,\mathrm{d}x\right| \\ &\quad + \left|\frac{\mu}{\phi^2}\int_{\Omega_\varepsilon} (u_\varepsilon\cdot\nabla)\tilde{u}_\varepsilon\,\varphi_\varepsilon \,\mathrm{d}x\right| + \left|\frac{\mu}{K_\varepsilon}\int_{\Omega_\varepsilon} u_\varepsilon\cdot\varphi_\varepsilon \,\mathrm{d}x\right|. \end{aligned} \tag{33}$$

Taking into account estimate (21) and Lemma 2, we have

$$\left|\mu_e \int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon \,\mathrm{d}x\right| \leq C\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}} \leq C\varepsilon^{\frac{1}{2}}\|p_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}.$$

Similarly, we obtain

$$\left|\int_{\Omega_\varepsilon} f\cdot\varphi_\varepsilon \,\mathrm{d}x\right| \leq C\varepsilon^{\frac{1}{2}}\|p_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}.$$

For the convective term, from estimate (21) and employing inequalities (17) and (18) and Lemma 2, we deduce

$$\begin{aligned}\left|\frac{\mu}{\phi^2}\int_{\Omega_\varepsilon} (u_\varepsilon\cdot\nabla)u_\varepsilon\,\varphi_\varepsilon \,\mathrm{d}x\right| &\leq C\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\|u_\varepsilon\|_{L^4(\Omega_\varepsilon)^3}\|\varphi_\varepsilon\|_{L^4(\Omega_\varepsilon)^3} \\ &\leq C\varepsilon^{\frac{5}{2}}\|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}}\|p_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}.\end{aligned}$$

Finally, we get

$$\left|\frac{\mu}{K_\varepsilon}\int_{\Omega_\varepsilon} u_\varepsilon\cdot\varphi_\varepsilon \,\mathrm{d}x\right| \leq C\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}.$$

Depending on the magnitude of K_ε with respect to ε, we conclude:

- if $K_\varepsilon \approx \varepsilon^2$, estimates (17), (19) and Lemma 2 yield

$$\left|\frac{\mu}{K_\varepsilon}\int_{\Omega_\varepsilon} u_\varepsilon\cdot\varphi_\varepsilon \,\mathrm{d}x\right| \leq C\varepsilon^{\frac{1}{2}}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

- if $K_\varepsilon \ll \varepsilon^2$, using estimate (19), we get

$$\left|\frac{\mu}{K_\varepsilon}\int_{\Omega_\varepsilon} u_\varepsilon\cdot\varphi_\varepsilon \,\mathrm{d}x\right| \leq C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

– if $K_\varepsilon \gg \varepsilon^2$, using estimate (20), we get

$$\left|\frac{\mu}{K_\varepsilon}\int_{\Omega_\varepsilon} u_\varepsilon \cdot \varphi_\varepsilon \, \mathrm{d}x\right| \leq C\frac{\varepsilon^{\frac{3}{2}}}{K_\varepsilon^{\frac{1}{2}}}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Thus, in view of (33), we deduce that if $K_\varepsilon \approx \varepsilon^2$ or $K_\varepsilon \gg \varepsilon^2$, we have

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}.$$

Finally, if $K_\varepsilon \ll \varepsilon^2$, we get

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon}.$$

□

Corollary 2 *For $\tilde{p}_\varepsilon$ satisfying system (11), (12),*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$ or $K_\varepsilon \gg \varepsilon^2$, we have

$$\|\tilde{p}_\varepsilon\|_{L^2(\Omega)} \leq C, \tag{34}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, we have

$$\|\tilde{p}_\varepsilon\|_{L^2(\Omega)} \leq C\frac{\varepsilon^2}{K_\varepsilon}. \tag{35}$$

Proof Estimates (34) are (35) are easily obtained from (34) and (35) by applying the change of variables (10). □

Some compactness results: From the a priori estimates of $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$, we can deduce the following compactness results:

Lemma 3 *For $\tilde{u}_\varepsilon$ satisfying system (11), (12), there exists $\tilde{u} \in H^1(h^-, h^+; L^2(\omega))^3$ where $\tilde{u}_3 = 0$ and $\tilde{u} = 0$ on $y_3 = h^-, h^+$, such that*

$$\frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \text{ in } H^1(h^-, h^+; L^2(\omega))^3 \quad \text{as } \varepsilon \to 0, \tag{36}$$

$$\begin{cases} \mathrm{div}_{x'}\left(\displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3)\mathrm{d}y_3\right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3)\mathrm{d}y_3\right) \cdot n = 0 & \text{on } \partial\omega. \end{cases} \tag{37}$$

Proof We divide the proof in three steps.

Step 1 Let us begin with the cases $K_\varepsilon \approx \varepsilon^2$ and $K_\varepsilon \ll \varepsilon^2$. Estimates (26) and (28) imply the existence of $\tilde{u} \in H^1(h^-, h^+; L^2(\omega)^3)$, such that, up to a subsequence, we have convergence (36). Consequently,

$$\frac{1}{\varepsilon^2}\mathrm{div}_{x'}\tilde{u}'_\varepsilon \rightharpoonup \mathrm{div}_{x'}\tilde{u}' \text{ in } H^1(h^-, h^+; H^{-1}(\omega)). \tag{38}$$

Since $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in Ω, multiplying by ε^{-2} we obtain

$$\frac{1}{\varepsilon^2}\mathrm{div}_{x'}\tilde{u}'_\varepsilon + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3} = 0, \quad \text{in } \Omega, \tag{39}$$

which, combined with (38) implies that $\partial_{y_3}\tilde{u}_{\varepsilon,3}/\varepsilon^3$ is bounded in $H^1(h^-, h^+; H^{-1}(\omega))$. This implies that $\partial_{y_3}\tilde{u}_{\varepsilon,3}/\varepsilon^2$ tends to zero in $H^1(h^-, h^+; H^{-1}(\omega))$. Also, from (36), we have that $\partial_{y_3}\tilde{u}_{\varepsilon,3}/\varepsilon^2$ tends to $\partial_{y_3}\tilde{u}_3$ in $L^2(\Omega)$. From the uniqueness of the limit, we have that $\partial_{y_3}\tilde{u}_3 = 0$, which implies that $\tilde{u}_3$ does not depend on y_3. Moreover, the continuity of the trace applications from the space of functions v such that $\|\tilde{v}\|_{L^2}$ and $\|\partial_{y_3}\tilde{v}\|_{L^2}$ are bounded to $L^2(\omega \times \{h^-\})$ and $L^2(\omega \times \{h^-\})$ implies that the values of $\tilde{u}_{\varepsilon,3}$ on the boundary $\omega \times \{h^-\}$ and $\omega \times \{h^+\}$ which are constant are preserved by letting ε tend to zero. Thus, $\tilde{u}_3 = 0$ on $\omega \times \{h^-\}$ and $\omega \times \{h^+\}$. This together with the fact that $\tilde{u}_3$ is independent of y_3 gives that $\tilde{u}_3 = 0$.

Step 2 In the case $K_\varepsilon \gg \varepsilon^2$, from the second estimate in (28), we can deduce that there exists $\tilde{u} \in H^1(h^-, h^+; L^2(\omega)^3)$ such that (36) holds.

Taking into account this convergence, estimate (27) and that $\varepsilon^{-1}K_\varepsilon^{-\frac{1}{2}}\tilde{u}_\varepsilon = \varepsilon\, K^{-\frac{1}{2}}\tilde{u}_\varepsilon/\varepsilon^2$, we deduce that $\varepsilon^{-1}K_\varepsilon^{-\frac{1}{2}}\tilde{u}_\varepsilon$ tends to zero.

Proceeding as the previous cases, we can deduce that $\tilde{u}_3 = 0$ in Ω.

Step 3 In this step, we prove (37). To do this, we consider $\varphi \in C_c^1(\omega)$ as test function in $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in Ω, which multiplying by ε^{-2} gives

$$\frac{1}{\varepsilon^2}\int_\Omega \mathrm{div}_{x'}\tilde{u}'_\varepsilon\, \varphi(x')\, \mathrm{d}x'\mathrm{d}y_3 = 0.$$

Now, from (36), we get (37). □

Lemma 4 *For $\tilde{p}_\varepsilon$ satisfying system (11), (12),*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, or $K_\varepsilon \gg \varepsilon^2$, there exists $\tilde{p} \in L^2_0(\Omega)$ such that

$$\tilde{p}_\varepsilon \rightharpoonup \tilde{p} \ \text{ in } \ L^2(\Omega), \quad \text{as } \varepsilon \to 0. \tag{40}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, then there exists $\tilde{p} \in L^2_0(\Omega)$ such that

$$\frac{K_\varepsilon}{\varepsilon^2}\tilde{p}_\varepsilon \rightharpoonup \tilde{p} \ \text{ in } \ L^2(\Omega), \quad \text{as } \varepsilon \to 0. \tag{41}$$

Proof Taking into account estimates (34) and (35), assertions (40) and (41) follow directly. □

We are now able to prove the main result of this section:

Proof (Proof of Theorem 1) First of all, applying the change of variables (10) to estimate (18), and using estimate (28), we deduce that

$$\left|\int_\Omega (\tilde{u}_\varepsilon \cdot \nabla_\varepsilon)\tilde{u}_\varepsilon\, v\, \mathrm{d}x'\mathrm{d}y_3\right| \leq \|\tilde{u}_\varepsilon\|_{L^4(\Omega)^3}\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|v_\varepsilon\|_{L^4(\Omega)^3}$$
$$\leq \varepsilon\|D_\varepsilon\tilde{u}_\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}\|D_\varepsilon v_\varepsilon\|_{L^2(\Omega)^{3\times 3}}$$
$$\leq \varepsilon^2\|Dv_\varepsilon\|_{L^2(\Omega)^{3\times 3}} = O_\varepsilon. \tag{42}$$

Now, we choose a test function $\varphi(x', y_3) \in \mathcal{D}(\Omega)^3$. Multiplying (11) by $\varphi(x', y_3)$, integrating by parts and taking into account the above estimate, we have

$$-\mu_e\int_\Omega D_{x'}\tilde{u}_\varepsilon : D_{x'}\varphi\,\mathrm{d}x'\mathrm{d}y_3 - \frac{\mu}{\varepsilon^2}\int_\Omega \partial_{y_3}\tilde{u}_\varepsilon : \partial_{y_3}\varphi\,\mathrm{d}x'\mathrm{d}y_3 + \int_\Omega \tilde{p}_\varepsilon\,\mathrm{div}_{x'}\varphi'\,\mathrm{d}x'\mathrm{d}y_3$$
$$+\frac{1}{\varepsilon}\int_\Omega \tilde{p}_\varepsilon\,\partial_{y_3}v_3\,\mathrm{d}x'\mathrm{d}y_3 - \frac{\mu}{K_\varepsilon}\int_\Omega \tilde{u}_\varepsilon\cdot v\,\mathrm{d}x'\mathrm{d}y_3 = \int_\Omega f'\cdot v'\,\mathrm{d}x'\mathrm{d}y_3 + O_\varepsilon. \tag{43}$$

The above variational formulation will be useful in the following.

We proceed in three steps depending on the magnitude of K_ε with respect to ε.

Step 1 Critical case $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$.

First, we prove that $\tilde{p}$ does not depend on the variable y_3. To do this, in (43) we consider as a test function $\varphi_\varepsilon(x', y_3) = (0, \varepsilon\varphi_3(x', y_3))$. Passing to the limit and using (36) and (40), we arrive at

$$\int_{\omega\times Y}\tilde{p}\,\partial_{y_3}\varphi_3\,\mathrm{d}x'\mathrm{d}y = 0$$

confirming that $\tilde{p}$ does not depend on y_3.

Now, we choose a test function $\varphi(x', y_3) = (\varphi'(x', y_3), 0)$ in (43) satisfying condition (37). By doing that, we can use convergences (36) and (40). When passing to the limit in (43), we take into account that $\frac{1}{K_\varepsilon^2} = \frac{\varepsilon^2}{K_\varepsilon}\frac{1}{\varepsilon^2}$ and that $\tilde{p}$ does not depend on y_3. As a result, we obtain

$$-\mu_e\int_\Omega \partial_{y_3}\tilde{u}'\cdot\partial_{y_3}\varphi'\,\mathrm{d}x'\mathrm{d}y_3 - \frac{\mu}{K}\int_\Omega \tilde{u}\cdot\varphi\,\mathrm{d}x'\mathrm{d}y_3 = \int_\Omega f'\cdot\varphi'\,\mathrm{d}x'\mathrm{d}y_3. \tag{44}$$

By density, (44) holds for every function v in the Hilbert space V defined by

$$V = \left\{ \begin{array}{ll} \varphi(x', y_3) \in H^1(h^-, h^+; L^2(\omega)^3), & \text{such that} \\ \mathrm{div}_{x'}\left(\displaystyle\int_{h^-(x')}^{h^+(x')} \varphi(x', y_3)\mathrm{d}y_3\right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{h^-(x')}^{h^+(x')} \varphi(x', y_3)\mathrm{d}y_3\right)\cdot n = 0 & \text{on } \omega \end{array} \right\}.$$

Due to Lax–Milgram lemma, variational formulation (44) in the Hilbert space V admits a unique solution $\tilde{u}$ in V. Following the same arguments as in [1], the orthogonal of V with respect to the usual scalar product in $H^1(h^-, h^+; L^2(\omega)^3)$ is made of gradients of the form $\nabla_{x'}\tilde{q}(x')$, with $\tilde{q}(x') \in L^2_0(\omega)$. Therefore, using the integration by parts, variational formulation (44) is equivalent to the effective system

$$\begin{cases} -\mu_e \partial^2_{y_3}\tilde{u}'(x', y_3) + \frac{\mu}{K}\tilde{u}'(x', y_3) = f'(x') - \nabla_{x'}\tilde{p}(x') & \text{in } \Omega, \\ \mathrm{div}_{x'}\left(\displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3)\mathrm{d}y_3\right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3)\mathrm{d}y_3\right)\cdot n = 0 & \text{on } \partial\omega. \end{cases} \tag{45}$$

It remains to prove that the pressure $\tilde{q}(x')$ arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div}_{x'}(\int_{h^-(x')}^{h^+(x')} \varphi(x', y_3)\mathrm{d}y_3) = 0$ is the same as the limit of the pressure $\tilde{p}_\varepsilon$. This can be easily done by multiplying Eq. (11) by a test function and identifying limits. Since (13) admits a unique solution, then the complete sequence $(\tilde{u}_\varepsilon/\varepsilon^2, \tilde{p}_\varepsilon)$ converges to the unique solution $(\tilde{u}(x', y_3), \tilde{p}(x'))$. This gives the desired result.

Since $(45)_1$ can be viewed as a linear second-order ODE (with constant coefficients) with respect to y_3, we get

$$\tilde{u}'(x', y_3) = \frac{K}{\mu}\left(A_1(x')e^{My_3} + A_2(x')e^{-My_3} - 1\right)\left(\nabla_{x'}\tilde{p}(x') - f'(x')\right).$$

Here, we introduce $M = \sqrt{\frac{\mu}{K\mu_e}}$, while the unknown functions $A_1(x')$, $A_2(x')$ are given by

$$A_1(x') = \frac{e^{-Mh^-(x')} - e^{-Mh^+(x')}}{e^{Mh^+(x') - Mh^-(x')} - e^{Mh^-(x') - Mh^+}},$$

$$A_2(x') = -\frac{e^{Mh^-(x')} - e^{Mh^+(x')}}{e^{Mh^+(x') - Mh^-(x')} - e^{Mh^-(x') - Mh^+}}.$$

By a simple integration with respect to y_3 from $h^-(x')$ to $h^+(x')$, we obtain (13).

Step 2 Subcritical case $K_\varepsilon \ll \varepsilon^2$. Similarly to the previous case, it can be proved that $\tilde{p}$ does not depend on y_3. In order to obtain the effective model, we take as a test function $\varphi_\varepsilon(x', y_3) = (K_\varepsilon/\varepsilon^2\varphi'(x', y_3), 0)$ in (43) with φ' satisfying (37). Thus, we can use convergences (36) and (41). When passing to the limit, we take into account that $K_\varepsilon/\varepsilon^2 \to 0$ to obtain

$$\mu \int_\Omega \tilde{u} \cdot \varphi \, dx' dy_3 = 0. \tag{46}$$

Again, by density, (46) holds for every function φ in the Hilbert space V. Proceeding as above, the variational formulation (46) is equivalent to the effective system

$$\begin{cases} \tilde{u}'(x', y_3) = -\frac{1}{\mu}\nabla_{x'}\tilde{p}(x') & \text{in } \Omega, \\ \mathrm{div}_{x'}\left(\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3) dy_3\right) = 0 & \text{in } \omega, \\ \left(\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3) dy_3\right) \cdot n = 0 & \text{on } \partial\omega, \end{cases} \tag{47}$$

which, after integrating with respect to y_3, gives (15).

Step 3 Supercritical case $K_\varepsilon \gg \varepsilon^2$. Similarly as above, it can be proved that $\tilde{p}$ does not depend on y_3. Proceeding as in Step 1 and taking into account that $\varepsilon^2/K_\varepsilon \to 0$, we obtain

$$-\mu_e \int_\Omega \partial_{y_3}\tilde{u}' : \partial_{y_3}\varphi' \, dx' dy_3 = \int_\Omega f' \cdot \varphi' \, dx, \tag{48}$$

which is equivalent to the effective problem

$$\begin{cases} -\mu_e \partial^2_{y_3}\tilde{u}'(x') = f' - \nabla_{x'}\tilde{p}(x') & \text{in } \omega, \\ \mathrm{div}_{x'}\left(\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3) dy_3\right) = 0 & \text{in } \omega, \\ \left(\int_{h^-(x')}^{h^+(x')} \tilde{u}'(x', y_3) dy_3\right) \cdot n = 0 & \text{on } \partial\omega \end{cases} \tag{49}$$

implying that

$$u(x', y_3) = \frac{(h^+(x') - y_3)(y_3 - h^-(x'))}{2\mu_e}(f'(x') - \nabla_{x'}\tilde{p}(x')),$$

which, after integrating with respect to y_3, gives (16). □

3 Oscillatory Case

In this section, we investigate the asymptotic behavior of a viscous fluid in the domain with highly oscillating boundaries, namely

$$\Lambda_\varepsilon = \left\{ (x', x_3) \in \mathbb{R}^3 \ : \ x' \in \omega, \ \varepsilon h^- \left(\frac{x'}{\varepsilon} \right) < x_3 < \varepsilon h^+ \left(\frac{x'}{\varepsilon} \right) \right\}.$$

In this setting, we consider h^-, h^+, with $h^- < h^+$, to be smooth functions defined for y' in $\mathbb{R}^2$, Y'-periodic, where $Y' = (-1/2, 1/2)^2$ is the 2D cell of periodicity. We define the 3D basic cell

$$Y = \left\{ y \in \mathbb{R}^3 \ : \ y' \in Y', \quad h^-(y') < y_3 < h^+(y') \right\}.$$

We define the values $h^-_{\min}$, $h^+_{\max}$ as

$$h^-_{\min} = \min_{y' \in Y'} h^-(y') \quad h^+_{\max} = \max_{y' \in Y'} h^+(y'). \tag{50}$$

We denote by $L^2_\sharp(Y)$, $H^1_\sharp(Y)$, the functional spaces

$$L^2_\sharp(Y) = \left\{ v \in L^2_{\mathrm{loc}}(Y) \ : \ \int_Y |v|^2 \mathrm{d}y < +\infty, \right.$$

$$\left. v(y' + k', y_3) = v(y) \forall k' \in \mathbb{Z}^2, \text{ a.e. } y \in Y \right\}$$

and

$$H^1_\sharp(Y) = \left\{ v \in H^1_{\mathrm{loc}}(Y) \cap L^2_\sharp(Y) : \int_Y |\nabla_y v|^2 \mathrm{d}y < +\infty \right\}.$$

As in Sect. 2, the porous medium flow is modeled by the Darcy–Lapwood–Brinkman equation (8), (9), now posed in Λ_ε. In view of that, let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Lambda_\varepsilon)^3 \times L^2(\Lambda_\varepsilon)$, which satisfies

$$\begin{cases} -\mu_e \Delta u_\varepsilon + \nabla p_\varepsilon + \dfrac{\mu}{K_\varepsilon} u_\varepsilon = f' - \dfrac{\rho}{\phi^2} (u_\varepsilon \cdot \nabla) u_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \end{cases} \tag{51}$$

endowed with

$$u_\varepsilon = 0 \ \text{ on } \ \partial \Lambda_\varepsilon. \tag{52}$$

As commented in Sect. 2, system (51), (52) has a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Lambda_\varepsilon)^3 \times L^2_0(\Lambda_\varepsilon)$.

Our goal is to find the asymptotic behavior of u_ε and p_ε when $\varepsilon \to 0$, depending on the magnitude of K_ε with respect to ε. For this purpose, we introduce the dilated variable $y_3 = \frac{x_3}{\varepsilon}$ in order to have the functions defined in an open set with a height of order one:

$$\widetilde{\Lambda}_\varepsilon = \left\{(x', y_3) \in \mathbb{R}^3 \,:\, x' \in \omega,\ h^-\left(\frac{x'}{\varepsilon}\right) < y_3 < h^+\left(\frac{x'}{\varepsilon}\right)\right\}.$$

In view of that, the functions $\tilde{u}_\varepsilon \in H_0^1(\widetilde{\Lambda}_\varepsilon)^3$, $\tilde{p}_\varepsilon \in L_0^2(\widetilde{\Lambda}_\varepsilon)$ are defined by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3),\ \ \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3),\ \ a.e.\ (x', y_3) \in \widetilde{\Lambda}_\varepsilon.$$

As a consequence, system (51) can be rewritten in $\widetilde{\Lambda}_\varepsilon$:

$$\begin{cases} -\mu_e \Delta_{x'} \tilde{u}_\varepsilon - \varepsilon^{-2}\mu_e \partial^2_{y_3}\tilde{u}_\varepsilon + \nabla_{x'}\tilde{p}_\varepsilon + \varepsilon^{-1}\partial_{y_3}\tilde{p}_\varepsilon e_3 + \dfrac{\mu}{K_\varepsilon}\tilde{u}_\varepsilon = f' - \dfrac{\rho}{\phi^2}(\tilde{u}_\varepsilon \cdot \nabla_\varepsilon)\tilde{u}_\varepsilon \\ \mathrm{div}_{x'}\, \tilde{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\tilde{u}_{\varepsilon,3} = 0 \end{cases} \tag{53}$$

with no-slip boundary condition on $\partial\widetilde{\Lambda}_\varepsilon$, i.e.,

$$\tilde{u}_\varepsilon = 0 \ \text{ on } \ \partial\widetilde{\Lambda}_\varepsilon. \tag{54}$$

It is essential to observe that the sequence of solutions $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon) \in H^1(\widetilde{\Lambda}_\varepsilon)^3 \times L_0^2(\widetilde{\Lambda}_\varepsilon)$ is not defined in a fixed domain (independent of ε) but in $\widetilde{\Lambda}_\varepsilon$ which varies with respect to ε. In order to pass the limit, the convergences in fixed Sobolev spaces (defined in an ε-independent set $\widetilde{\Lambda}$) are to be employed which requires first that $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ be extended to the whole domain

$$\widetilde{\Lambda} = \left\{(x', y_3) \in \mathbb{R}^3 \,:\, x' \in \omega,\ h^-_{\min} < y_3 < h^+_{\max}\right\}.$$

Here $h^-_{\min}$ and $h^+_{\max}$ are defined by (50).
The main result of the paper can be formulated as follows:

Theorem 2 *We distinguish the three characteristic cases:*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, then the extension $(\tilde{v}_\varepsilon/\varepsilon^2, \tilde{P}_\varepsilon)$ converges weakly, as ε tends to zero, in $H^1(h^-_{\min}, h^+_{\max}; L^2(\omega)^3) \times L_0^2(\omega)$ to $(\tilde{v}, \tilde{P})$, with $\tilde{v}_3 = 0$ and $\tilde{v}' = 0$ on $y_3 = h^-_{\min}, h^+_{\max}$. Moreover, $\tilde{P} \in H^1(\omega)$ and $(\widetilde{V}'(x'), \tilde{P}(x'))$ is the solution of the effective problem

$$\begin{cases} \tilde{V}'(x') = \frac{K}{\mu} A_M \left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) & \text{in } \omega, \\ \tilde{V}_3(x') = 0 & \\ \mathrm{div}_{x'}\, \tilde{V}'(x', y_3) = 0 & \text{in } \omega, \\ \tilde{V}'(x', y_3) \cdot n = 0 & \text{on } \partial\omega. \end{cases} \tag{55}$$

where $\tilde V(x') = \int_{h^-_{\min}}^{h^+_{\max}} \tilde v(x', y_3)\,\mathrm{d}y_3$ *and* $A_M \in \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ *is symmetric, positive definite and defined by its entries:*

$$(A_M)_{ij} = \int_Y Dw^i(y) : D_y w^j(y)\,\mathrm{d}y, \quad i, j = 1, 2. \tag{56}$$

Here, $w^i(y)$ $(i = 1, 2)$ *denote the unique solutions in* $H^1_\sharp(Y)^3$ *of the local Darcy–Brinkman problems in 3D:*

$$\begin{cases} -\frac{1}{M^2}\Delta_y w^i + \frac{1}{M^2\mu_e}\nabla_y q^i + w^i = e_i & \text{in } Y, \\ \mathrm{div}_y w^i = 0 & \text{in } Y, \\ w^i = 0 & \text{in } y_3 = h^-(y'), h^+(y'), \\ w^i, \pi^i \ Y' - \text{periodic}, & \end{cases} \tag{57}$$

with $M = \sqrt{\frac{\mu}{K\mu_e}}$.

(ii) if $K_\varepsilon \ll \varepsilon^2$*, then the extension* $(\tilde v_\varepsilon/\varepsilon^2, K_\varepsilon/\varepsilon^2 \tilde P_\varepsilon)$ *converges weakly, as* ε *tends to zero, in* $H^1(h^-_{\min}, h^+_{\max}; L^2(\omega)^3) \times L^2_0(\omega)$ *to* $(\tilde v, \tilde P)$*, with* $\tilde v_3 = 0$ *and* $\tilde v' = 0$ *on* $y_3 = h^-_{\min}, h^+_{\max}$*. Moreover,* $\tilde P \in H^1(\omega)$ *and* $(\widetilde{V}'(x'), \tilde P(x'))$ *is the solution of the effective problem*

$$\begin{cases} \tilde V'(x') = -\frac{A_0}{\mu}\nabla_{x'}\tilde P(x') & \text{in } \omega, \\ \tilde V_3(x') = 0, & \\ \mathrm{div}_{x'} \tilde V'(x') = 0 & \text{in } \omega, \\ \tilde V'(x')\cdot n = 0 & \text{in } \partial\omega, \end{cases} \tag{58}$$

where $\tilde V(x') = \displaystyle\int_{h^-_{\min}}^{h^+_{\max}} \tilde v(x', y_3)\,\mathrm{d}y_3$ *and* $A_0 \in \mathbb{R}^{2\times 2}$ *is a symmetric tensor defined by its entries:*

$$(A_0)_{ij} = \int_{Y'} (e^i + \nabla_{y'} q^i) e_j \,\mathrm{d}y', \quad i, j = 1, 2.$$

Here $q^i(y')$ $(i = 1, 2)$ *denote the unique solutions in* $H^1_\sharp(Y'_f)^2$ *of the local Hele-Shaw problems in 2D:*

$$\begin{cases} \Delta_{y'} q^i = 0 & \text{in } Y'_f, \\ (\nabla_{y'} q^i + e_i)\cdot n = 0 & \text{in } \partial Y', \\ q^i \ Y' - \text{periodic}. & \end{cases} \tag{59}$$

(iii) if $K_\varepsilon \gg \varepsilon^2$*, then the extension* $(\tilde v_\varepsilon/\varepsilon^2, \tilde P_\varepsilon)$ *converges weakly, as* ε *tends to zero, in* $H^1(h^-_{\min}, h^+_{\max}; L^2(\omega)^3) \times L^2_0(\omega)$ *to* $(\tilde v, \tilde P)$*, with* $\tilde v_3 = 0$ *and* $\tilde v' = 0$ *on* $y_3 = h^-_{\min}, h^+_{\max}$*. Moreover,* $\tilde P \in H^1(\omega)$ *and* $(\widetilde{V}'(x'), \tilde P(x'))$ *is the solution of the effective problem*

$$\begin{cases} \tilde{V}'(x') = \dfrac{A_\infty}{\mu_e}\Big(f'(x') - \nabla_{x'}\tilde{P}(x')\Big) & \text{in } \omega, \\ \tilde{V}_3(x') = 0 & \\ \mathrm{div}_{x'}\,\tilde{V}'(x', y_3) = 0 & \text{in } \omega, \\ \tilde{V}'(x', y_3)\cdot n = 0 & \text{on } \partial\omega. \end{cases} \tag{60}$$

where $\tilde{V}(x') = \int_{h^-_{\min}}^{h^+_{\max}} \tilde{v}(x', y_3)\,\mathrm{d}y_3$ *and* $A_\infty \in \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ *is symmetric, positive definite and defined by its entries:*

$$(A_\infty)_{ij} = \int_Y Dw^i(y) : D_y w^j(y)\,\mathrm{d}y, \quad \forall\, i, j = 1, 2. \tag{61}$$

Here, $w^i(y)$ $(i = 1, 2)$ *denote the unique solutions in* $H^1_\sharp(Y)^3$ *of the local Stokes problems in 3D*

$$\begin{cases} -\Delta_y w^i + \nabla_y q^i = e_i & \text{in } Y, \\ \mathrm{div}_y w^i = 0 & \text{in } Y, \\ w^i = 0 & \text{in } y_3 = h^-(y'), h^+(y'), \\ w^i, \pi^i \ Y' - \text{periodic}. & \end{cases} \tag{62}$$

3.1 Proof of the Main Result

A priori estimates: Using the same arguments as in Sect. 2, we derive the a priori estimates for $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$ in $\widetilde{\Lambda}_\varepsilon$.

Lemma 5 *For* u_ε *satisfying system* (53), (54)*,*

(i) if $K_\varepsilon \approx \varepsilon^2$*, with* $K_\varepsilon/\varepsilon^2 \to K$*,* $0 < K < +\infty$ *or* $K_\varepsilon \ll \varepsilon^2$*, the following estimate holds:*

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Lambda}_\varepsilon)^3} \le C\varepsilon^2. \tag{63}$$

(ii) if $K_\varepsilon \gg \varepsilon^2$*, the following estimate holds:*

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Lambda}_\varepsilon)^3} \le C\varepsilon K_\varepsilon^{\frac{1}{2}}. \tag{64}$$

Moreover, in every cases, it holds

$$\|D_{x'}\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Lambda}_\varepsilon)^{3\times 2}} \le C\varepsilon, \quad \|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Lambda}_\varepsilon)^3} \le C\varepsilon^2. \tag{65}$$

Now, we turn our attention to the pressure. As in the previous section, from Eq. (53) we can obtain the estimate for the pressure p_ε. However, now the constant C appearing in (30) depends on the domain Λ_ε, and, thus, the estimate for the corresponding

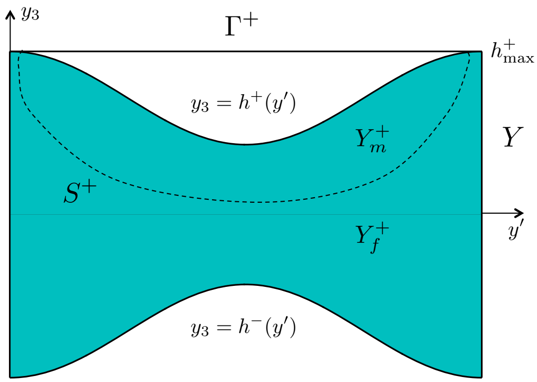


Fig. 1 Basic cell Y

pressure $\tilde{p}_\varepsilon$ may not be uniformly bounded when $\varepsilon \to 0$. For that reason, the idea is to extend the pressure $\tilde{p}_\varepsilon$ to the ε-independent domain $\widetilde{\Lambda}$.

The Extension of $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to the domain $\widetilde{\Lambda}$: It is easy to extend the velocity by zero in $\widetilde{\Lambda} \setminus \widetilde{\Lambda}_\varepsilon$ (this is compatible with its Dirichlet boundary condition on $\partial \widetilde{\Lambda}_\varepsilon$). We will denote by $\tilde{v}_\varepsilon$ the continuation of $\tilde{u}_\varepsilon$ in $\widetilde{\Lambda}$. It is well known that extension by zero preserves L^2 and H^1_0 norms. We note that the extension $\tilde{v}_\varepsilon$ belongs to $H^1_0(\widetilde{\Lambda})^3$.

However, extending the pressure is a much more difficult task. Tartar [29] introduced a continuation of the pressure for a flow in porous media. This construction applies to periodic holes in a domain $\widetilde{\Lambda}_\varepsilon$ when each hole is strictly contained into the periodic cell. In this context, we cannot use directly this result because the “holes” are along the top and bottom boundaries of Λ_ε, and moreover the scale of the vertical direction is smaller than the scales of the horizontal directions. This fact will induce several limitations in the results obtained by using the method, especially in view of the convergence for the pressure. In this sense, for the case of Newtonian fluids in a domain with a top boundary with roughness, Bayada and Chambat [8] and Mikelić [22] introduced an operator R^ε generalizing the results of Tartar [29] to this context. In our case, we need an operator R^ε between $H^1_0(Q_\varepsilon)^3$ and $H^1_0(\Lambda_\varepsilon)^3$ with similar properties, where $Q_\varepsilon = \omega \times (\varepsilon h^-_{\min}, \varepsilon h^+_{\max})$.

Following [8], we make a few more assumptions on the geometrical structure:

H1 The surface roughness is made of detached smooth humps periodically given on the upper (resp. the lower) part of the gap.

H2 We consider that the domain ω is covered by a finite number of periodic cells $Y'_{k',\varepsilon}$, of size ε, where for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$, with $Y' = (-1/2, 1/2)^2$. We define $T_\varepsilon = \left\{ k' \in \mathbb{Z}^2 \; : \; \omega \cap Y'_{k',\varepsilon} \neq \emptyset \right\}$.

We consider a smooth surface included in Y and surrounding the hump (in the top) such that Y is split into two areas Y^+_f and Y^+_m (see Fig. 1 for more details).

H3 ∂Y^+_m is a C^1 manifold.

We note

$$\Pi^+ = Y' \times (h^-(y'), h^+_{\max}), \quad S^+ = \partial Y^+_m \cap \partial Y^+_f.$$

We obtain the following result.

Lemma 6 *For given* $\tilde{\varphi} \in H^1(\Pi^+)^3$ *such that* $\tilde{\varphi} = 0$ *on* Γ^+, *there exists* $\tilde{w}^+ \in H^1(Y_m^+)^3$ *such that:*

$$\tilde{w}^+_{|_{S^+}} = \tilde{\varphi}_{|_{S^+}} \quad \text{and} \quad \tilde{w}^+_{|_{\partial Y_m^+ \setminus S^+}}.$$

Moreover, there exists a constant C *which does not depend on* $\tilde{\varphi}$ *such that:*

$$\begin{cases} \|\tilde{w}^+\|_{H^1(Y_m^+)^3} \leq C\|\tilde{\varphi}\|_{H^1(\Pi^+)^3}, \\ \mathrm{div}_\varepsilon \tilde{\varphi} = 0 \Rightarrow \mathrm{div}_\varepsilon \tilde{w}^+ = 0. \end{cases} \tag{66}$$

Proof It is analogous to the proof of Lemma 3.1 in [8]. □

Lemma 7 *There exists an operator* $R_\varepsilon^+ : H_0^1(Q_\varepsilon^+) \to H_0^1(\Lambda_\varepsilon)$ *such that*

1. $\varphi \in H_0^1(\Lambda_\varepsilon)^3 \Rightarrow R_\varepsilon^+(\varphi) = \varphi$,
2. $\mathrm{div}\, \varphi = 0 \Rightarrow \mathrm{div} R_\varepsilon^+(\varphi) = 0$.
3. *For any* $\varphi \in H_0^1(Q_\varepsilon)^3$, *we have*

$$\|R_\varepsilon^+(\varphi)\|_{L^2(\Lambda_\varepsilon)^3} \leq C\Big(\|\varphi\|_{L^2(Q_\varepsilon^+)^3} + \varepsilon\|D\varphi\|_{L^2(Q_\varepsilon^+)^{3\times 3}}\Big),$$

$$\|DR_\varepsilon^+(\varphi)\|_{L^2(\Lambda_\varepsilon)^{3\times 3}} \leq C\Big(\frac{1}{\varepsilon}\|\varphi\|_{L^2(Q_\varepsilon^+)^3} + \|D\varphi\|_{L^2(Q_\varepsilon^+)^{3\times 3}}\Big),$$

with constant C *independent of* φ *and* ε.

Proof For any $\tilde{\varphi} \in H_0^1(\Pi^+)^3$ such that $\tilde{\varphi} = 0$ on Γ^+, Lemma 8 allows us to define $R^+(\tilde{\varphi}) \in H^1(\Pi^+)^3$ by

$$R^+(\tilde{\varphi}) = \begin{cases} \tilde{\varphi} & \text{if } y \in Y_f^+, \\ \tilde{w}^+ & \text{if } y \in Y_m^+, \\ 0 & \text{if } y \in Y_s^+, \end{cases}$$

which satisfies

$$\int_{\Pi^+} |R^+(\tilde{\varphi})|^2\, \mathrm{d}y + \int_{\Pi^+} |D_y R^+(\tilde{\varphi})|^2\, \mathrm{d}y \leq C\left(\int_{\Pi^+} |\tilde{\varphi}|^2\, \mathrm{d}y + \int_{\Pi^+} |D_y\tilde{\varphi}|^2\, \mathrm{d}y\right) \tag{67}$$

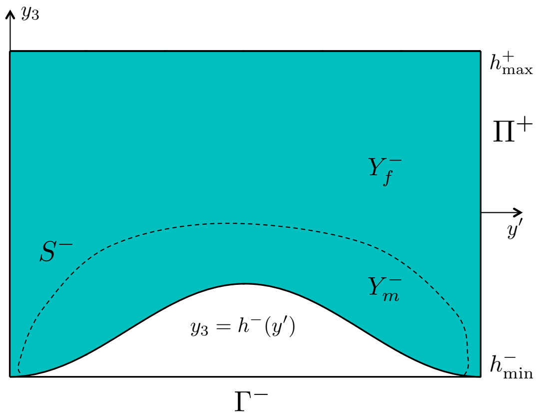
For every $k' \in T_\varepsilon$, by the change of variables

$$k' + y' = \frac{x'}{\varepsilon},\ y_3 = \frac{x_3}{\varepsilon},\ \mathrm{d}y = \frac{\mathrm{d}x}{\varepsilon^3},\ \partial_y = \varepsilon\, \partial_x, \tag{68}$$

we rescale (71) from Π^+ to $Q^+_{k',\varepsilon}$. This yields that, for every function $\varphi \in H^1(Q^+_{k',\varepsilon})^3$, one has

$$\int_{Q^+_{k',\varepsilon}} |R^+(\varphi)|^2\, \mathrm{d}x + \varepsilon^2 \int_{Q^+_{k',\varepsilon}} |D_x R^+(\varphi)|^2\, \mathrm{d}x$$

Fig. 2 Basic cell Π^+



$$\leq C\left(\int_{Q^+_{k',\varepsilon}} |\varphi|^2\,\mathrm{d}x + \varepsilon^2 \int_{Q^+_{k',\varepsilon}} |D_{x'}\varphi|^2\,\mathrm{d}x.\right)$$

We define R^+_ε by applying R^+ to each period $Q^+_{k',\varepsilon}$. Summing the previous inequalities for all the periods $Q_{k',\varepsilon}$, and taking into account that from (H2) we have $Q_\varepsilon = \cup_{k'\in T_\varepsilon} Q_{k',\varepsilon}$, gives

$$\int_{Q^+_\varepsilon} |R^+_\varepsilon(\varphi)|^2\,\mathrm{d}x + \varepsilon^2 \int_{Q^+_\varepsilon} |D_x R^+_\varepsilon(\varphi)|^2\,\mathrm{d}x$$
$$\leq C\left(\int_{Q^+_\varepsilon} |\varphi|^2\,\mathrm{d}x + \varepsilon^2 \int_{Q^+_\varepsilon} |D_x\varphi|^2\,\mathrm{d}x\right). \tag{69}$$

Obviously, $R^+_\varepsilon(\varphi)$ lies in $H^1_0(\Lambda_\varepsilon)^3$ and is equal to φ if φ is zero on $Q^+_\varepsilon \setminus \Lambda_\varepsilon$, so we get the estimates in 3. Moreover, the second item is obvious from $(66)_2$ and the definition of R^+_ε. □

We make a few more assumptions on the geometrical structure. Thus, we consider a smooth surface included in Π^+ and surrounding the hump such that Π^+ is split into two areas Y^-_f and Y^-_m (see Fig. 2 for more details). We also assume that ∂Y^-_m is a C^1 manifold.

We note

$$\Pi = Y' \times (h^-_{\min}, h^+_{\max}), \quad S^- = \partial Y^-_m \cap \partial Y^-_f.$$

Analogously, we have the following result.

Lemma 8 *For given $\tilde{\varphi} \in H^1(\Pi)^3$ such that $\tilde{\varphi} = 0$ on Γ^-, there exists $\tilde{w}^- \in H^1(Y^-_m)^3$ such that:*

$$\tilde{w}^-_{|_{S^-}} = \tilde{\varphi}_{|_{S^-}} \quad \textit{and} \quad \tilde{w}^-_{|_{\partial Y^-_m \setminus S^-}}.$$

Moreover, there exists a constant C which does not depend on $\tilde{\varphi}$ such that:

$$\begin{cases} \|\tilde{w}^-\|_{H^1(Y_m^-)^3} \leq C\|\tilde{\varphi}\|_{H^1(\Pi)^3}, \\ \mathrm{div}_\varepsilon\tilde{\varphi} = 0 \Rightarrow \mathrm{div}_\varepsilon\tilde{w}^- = 0. \end{cases} \tag{70}$$

Finally, we give the properties of the operator R_ε.

Lemma 9 *There exists an operator $R_\varepsilon : H_0^1(Q_\varepsilon) \to H_0^1(\Lambda_\varepsilon)$ such that*

1. $\varphi \in H_0^1(\Lambda_\varepsilon)^3 \Rightarrow R_\varepsilon(\varphi) = \varphi$,
2. $\mathrm{div}\,\varphi = 0 \Rightarrow \mathrm{div} R_\varepsilon(\varphi) = 0$.
3. *For any $\varphi \in H_0^1(Q_\varepsilon)^3$, we have*

$$\|R_\varepsilon(\varphi)\|_{L^2(\Lambda_\varepsilon)^3} \leq C\Big(\|\varphi\|_{L^2(Q_\varepsilon)^3} + \varepsilon\|D\varphi\|_{L^2(Q_\varepsilon)^{3\times3}}\Big),$$
$$\|DR_\varepsilon(\varphi)\|_{L^2(\Lambda_\varepsilon)^{3\times3}} \leq C\Big(\frac{1}{\varepsilon}\|\varphi\|_{L^2(Q_\varepsilon)^3} + \|D\varphi\|_{L^2(Q_\varepsilon)^{3\times3}}\Big),$$

with constant C independent of φ and ε.

Proof For any $\tilde{\varphi} \in H_0^1(\Pi)^3$ such that $\tilde{\varphi} = 0$ on Γ^-, Lemma 8 allows us to define $R(\tilde{\varphi}) \in H^1(\Pi)^3$ by

$$R(\tilde{\varphi}) = \begin{cases} R^+(\tilde{\varphi}) & \text{if } y \in Y_f^-, \\ \tilde{w}^- & \text{if } y \in Y_m^-, \\ 0 & \text{if } y \in Y_s^-, \end{cases}$$

which satisfies

$$\int_\Pi |R(\tilde{\varphi})|^2\,\mathrm{d}y + \int_\Pi |D_y R(\tilde{\varphi})|^2\,\mathrm{d}y \leq C\left(\int_\Pi |R^+(\tilde{\varphi})|^2\,\mathrm{d}y + \int_\Pi |D_y R^+(\tilde{\varphi})|^2\,\mathrm{d}y\right). \tag{71}$$

For every $k' \in T_\varepsilon$, by the change of variables (68), we rescale (71) from Π to $Q_{k',\varepsilon}$. This yields that, for every function $\varphi \in H^1(Q_{k',\varepsilon})^3$, one has

$$\int_{Q_{k',\varepsilon}} |R(\varphi)|^2\,\mathrm{d}x + \varepsilon^2\int_{Q_{k',\varepsilon}} |D_x R(\varphi)|^2\,\mathrm{d}x$$
$$\leq C\left(\int_{Q_{k',\varepsilon}} |R^+(\varphi)|^2\,\mathrm{d}x + \varepsilon^2\int_{Q_{k',\varepsilon}} |D_x R^+(\varphi)|^2\,\mathrm{d}x.\right)$$

We define R_ε by applying R to each period $Q_{k',\varepsilon}$. Summing the previous inequalities for all the periods $Q_{k',\varepsilon}$, and taking into account that from (H2) we have $Q_\varepsilon = \cup_{k'\in T_\varepsilon} Q_{k',\varepsilon}$, gives

$$\int_{Q_\varepsilon} |R_\varepsilon(\varphi)|^2\,dx + \varepsilon^2 \int_{Q_\varepsilon} |D_x R_\varepsilon(\varphi)|^2\,dx$$
$$\leq C\left(\int_{Q_\varepsilon} |R_\varepsilon^+(\varphi)|^2\,dx + \varepsilon^2 \int_{Q_\varepsilon} |D_x R_\varepsilon^+(\varphi)|^2\,dx\right),$$

which thanks to (69) gives

$$\int_{Q_\varepsilon} |R_\varepsilon(\varphi)|^2\,dx + \varepsilon^2 \int_{Q_\varepsilon} |D_x R_\varepsilon(\varphi)|^2\,dx \leq C\left(\int_{Q_\varepsilon} |\varphi|^2\,dx + \varepsilon^2 \int_{Q_\varepsilon} |D_x \varphi|^2\,dx\right),$$

Obviously, $R_\varepsilon(\varphi)$ lies in $H_0^1(\Lambda_\varepsilon)^3$ and is equal to φ if φ is zero on $Q_\varepsilon \setminus \Lambda_\varepsilon$, so we get the estimates in 3. Moreover, the second item is obvious from $(70)_2$ and the definition of R_ε. □

We obtain the following a priori estimates for the extension $(v_\varepsilon, P_\varepsilon)$ in the domain Q_ε.

Lemma 10 *There exists a constant C independent of ε, such that the extension $(v_\varepsilon, P_\varepsilon) \in H_0^1(Q_\varepsilon)^3 \times L_0^2(Q_\varepsilon)$ of a solution $(u_\varepsilon, p_\varepsilon)$ of problem (51), (52) satisfies*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, the following estimates hold

$$\|v_\varepsilon\|_{L^2(Q_\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}, \tag{72}$$
$$\|P_\varepsilon\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}. \tag{73}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, the following estimates hold

$$\|v_\varepsilon\|_{L^2(Q_\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}, \tag{74}$$
$$\|P_\varepsilon\|_{L^2(Q_\varepsilon)} \leq C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon}. \tag{75}$$

(iii) if $K_\varepsilon \gg \varepsilon^2$, the following estimates hold

$$\|v_\varepsilon\|_{L^2(Q_\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}}, \tag{76}$$
$$\|P_\varepsilon\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}. \tag{77}$$

Moreover, in every case it holds

$$\|Dv_\varepsilon\|_{L^2(Q_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3}{2}}. \tag{78}$$

Proof We first estimate the velocity. Taking into account Lemma 5, it is clear that, after extension, (72), (74), (76) and (78) hold.

The mapping R^ε defined in Lemma 7 allows us to extend the pressure p_ε to Q_ε introducing F_ε in $H^{-1}(Q_\varepsilon)^3$:

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = \langle \nabla p_\varepsilon, R^\varepsilon(\varphi)\rangle_{\Lambda_\varepsilon}, \quad \text{for any } \varphi \in H_0^1(Q_\varepsilon)^3. \tag{79}$$

We calculate the right-hand side of (79) by using (51) to obtain

$$\begin{aligned}\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = &-\mu_e \int_{\Lambda_\varepsilon} Du_\varepsilon : DR^\varepsilon(\varphi)\,\mathrm{d}x - \frac{\mu}{K_\varepsilon}\int_{\Lambda_\varepsilon} u_\varepsilon \cdot R^\varepsilon(\varphi)\,\mathrm{d}x \\ &+ \int_{\Lambda_\varepsilon} f' \cdot R_p^\varepsilon(\varphi)\,\mathrm{d}x - \frac{\rho}{\phi^2}\int_{\Lambda_\varepsilon} (u_\varepsilon\cdot\nabla)u_\varepsilon\, R^\varepsilon(\varphi)\,\mathrm{d}x.\end{aligned} \tag{80}$$

Moreover, $\mathrm{div}\varphi = 0$ implies

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = 0,$$

and the DeRham theorem (see, e.g., [16]) gives the existence of P_ε in $L_0^2(Q_\varepsilon)$ with $F_\varepsilon = \nabla P_\varepsilon$.

We introduce φ_ε as the function solution of the auxiliary problem

$$\mathrm{div}\,\varphi_\varepsilon = P_\varepsilon \in L_0^2(Q_\varepsilon) \quad \text{in } Q_\varepsilon, \quad \varphi_\varepsilon = 0 \quad \text{on } \partial Q_\varepsilon.$$

According to Lemma 2, such problem has at least one solution such that

$$\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\|P_\varepsilon\|_{L^2(Q_\varepsilon)}, \quad \|D\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le \frac{C}{\varepsilon}\|P_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

Thus, we get

$$\begin{aligned}\|P_\varepsilon\|_{L^2(Q_\varepsilon)} &= \left|\int_{Q_\varepsilon} P_\varepsilon\,\mathrm{div}\,\varphi_\varepsilon\,\mathrm{d}x\right| \\ &\le \mu_e\left|\int_{\Lambda_\varepsilon} Du_\varepsilon : DR^\varepsilon(\varphi)\,\mathrm{d}x\right| + \left|\frac{\mu}{K_\varepsilon}\int_{\Lambda_\varepsilon} u_\varepsilon\cdot R^\varepsilon(\varphi_\varepsilon)\,\mathrm{d}x\right| \\ &\quad + \left|\int_{\Lambda_\varepsilon} f'\cdot R^\varepsilon(\varphi)\,\mathrm{d}x\right| + \left|\frac{\rho}{\phi^2}\int_{\Lambda_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\, R^\varepsilon(\varphi)\,\mathrm{d}x\right|.\end{aligned} \tag{81}$$

Taking into account Lemma 7 iii) and Lemma 2 applied to the domain Q_ε, we conclude

$$\begin{aligned}&\|R^\varepsilon(\varphi_\varepsilon)\|_{L^2(\Lambda_\varepsilon)^3} \le C\Big(\|\varphi_\varepsilon\|_{L^2(Q_\varepsilon)^3} + \varepsilon\|D\varphi_\varepsilon\|_{L^2(Q_\varepsilon)^{3\times3}}\Big) \le C\|P_\varepsilon\|_{L^2(Q_\varepsilon)}, \\ &\|DR^\varepsilon(\varphi_\varepsilon)\|_{L^2(\Lambda_\varepsilon)^{3\times3}} \le C\Big(\frac{1}{\varepsilon}\|\varphi_\varepsilon\|_{L^2(Q_\varepsilon)^3} + \|D\varphi_\varepsilon\|_{L^2(Q_\varepsilon)^{3\times3}}\Big) \le \frac{C}{\varepsilon}\|P_\varepsilon\|_{L^2(Q_\varepsilon)}.\end{aligned}$$

Finally, proceeding as in the proof of Lemma 2, we deduce the desired estimates of the pressure in every case. □

Applying dilatation (10), we obtain the following a priori estimates for the extension $(\tilde{v}_\varepsilon, \tilde{P}_\varepsilon)$ in $\widetilde{\Lambda}$.

Corollary 3 *For the extension $(\tilde{v}_\varepsilon, \tilde{P}_\varepsilon)$ satisfying system (53), (54), we have*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, the following estimates hold

$$\|\tilde{v}_\varepsilon\|_{L^2(\widetilde{\Lambda})^3} \leq C\varepsilon^2, \tag{82}$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\widetilde{\Lambda})} \leq C. \tag{83}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, the following estimates hold

$$\|\tilde{v}_\varepsilon\|_{L^2(\widetilde{\Lambda})^3} \leq C\varepsilon^2, \tag{84}$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\widetilde{\Lambda})} \leq C\frac{\varepsilon^2}{K_\varepsilon}. \tag{85}$$

(iii) if $K_\varepsilon \gg \varepsilon^2$, the following estimates hold

$$\|\tilde{v}_\varepsilon\|_{L^2(\widetilde{\Lambda})^3} \leq C\varepsilon K_\varepsilon^{\frac{1}{2}}, \tag{86}$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\widetilde{\Lambda})} \leq C. \tag{87}$$

Moreover, in every case it holds

$$\|D_{x'}\tilde{v}_\varepsilon\|_{L^2(\widetilde{\Lambda})^{3\times 2}} \leq C\varepsilon, \quad \|\partial_{y_3}\tilde{v}_\varepsilon\|_{L^2(\widetilde{\Lambda})^3} \leq C\varepsilon^2. \tag{88}$$

Adaptation of the Unfolding Method: The change of variable (10) does not provide the information we need about the behavior of $\tilde{u}_\varepsilon$ in the microstructure associated with $\widetilde{\Lambda}_\varepsilon$. To solve this difficulty, we introduce an adaptation of the unfolding method (see [6,14] for more details). First, we explain the notation used in the sequel. Recalling that $Y' = (-1/2, 1/2)^2$, $Y_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$, for every $k' \in T_\varepsilon$, and that the basic cell is given by

$$Y = \left\{ y \in \mathbb{R}^3 \,:\, y' \in Y', \quad h^-(y') < y_3 < h^+(y') \right\},$$

we define $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (h^-(y'), h^+(y'))$ for every $k' \in T_\varepsilon$. We also define the extension of the basic cell by

$$\Pi = Y' \times (h^-_{\min}, h^+_{\max}).$$

The corresponding cubes of size ε and height $\varepsilon(h^+_{\max} - h^-_{\min})$ are given by $Q_{k',\varepsilon} = Y'_{k',\varepsilon} \times (\varepsilon h^-_{\min}, \varepsilon h^+_{\max})$ and $\widetilde{Q}_{k',\varepsilon} = Y'_{k',\varepsilon} \times (h^-_{\min}, h^+_{\max})$.

Given $\tilde{u}_\varepsilon \in H_0^1(\widetilde{\Lambda}_\varepsilon)^3$ a solution of the rescaled system (53), extended by zero outside of $\widetilde{\Lambda}_\varepsilon$, we define $\hat{u}_\varepsilon$, by

$$\hat{u}_\varepsilon(x', y) = \tilde{u}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right) + \varepsilon y', y_3\right), \text{ a.e. } (x', y) \in \omega \times Y. \tag{89}$$

Here, the function κ is defined as follows: For $k' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \Longleftrightarrow x' \in Y'_{k',1}.$$

Note that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial Y'_{k',1}$). Moreover, for every $\varepsilon > 0$, we have

$$\kappa\left(\frac{x'}{\varepsilon}\right) = k' \Longleftrightarrow x' \in Y'_{k',\varepsilon}.$$

In the same sense, given the extension of the pressure $\tilde{P}_\varepsilon \in L_0^2(\widetilde{\Lambda})$, we define $\hat{P}_\varepsilon$ by

$$\hat{P}_\varepsilon(x', y) = \tilde{P}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right) + \varepsilon y', y_3\right), \text{ a.e. } (x', y) \in \omega \times \Pi. \tag{90}$$

Remark 1 For $k' \in T_\varepsilon$, the restrictions of $\hat{u}_\varepsilon$ to $Y'_{k',\varepsilon} \times Y$ and $\hat{P}_\varepsilon$ to $Y'_{k',\varepsilon} \times \Pi$ do not depend on x'. As a function of y, it is obtained from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ by using the change of variables

$$y' = \frac{x' - \varepsilon k'}{\varepsilon}, \tag{91}$$

transforming $Y_{k',\varepsilon}$ into Y and $\widetilde{Q}_{k',\varepsilon}$ into Π, respectively.

Lemma 11 *There exists a constant C independent of ε, such that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (89), (90) satisfies*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, the following estimates hold

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon^2, \tag{92}$$

$$\|\hat{P}_\varepsilon\|_{L^2(\omega\times \Pi)} \leq C. \tag{93}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, the following estimates hold

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon^2, \tag{94}$$

$$\|\hat{P}_\varepsilon\|_{L^2(\omega\times \Pi)} \leq C\frac{\varepsilon^2}{K_\varepsilon}. \tag{95}$$

(iii) if $K_\varepsilon \gg \varepsilon^2$, the following estimates hold

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon K_\varepsilon^{\frac{1}{2}}, \tag{96}$$
$$\|\hat{P}_\varepsilon\|_{L^2(\omega\times \Pi)} \leq C. \tag{97}$$

Moreover, in every case it holds

$$\|D_{y'}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 2}} \leq C\varepsilon^2, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon^2. \tag{98}$$

Proof Let us first derive some estimates for the sequence $\hat{u}_\varepsilon$ defined by (89). We obtain

$$\begin{aligned}
&\int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(x',y)\right|^2 dx'dy\\
&= \sum_{k'\in T_\varepsilon} \int_{Y'_{k',\varepsilon}} \int_Y \left|D_{y'}\hat{u}_\varepsilon(x',y)\right|^2 dx'dy\\
&= \sum_{k'\in T_\varepsilon} \int_{Y'_{k',\varepsilon}} \int_{Y'} \int_{h^-(y')}^{h^+(y')} \left|D_{y'}\tilde{u}_\varepsilon(\varepsilon k'+\varepsilon y', y_3)\right|^2 dx'dy'dy_3.
\end{aligned}$$

We observe that $\tilde{u}_\varepsilon$ does not depend on x' so we deduce

$$\begin{aligned}
&\int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(x',y)\right|^2 dx'dy\\
&\quad= \varepsilon^2 \sum_{k'\in T_\varepsilon} \int_{Y'} \int_{h^-(y')}^{h^+(y')} \left|D_{y'}\tilde{u}_\varepsilon(\varepsilon k'+\varepsilon y', y_3)\right|^2 dy'dy_3.
\end{aligned}$$

Using the change of variables (91) and the Y'-periodicity of h^- and h^+, we get

$$\begin{aligned}
&\int_{\omega\times Y} \left|D_{y'}\hat{u}_\varepsilon(x',y)\right|^2 dx'dy\\
&\quad= \varepsilon^2 \sum_{k'\in T_\varepsilon} \int_{Y'_{k',\varepsilon}} \int_{h^-(\frac{x'}{\varepsilon}-k')}^{h^+(\frac{x'}{\varepsilon}-k')} \left|D_{x'}\tilde{u}_\varepsilon(x',y_3)\right|^2 dx'dy_3\\
&\quad= \varepsilon^2 \sum_{k'\in T_\varepsilon} \int_{Y'_{k',\varepsilon}} \int_{h^-(\frac{x'}{\varepsilon})}^{h^+(\frac{x'}{\varepsilon})} \left|D_{x'}\tilde{u}_\varepsilon(x',y_3)\right|^2 dx'dy_3\\
&\quad= \varepsilon^2 \int_{\widetilde{\Lambda}_\varepsilon} \left|D_{x'}\tilde{u}_\varepsilon(x',y_3)\right|^2 dx'dy_3.
\end{aligned}$$

Employing estimate $(65)_1$, we deduce $(98)_1$.

Similarly, using Remark 1 and definition (89), we have

$$\int_{\omega\times Y}\left|\partial_{y_3}\hat{u}_\varepsilon(x',y)\right|^2 \mathrm{d}x'\mathrm{d}y \le \varepsilon^2\sum_{k'\in T_\varepsilon}\int_Y\left|\partial_{y_3}\tilde{u}_\varepsilon(\varepsilon k'+\varepsilon y',y_3)\right|^2\mathrm{d}y.$$

Using the change of variables (91) and estimate $(65)_2$, we obtain

$$\int_{\omega\times Y}\left|\partial_{y_3}\hat{u}_\varepsilon(x',y)\right|^2 \mathrm{d}x'\mathrm{d}y \le \int_{\widetilde{\Lambda}_\varepsilon}\left|\partial_{y_3}\tilde{u}_\varepsilon(x',y_3)\right|^2\mathrm{d}x'\mathrm{d}y_3 \le C\varepsilon^4,$$

proving $(98)_2$.

Similarly, using definition (89), the change of variables (91) and estimates (63) and (64), we have in the cases $K_\varepsilon \approx \varepsilon^2$ and $K_\varepsilon \ll \varepsilon^2$ that

$$\int_{\omega\times Y}\left|\hat{u}_\varepsilon(x',y)\right|^2 \mathrm{d}x'\mathrm{d}y \le C\varepsilon^4,$$

whereas, in the case $K_\varepsilon \gg \varepsilon^2$, it holds

$$\int_{\omega\times Y}\left|\hat{u}_\varepsilon(x',y)\right|^2 \mathrm{d}x'\mathrm{d}y \le C\varepsilon^2 K_\varepsilon,$$

implying (92), (94) and (96).

Finally, let us obtain some estimates for the sequence $\hat{P}_\varepsilon$ defined by (90). We observe that using definition (90) of $\hat{P}_\varepsilon$, we obtain

$$\int_{\omega\times\Pi}\left|\hat{P}_\varepsilon(x',y)\right|^{p'} \mathrm{d}x'\mathrm{d}y \le \sum_{k'\in T_\varepsilon}\int_{Y'_{k',\varepsilon}}\int_{Y'}\int_{h^-_{\min}}^{h^+_{\max}}\left|\tilde{P}_\varepsilon(\varepsilon k'+\varepsilon y',y_3)\right|^2\mathrm{d}x'\mathrm{d}y.$$

We also note that $\tilde{P}_\varepsilon$ does not depend on x' so we have

$$\int_{\omega\times\Pi}\left|\hat{P}_\varepsilon(x',y)\right|^{2} \mathrm{d}x'\mathrm{d}y \le \varepsilon^2\sum_{k'\in T_\varepsilon}\int_{Y'}\int_{h^-_{\min}}^{h^+_{\max}}\left|\tilde{P}_\varepsilon(\varepsilon k'+\varepsilon y',y_3)\right|^2\mathrm{d}y'\mathrm{d}y_3.$$

By the change of variables (91), we get

$$\int_{\omega\times\Pi}\left|\hat{P}_\varepsilon(x',y)\right|^{2} \mathrm{d}x'\mathrm{d}y \le \int_{\widetilde{\Lambda}}\left|\tilde{P}_\varepsilon(x',y_3)\right|^2\mathrm{d}x'\mathrm{d}y_3.$$

Taking into account (83), (85) and (87), we deduce (93), (95) and (97), respectively. □

Some compactness results: From the a priori estimates of the extension $(\tilde{v}_\varepsilon, \tilde{P}_\varepsilon)$, we can deduce the following compactness results.

Lemma 12 *Consider the extension $\tilde{v}_\varepsilon$ of $\widetilde{u}_\varepsilon$ satisfying system (53), (54). Then, there exists $\tilde{v} \in H^1(h^-_{\min}, h^+_{\max}; L^2(\omega))^3$ where $\tilde{u}_3 = 0$ and $\tilde{u} = 0$ on $y_3 = h^-_{\min}, h^+_{\max}$, such that*

$$\frac{\tilde{v}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{v}', 0) \text{ in } H^1(h^-_{\min}, h^+_{\max}; L^2(\omega))^3, \quad \text{as } \varepsilon \to 0, \tag{99}$$

$$\mathrm{div}_{x'}\left(\int_{h^-_{\min}}^{h^+_{\max}} \tilde{v}'(x', y_3)\mathrm{d}y_3\right) = 0 \text{ in } \omega, \quad \left(\int_{h^-_{\min}}^{h^+_{\max}} \tilde{v}'(x', y_3)\mathrm{d}y_3\right) \cdot n = 0 \text{ on } \partial\omega. \tag{100}$$

We omit the proof since it is similar to the proof of Lemma 3 considering the domain $\widetilde{\Lambda}$ instead of Ω.

Lemma 13 *Consider the extension $\tilde{P}_\varepsilon$ of $\tilde{p}_\varepsilon$ satisfying system (53), (54). Then,*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$ of $K_\varepsilon \gg \varepsilon^2$, then there exists $\tilde{P} \in L^2_0(\widetilde{\Lambda})$ such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \ \text{ in } \ L^2(\widetilde{\Lambda}), \quad \text{as } \varepsilon \to 0. \tag{101}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, then there exists $\tilde{P} \in L^2_0(\widetilde{\Lambda})$ such that

$$\frac{K_\varepsilon}{\varepsilon^2}\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \ \text{ in } \ L^2(\widetilde{\Lambda}), \quad \text{as } \varepsilon \to 0. \tag{102}$$

Again we omit the proof since it is similar to the proof of Lemma 4 considering the domain $\widetilde{\Lambda}$ instead of Ω.
Next, from the a priori estimates of $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$, we can prove the following compactness results:

Lemma 14 *For a subsequence of ε still denoted by ε, there exists $\hat{u} \in L^2(\omega; H^1_\sharp(Y)^3)$, with $\int_Y \hat{u}_3 \,\mathrm{d}y = 0$ and $\hat{u} = 0$ on $y_3 = h^-, h^+$, such that*

$$\varepsilon^{-2}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\omega; H^1(Y)^3), \quad \text{as } \varepsilon \to 0, \tag{103}$$

$$\mathrm{div}_y \hat{u} = 0 \text{ in } \omega \times Y. \tag{104}$$

Moreover, we have

$$\mathrm{div}_{x'}\left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) = 0 \text{ in } \omega, \quad \left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) \cdot n = 0 \text{ on } \partial\omega. \tag{105}$$

Proof We start with the case $K_\varepsilon \approx \varepsilon^2$. Estimates (92) and (98) imply the existence of $\hat{u} : \omega \times Y \to \mathbb{R}^3$, such that, up to a subsequence, (103) holds. By the semicontinuity and estimates given in (92) and (98), we have

$$\int_{\omega\times Y} \left|\hat{u}\right|^2 \mathrm{d}x'\mathrm{d}y \leq C, \quad \int_{\omega\times Y} \left|D_y\hat{u}\right|^2 \mathrm{d}x'\mathrm{d}y \leq C,$$

confirming that $\hat{u} \in L^2(\omega; H^1(Y)^3)$.

It remains to prove the Y'-periodicity of $\hat{u}$ in y'. This can be accomplished by proceeding as in Lemma 5.4 from Casado-Díaz et al. [11].

Since $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ in $\widetilde{\Lambda}_\varepsilon$, by definition of $\hat{u}_\varepsilon$ we have $\varepsilon^{-1}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-1}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0$. Multiplying by ε^{-1}, we obtain

$$\varepsilon^{-2}\mathrm{div}_{y'}\hat{u}'_\varepsilon + \varepsilon^{-2}\partial_{y_3}\hat{u}_{\varepsilon,3} = 0, \quad \text{in } \omega \times Y,$$

which, combined with (103), proves (104).
Observe that the case $K_\varepsilon \ll \varepsilon^2$ is analogous to the previous one, so we omit the proof.

For the case $K_\varepsilon \gg \varepsilon^2$, estimates (96) and (98) imply that $\varepsilon^{-1}K_\varepsilon^{-1/2}\hat{u}_\varepsilon$ tends to zero and we have the same result as in the previous cases.

Finally, in order to prove (105), let us first prove the following relation between $\tilde{v}$ and $\hat{u}$,

$$\frac{1}{|Y'|}\int_Y \hat{u}(x', y)\mathrm{d}y = \int_{h_{\min}^-}^{h_{\max}} \tilde{v}(x', y_3)\mathrm{d}y_3. \tag{106}$$

For this, let us consider $\varphi \in C_c^1(\omega)$. We observe that, using definition (89) of $\hat{u}_\varepsilon$, we obtain

$$\begin{aligned}&\varepsilon^{-2}\int_\omega\int_Y \hat{u}_\varepsilon(x', y)\varphi(x')\mathrm{d}y\mathrm{d}x'\\ &\quad= \varepsilon^{-2}\sum_{k'\in T_\varepsilon}\int_{Y'_{k',\varepsilon}}\int_Y \tilde{u}_\varepsilon(\varepsilon k' + \varepsilon y', y_3)\,\varphi(\varepsilon k' + \varepsilon y')\mathrm{d}y\mathrm{d}x' + O_\varepsilon.\end{aligned}$$

Further, we note that $\tilde{u}_\varepsilon$ and φ do not depend on x' so we have

$$\begin{aligned}&\varepsilon^{-2}\int_\omega\int_Y \hat{u}_\varepsilon(x', y)\varphi(x')\mathrm{d}y\mathrm{d}x'\\ &\quad= |Y'|\sum_{k'\in T_\varepsilon}\int_{Y'}\int_{h^-(y')}^{h^+(y')} \tilde{u}_\varepsilon(\varepsilon k' + \varepsilon y', y_3)\,\varphi(\varepsilon k' + \varepsilon y')\mathrm{d}y_3\mathrm{d}y' + O_\varepsilon.\end{aligned}$$

Using the change of variables (91) and the Y'-periodicity of $h^\pm$, we get

$$\begin{aligned}&\varepsilon^{-2}\int_\omega\int_Y \hat{u}_\varepsilon(x', y)\varphi(x')\mathrm{d}y\mathrm{d}x'\\ &\quad= \varepsilon^{-2}|Y'|\sum_{k'\in T_\varepsilon}\int_{Y'_{k',\varepsilon}}\int_{h^-(x'/\varepsilon)}^{h^+(x'/\varepsilon)} \tilde{u}_\varepsilon(x', y_3)\,\varphi(x')\mathrm{d}y_3\mathrm{d}x' + O_\varepsilon\end{aligned}$$

$$= \varepsilon^{-2}|Y'| \sum_{k' \in T_\varepsilon} \int_{Y'_{k',\varepsilon}} \int_{h^-_{\min}}^{h_{\max}} \tilde{v}_\varepsilon(x', y_3)\, \varphi(x')\mathrm{d}y_3\mathrm{d}x' + O_\varepsilon$$
$$= \varepsilon^{-2}|Y'| \int_\Omega \tilde{v}_\varepsilon(x', y_3)\, \varphi(x')\mathrm{d}y_3\mathrm{d}x' + O_\varepsilon.$$

Taking into account (99) and (103), we obtain (106). Since $\tilde{v}_3 = 0$, we deduce that $\int_Y \hat{u}_3\, \mathrm{d}y = 0$ a.e. in ω. Finally, relation (106) together with (100) yields (105). □

Lemma 15 *For a subsequence of ε still denoted by ε,*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$ of $K_\varepsilon \gg \varepsilon^2$, then there exists $\hat{P} \in L^2(\omega \times \Pi)$ such that

$$\hat{P}_\varepsilon \rightharpoonup \hat{P} \ \text{ in } \ L^2(\omega \times \Pi), \quad \text{as } \varepsilon \to 0. \tag{107}$$

(ii) if $K_\varepsilon \ll \varepsilon^2$, then there exists $\hat{P} \in L^2(\omega \times \Pi)$ such that

$$\frac{K_\varepsilon}{\varepsilon^2}\hat{P}_\varepsilon \rightharpoonup \tilde{P} \ \text{ in } \ L^2(\omega \times \Pi), \quad \text{as } \varepsilon \to 0. \tag{108}$$

Effective problems: The idea is to multiply system (53) by a test function having the form of the limit $\hat{u}$ (as explicated in Lemma 14).

Theorem 3 *It holds:*

(i) if $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$, then $(\varepsilon^{-2}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges, as ε tends to zero, to the unique solution $(\hat{u}(x', y), \tilde{P}(x'))$ in $L^2(\omega; H^1(Y)^3) \times L^2_0(\omega)$, with $\int_Y \hat{u}_3\, \mathrm{d}y = 0$, of the effective problem

$$\begin{cases} -\mu_e \Delta_y \hat{u} + \nabla_y \hat{q} + \frac{\mu}{K}\hat{u} = f' - \nabla_{x'}\tilde{P} & \text{in } \omega \times Y, \\ \mathrm{div}_y \hat{u} = 0 & \text{in } \omega \times Y, \\ \hat{u} = 0 & \text{on } y_3 = h^-(y'), h^+(y'), \\ \mathrm{div}_{x'}\left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) = 0 & \text{in } \omega, \\ \left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) \cdot n = 0 & \text{on } \partial\omega, \\ y' \to \hat{u}, \hat{q} & Y' - \text{periodic}. \end{cases} \tag{109}$$

(ii) if $K_\varepsilon \ll \varepsilon$, then $(\varepsilon^{-2}\hat{u}_\varepsilon, K_\varepsilon/\varepsilon^2\hat{P}_\varepsilon)$ converges, as ε tends to zero, to the unique solution $(\hat{u}(x', y), \tilde{P}(x'))$ in $L^2(\omega; H^1(Y)^3) \times L^2_0(\omega)$, with $\int_Y \hat{u}_3 \mathrm{d}y = 0$, of the effective problem

$$\begin{cases} \mu\, \hat{u} + \nabla_y \hat{q} = -\nabla_{x'}\tilde{P} & \text{in } \omega \times Y \\ \mathrm{div}_y \hat{u}' = 0 & \text{in } \omega \times Y, \\ \hat{u}' = 0 & \text{on } y_3 = h^-(y'), h^+(y'), \\ \mathrm{div}_{x'}\left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) = 0 & \text{in } \omega, \\ \left(\int_Y \hat{u}'(x', y)\mathrm{d}y\right) \cdot n = 0 & \text{on } \partial\omega, \\ y' \to \hat{u}', \hat{q} & Y' - \text{periodic}. \end{cases} \tag{110}$$

(iii) if $K_\varepsilon \gg \varepsilon^2$, then $(\varepsilon^{-2}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges, as ε tends to zero, to the unique solution $(\hat{u}(x', y), \tilde{P}(x'))$ in $L^2(\omega; H^1(Y)^3) \times L^2_0(\omega)$, with $\int_Y \hat{u}_3\, \mathrm{d}y = 0$, of the effective problem

$$\begin{cases} -\mu_e \Delta_y \hat{u} + \nabla_y \hat{q} = f' - \nabla_{x'} \tilde{P} & \text{in } \omega \times Y, \\ \mathrm{div}_y \hat{u} = 0 & \text{in } \omega \times Y, \\ \hat{u} = 0 & \text{on } y_3 = h^-(y'), h^+(y'), \\ \mathrm{div}_{x'} \left(\int_Y \hat{u}'(x', y) \mathrm{d}y \right) = 0 & \text{in } \omega, \\ \left(\int_Y \hat{u}'(x', y) \mathrm{d}y \right) \cdot n = 0 & \text{on } \partial\omega, \\ y' \to \hat{u}, \hat{q} & Y' - \text{periodic}. \end{cases} \tag{111}$$

Proof First of all, we choose a test function $\varphi(x', y) \in \mathcal{D}(\omega; C^\infty_\sharp(Y)^3)$. Multiplying (53) by $\varphi(x', x'/\varepsilon, y_3)$, integrating by parts and taking into account the same reasoning as in (42), we get

$$\int_{\widetilde{\Lambda}_\varepsilon} (\tilde{u}_\varepsilon \cdot \nabla_\varepsilon) \tilde{u}_\varepsilon\, \varphi\, \mathrm{d}x' \mathrm{d}y_3 = O_\varepsilon.$$

As a result,

$$\mu_e \int_{\widetilde{\Lambda}_\varepsilon} D_\varepsilon \tilde{u}_\varepsilon : \left(D_{x'} \varphi + \frac{1}{\varepsilon} D_y \varphi \right) \mathrm{d}x' \mathrm{d}y_3 + \int_{\widetilde{\Lambda}_\varepsilon} \nabla_\varepsilon \tilde{p}_\varepsilon\, \varphi\, \mathrm{d}x' \mathrm{d}y_3$$
$$+ \frac{\mu}{K_\varepsilon} \int_{\widetilde{\Lambda}_\varepsilon} \tilde{u}_\varepsilon \cdot \varphi\, \mathrm{d}x' \mathrm{d}y_3 = \int_{\widetilde{\Lambda}_\varepsilon} f' \cdot \varphi'\, \mathrm{d}x' \mathrm{d}y_3 + O_\varepsilon.$$

Taking into account the prolongation of the pressure, we have

$$\int_{\widetilde{\Lambda}_\varepsilon} \nabla_\varepsilon \tilde{p}_\varepsilon\, \varphi'\, \mathrm{d}x' \mathrm{d}y_3 = \int_{\widetilde{\Lambda}} \nabla_\varepsilon \tilde{P}_\varepsilon\, \varphi\, \mathrm{d}x' \mathrm{d}y_3,$$

and so

$$\mu_e \int_{\widetilde{\Lambda}_\varepsilon} D_\varepsilon \tilde{u}_\varepsilon : \left(D_{x'} \varphi + \frac{1}{\varepsilon} D_y \varphi \right) \mathrm{d}x' \mathrm{d}y_3 - \int_{\widetilde{\Lambda}} \tilde{P}_\varepsilon\, \mathrm{div}_{x'} \varphi'\, \mathrm{d}x' \mathrm{d}y_3$$
$$- \frac{1}{\varepsilon} \int_{\widetilde{\Lambda}} \tilde{P}_\varepsilon\, \mathrm{div}_y \varphi\, \mathrm{d}x' \mathrm{d}y_3 + \frac{\mu}{K_\varepsilon} \int_{\widetilde{\Lambda}_\varepsilon} \tilde{u}_\varepsilon \cdot \varphi\, \mathrm{d}x' \mathrm{d}y_3 = \int_{\widetilde{\Omega}_\varepsilon} f' \cdot \varphi'\, \mathrm{d}x' \mathrm{d}y_3 + O_\varepsilon. \tag{112}$$

By the change of variables given in Remark 1, we obtain

$$\frac{\mu_e}{\varepsilon^2} \int_{\omega \times Y} D_y \hat{u}_\varepsilon : D_y \varphi\, \mathrm{d}x' \mathrm{d}y - \int_{\omega \times \Pi} \hat{P}_\varepsilon\, \mathrm{div}_{x'} \varphi'\, \mathrm{d}x' \mathrm{d}y - \frac{1}{\varepsilon} \int_{\omega \times \Pi} \hat{P}_\varepsilon\, \mathrm{div}_y \varphi\, \mathrm{d}x' \mathrm{d}y$$
$$- \frac{\mu}{K_\varepsilon} \int_{\omega \times Y} \hat{u}_\varepsilon \cdot \varphi\, \mathrm{d}x' \mathrm{d}y = \int_{\omega \times Y} f' \cdot \varphi'\, \mathrm{d}x' \mathrm{d}y + O_\varepsilon. \tag{113}$$

This variational formulation will prove useful in the following steps.

Step 1 Case $K_\varepsilon \approx \varepsilon^2$, with $K_\varepsilon/\varepsilon^2 \to K$, $0 < K < +\infty$.

First, we prove that $\hat{P}$ does not depend on the microscopic variable y. To do this, we consider as the test function $\varphi_\varepsilon(x', y_3) = \varepsilon\varphi(x', x'/\varepsilon, y_3)$ in (113). Passing to the limit and using convergences (103) and (107), we get

$$\int_{\omega\times\Pi} \hat{P}\,\mathrm{div}_y\varphi\,\mathrm{d}x'\mathrm{d}y = 0,$$

implying that $\hat{P}$ does not depend on y.

We consider now $\varphi \in \mathcal{D}(\omega; C_\sharp^\infty(Y)^3)$ with $\mathrm{div}_{y'}\varphi = 0$ in $\omega \times Y$ and $\mathrm{div}_{x'}(\int_Y \varphi'\,\mathrm{d}y) = 0$ in ω. Then, when passing to the limit in (113) and taking into account (103), (107), the second term contributes nothing because the limit of $\hat{P}_\varepsilon$ does not depend on y and $\hat{u}'$ satisfies $\mathrm{div}_{x'}(\int_Y \varphi'\,\mathrm{d}y) = 0$. Now, if we pass to the limit in (113), we get

$$\mu_e \int_{\omega\times Y} D_y\hat{u} : D_y\varphi\,\mathrm{d}x'\mathrm{d}y + \frac{\mu}{K}\int_{\omega\times Y} \hat{u}\cdot\varphi\,\mathrm{d}x'\mathrm{d}y = \int_{\omega\times Y} f'\cdot\varphi'\,\mathrm{d}x'\mathrm{d}y. \quad (114)$$

By density, this holds for every function φ in the Hilbert space V defined by

$$V = \left\{ \begin{array}{lll} \varphi(x', y) \in L^2(\omega; H_\sharp^1(Y)^3), & \text{such that} & \\ \mathrm{div}_y\varphi(x', y) = 0 & \text{in } \omega\times Y,\ \mathrm{div}_{x'}\left(\int_Y \varphi(x', y)\,\mathrm{d}y\right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_Y \varphi(x', y)\,\mathrm{d}y\right)\cdot n = 0 & \text{on } \omega & \end{array} \right\}.$$

By Lax–Milgram lemma, variational formulation (114) in the Hilbert space V admits a unique solution $\hat{u}$ in V. Reasoning as in [1], the orthogonal of V with respect to the usual scalar product in $L^2(\omega\times Y)$ is made of gradients of the form $\nabla_{x'}q(x') + \nabla_y\hat{q}(x', y)$, with $q(x') \in L_0^2(\omega)$ and $\hat{q}(x', y) \in L^2(\omega; H_\sharp^1(Y))$. Therefore, by integration by parts, variational formulation (114) is equivalent to the effective system (109). It remains to prove that the pressure $\tilde{P}(x')$, arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div}_{x'}(\int_Y \hat{u}(x', y)\mathrm{d}y) = 0$, is the same as the limit of the pressure $\tilde{P}_\varepsilon$. This can be easily done by multiplying Eq. (53) by a test function with div_y equal to zero, and identifying limits. Since (109) admits a unique solution, then the complete sequence $(\varepsilon^{-2}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges to the solution $(\hat{u}(x', y), \hat{P}(x'))$.

Step 2 Case $K_\varepsilon \ll \varepsilon^2$. First, taking as test $\varphi_\varepsilon(x', y_3) = K_\varepsilon/\varepsilon\varphi(x', x'/\varepsilon, y_3)$ in (113) and passing to the limit, taking into account convergences (103) and (108), we prove that $\hat{P}_\varepsilon$ does not depend on y. Moreover, similarly to the Step 1, if we pass to the limit in (113), we get

$$\mu\int_{\omega\times Y} \hat{u}\cdot\varphi\,\mathrm{d}x'\mathrm{d}y = 0. \quad (115)$$

By density, and reasoning as in Step 1, this problem is equivalent to the effective system (110).

Step 3 Case $K_\varepsilon \gg \varepsilon^2$. First, taking as test function of the form $\varphi_\varepsilon(x', y_3) = \varepsilon\varphi(x', x'/\varepsilon, y_3)$ in (113) and passing to the limit taking into account convergences (103) and (107), we prove that $\hat{P}_\varepsilon$ does not depend on y. Moreover, similarly to the Step 1, if we pass to the limit in (113), we get

$$\mu_e \int_{\omega\times Y} D_y\hat{u} : D_y\varphi \, \mathrm{d}x'\mathrm{d}y = \int_{\omega\times Y} f' \cdot \varphi' \, \mathrm{d}x'\mathrm{d}y. \tag{116}$$

By density, and reasoning as in Step 1, this problem is equivalent to the effective system (111).

Now, the proof of Theorem 3.1 can be conducted as follows:

Proof (Proof of Theorem 2) The derivation of (55), (58) and (60) from the effective problems (109), (110) and (111), respectively, is straightforward, and we leave it to the reader as an easy algebra exercise. It should be mentioned that the problems (55), (58) and (60) are well-posed problems since they consist of simple second-order elliptic equations for the pressure $\tilde{P}$ (endowed with Neumann boundary condition).

As it is well known (see [29]), the corresponding local problems are also well posed with periodic boundary condition, and it can be easily checked, by integration by parts that

$$(A_M)_{ij} = \int_Y D_y w^i(y) : D_y w^j(y)\, \mathrm{d}y = \int_Y w^i(y) e_j \mathrm{d}y, \quad i = 1, 2, \; j = 1, 2, 3.$$

The definition implies that A_M is symmetric and positive definite. Observe that the condition $\int_Y w_3^i \, \mathrm{d}y = 0$, $i = 1, 2$, implies that $(A_M)_{i3} = 0$ and since A_M is symmetric, we have that $(A_M)_{3i} = 0$. Then, $A_M \in \mathbb{R}^{2\times 2}$. It happens the same with A_∞ and A_0. □

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