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Pullback Attractor for the 2D Micropolar Fluid Flows with Delay on Unbounded Domains

Wenlong Sun[1] · **Guowei Liu**[2]

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Abstract In this paper, we investigate the pullback asymptotic behavior of micropolar fluid flows with delay on 2D unbounded domains. Firstly, the existence of pullback attractor for the universe given by a tempered condition is established. Then we obtain the consistency of the pullback attractor with that for the universe of fixed bounded sets. Furthermore, the tempered behavior of the pullback attractor is given. Here we develop the energy method with the technique of decomposition of spatial domain to overcome the lack of compactness due to unbounded domains.

Keywords Micropolar fluid flow · Unbounded domains · Pullback attractor · Tempered property

Mathematics Subject Classification 35B40 · 35Q35 · 35B41

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✉ Guowei Liu
guoweiliu19881203@163.com

Wenlong Sun
wenlongsun1988@163.com

1 Department of Mathematics, East China University of Science and Technology, Shanghai 200237, People's Republic of China

2 School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

1 Introduction

1.1 Motivation

The classical 3D incompressible micropolar fluid model was firstly derived by Eringen [13], which was used to describe the fluids consisting of randomly oriented particles suspended in a viscous medium. The model is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \mathrm{rot}\omega + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \\ \dfrac{\partial \omega}{\partial t} - (c_a + c_d)\Delta\omega + 4\nu_r\omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla \mathrm{div}\omega - 2\nu_r \mathrm{rot} u = \tilde{f}, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity field of rotation of particles, p represents the pressure, and $f = (f_1, f_2, f_3)$ and $\tilde{f} = (\tilde{f_1}, \tilde{f_2}, \tilde{f_3})$ stand for the external force and moment, respectively. The positive parameters ν, ν_r, c_0, c_a and c_d are viscous coefficients. Actually, ν represents the usual Newtonian viscosity and ν_r is called microrotation viscosity.

Micropolar fluid model plays an important role in the fields of applied and computational mathematics. There is a wide literature on the mathematical theory of micropolar fluid model (1.1). The existence, uniqueness and regularity of solutions for the micropolar fluids have been investigated in [11,20]. Also, lots of works are devoted to the long time behavior of solutions for the micropolar fluids. More precisely, in the case of 2D bounded domains, Chen, Chen and Dong proved the existence of H^2-global attractor in [6] and verified the existence of uniform attractor in [7]. Łukaszewicz and Tarasińska [22] proved the existence of H^1-pullback attractor. Recently, Zhao and Sun et al. [36] established the L^2-pullback attractor and H^1-pullback attractor of solutions for the universe given by a temper condition, respectively. For the case of 2D unbounded domains, Dong and Chen [9] investigated the existence and regularity of global attractors. Later, they [10] obtained the L^2 time decay rate for global solutions of the 2D micropolar equations via the Fourier splitting method. Zhao, Zhou and Lian [34] established the existence of H^1-uniform attractor and further proved the L^2-uniform attractor belongs to the H^1-uniform attractor. Also some efforts are focused on the 2D micropolar equations with partial dissipation. For example, Dong and Zhang [11] examined the microrotation viscosity, namely $c_a + c_d = 0$. The global regularity problem for this partial dissipation case is not trivial due to the presence of the term $\nabla \times \omega$ in the velocity equation. Dong and Zhang overcame the difficulty by making full use of the quantity $\nabla \times u - \frac{2\nu_r}{\nu+\nu_r}\omega$, which obeys a transport–diffusion equation. When the parameters $\nu = 0$ and $\nu_r \neq c_a + c_d$, the global well posedness of the micropolar fluid equations was obtained in the frame work of Besov spaces in [33]. More recently, Dong, Li and Wu [12] studied the global regularity and large time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation.

In the real world, delay terms appear naturally, for instance as effects in wind tunnel experiments (see [26]), in the equations describing the motions of the fluids. The delay situations may also occur, for example, when we want to control the system via applying a force which considers not only the present state but also the history state of the system. To the best of our knowledge, the delays for ordinary differential equations (ODE) were first studied by Hale (see [17,18]). As regards the partial differential equations (PDE) with delays including finite delays (constant, variable, distributed, etc.) and infinite delays. Different types of delays need to be treated by different approaches. To this respect, there are lots of important foundational works, particularly in the case of random dynamical systems. For the case where the delays are finite, one can refer to [3–5,8,21]. For the other case where the delays are infinite, one can see [2,19,23], etc.

However, to our knowledge, there is little literature for micropolar fluid with delay. Zhao and Sun [37] established the global well posedness of the weak solutions and proved the existence of pullback attractors for the micropolar fluid flows with infinite delay on 2D bounded domains. Furthermore, Zhou et al. [38] verified the H^2-boundedness of the pullback attractors obtained in [37]. Nowadays, Sun [30] proved the global well posedness for the micropolar fluid flows with delay on 2D unbounded domains.

The purpose of this paper is to study the pullback asymptotic properties of the global solutions obtained by Sun [30]. The main objective is to show the existence of pullback attractor for the universe given by a tempered condition. As a consequence, the existence of pullback attractor for the universe of fixed bounded sets follows. Giving suitable assumptions for external force, the consistent relationship between two pullback attractors and the tempered property of the pullback attractors is easily obtained by the means of [16,27].

We mention that the asymptotic compactness is needed to achieve our goal and the method we want to address here is called energy equation method. For many physical systems there are energy equations (or their analogues) in the sense that the changing rate of energy equals the rate that energy is pumped into the system minus the energy dissipation rate due to various dissipation mechanisms. As far as we know, the method was first observed in 1922 by Ball (for weakly damped, driven semilinear wave equations) that such energy equations may be used to derive the asymptotic compactness of the solution semigroup, and he then wrote it up and published in [1]. The method has now been used in a variety of applications. For example, it was applied to a weakly damped, driven Korteweg–de Vries (KdV) equation by Ghidaglia [14], to weakly damped, driven hyperbolic-type equations by Wang [31], to parabolic-type problem by Rosa [28]. At almost the same time, Moise et al. [24] used this method to derive the asymptotic compactness property of the semigroup and presented a general formulation that can handle a number of weakly damped hyperbolic equations and parabolic equations on either bounded or unbounded spatial domains. In this paper, for the lack of compactness of the usual Sobolev embedding in unbounded domains. Inspired by [15,25,37], we apply the technique of the decomposition of spatial domain to overcome the difficulty. It also is worth mentioning that some new techniques are needed to deal with the term of delay, which is more complex than the case without delay.

1.2 Formation of Problem

In this paper, we consider the special situation that the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3 axis. That is, $u = (u_1, u_2, 0)$, $\omega = (0, 0, \omega_3)$, $f = (f_1, f_2, 0)$, $\tilde{f} = (0, 0, \tilde{f}_3)$, $g = (g_1, g_2, 0)$ and $\tilde{g} = (0, 0, \tilde{g}_3)$. Let $\Omega \subseteq \mathbb{R}^2$ be an open set with boundary Γ that is not necessarily bounded but satisfies the following Poincaré inequality:

$$\text{There exists } \lambda_1 > 0 \text{ such that } \lambda_1 \|\varphi\|^2_{L^2(\Omega)} \leqslant \|\nabla\varphi\|^2_{L^2(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega). \tag{1.2}$$

Then, we discuss the following 2D incompressible micropolar fluid model with delay

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p \\ \quad = f(t, x) + g(t, u_t), \quad \text{in } (\tau, +\infty) \times \Omega, \\ \dfrac{\partial \omega}{\partial t} - \bar{\alpha}\Delta\omega + 4\nu_r\omega - 2\nu_r \nabla \times u + (u \cdot \nabla)\omega \\ \quad = \tilde{f}(t, x) + \tilde{g}(t, \omega_t), \quad \text{in } (\tau, +\infty) \times \Omega, \\ \nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \Omega, \\ u = 0, \ \omega = 0, \quad \text{on } (\tau, +\infty) \times \Gamma, \\ (u(\tau), \omega(\tau)) = (u^{\text{in}}, \omega^{\text{in}}), \ (u(t), \omega(t)) \\ \quad = (\phi_1^{\text{in}}(t - \tau), \phi_2^{\text{in}}(t - \tau)), \ \ t \in (\tau - h, \tau), \end{cases} \tag{1.3}$$

where $\bar{\alpha} := c_a + c_d$, $x := (x_1, x_2) \in \Omega$, g and $\tilde{g}$ stand for the external force containing some hereditary characteristics u_t and ω_t, which are defined on $(-h, 0)$ as follows

$$u_t(s) := u(t + s), \ \omega_t(s) := \omega(t + s), \ \forall t \geqslant \tau, \ s \in (-h, 0). \tag{1.4}$$

$(\phi_1^{\text{in}}, \phi_2^{\text{in}})$ represents the initial data in the interval of time $(-h, 0)$, where h is a positive fixed number, and

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times \omega := \left(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}\right).$$

For the sake of convenience, we introduce the following useful operators

$$\begin{cases} \langle Aw, \phi \rangle := (\nu + \nu_r)(\nabla u, \nabla \Phi) + \bar{\alpha}(\nabla\omega, \nabla\phi_3), \ \forall w = (u, \omega), \phi = (\Phi, \phi_3) \in \widehat{V}, \\ \langle B(u, w), \phi \rangle := ((u \cdot \nabla)w, \phi), \ \forall u \in V, \ w = (u, \omega) \in \widehat{V}, \ \forall \phi \in \widehat{V}, \\ N(w) := (-2\nu_r \nabla \times \omega, -2\nu_r \nabla \times u + 4\nu_r\omega), \ \forall w = (u, \omega) \in \widehat{V}. \end{cases} \tag{1.5}$$

Then, we can formulate the weak version of Eq. (1.3) as follows

$$\begin{cases} \dfrac{\partial w}{\partial t} + Aw + B(u, w) + N(w) = F(t, x) + G(t, w_t), \text{ in } (\tau, +\infty) \times \Omega, \\ \nabla \cdot u = 0, \text{ in } (\tau, +\infty) \times \Omega, \\ w = (u, \omega) = 0, \quad \text{on } (\tau, +\infty) \times \Gamma, \\ w(\tau) = w^{\text{in}} = (u^{\text{in}}, \omega^{\text{in}}), \ w(t) = \phi^{\text{in}}(t - \tau) \\ \quad = (\phi_1^{\text{in}}(s), \phi_2^{\text{in}}(s)), \ \ t \in (\tau - h, \tau), \ s \in (-h, 0), \end{cases} \tag{1.6}$$

where

$$\begin{cases} w(t, x) := (u(t, x), \omega(t, x)), \\ F(t, x) := (f(t, x), \tilde{f}(t, x)) \\ G(t, w_t) := (g(t, u_t), \tilde{g}(t, \omega_t)). \end{cases}$$

1.3 Notation

Throughout this paper, we denote the usual Lebesgue space and Sobolev space by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. Particularly, we denote $H^m(\Omega) := W^{m,2}(\Omega)$.

$$\begin{aligned} \mathcal{V} :=& \mathcal{V}(\Omega) := \left\{\varphi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) | \, \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0\right\}, \\ \widehat{\mathcal{V}} :=& \widehat{\mathcal{V}}(\Omega) := \mathcal{V} \times \mathcal{C}_0^\infty(\Omega), \\ H :=& H(\Omega) := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega) \\ & \quad \times L^2(\Omega), \ \text{with norm } \|\cdot\|_H \text{ and dual space } H^*, \\ V :=& V(\Omega) := \text{closure of } \mathcal{V} \text{ in } H^1(\Omega) \\ & \quad \times H^1(\Omega), \ \text{with norm } \|\cdot\|_V \text{ and dual space } V^*, \\ \widehat{H} :=& \widehat{H}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } L^2(\Omega) \\ & \quad \times L^2(\Omega) \times L^2(\Omega), \ \text{with norm } \|\cdot\|_{\widehat{H}} \text{ and dual space } \widehat{H}^*, \\ \widehat{V} :=& \widehat{V}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } H^1(\Omega) \\ & \quad \times H^1(\Omega) \times H^1(\Omega), \ \text{with norm } \|\cdot\|_{\widehat{V}} \text{ and dual space } \widehat{V}^*. \end{aligned}$$

$(\cdot, \cdot)$– the inner product in $L^2(\Omega)$, H or $\widehat{H}$, $\langle \cdot, \cdot \rangle$– the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$. Throughout this article, we simplify the notations $\|\cdot\|_2$, $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion. Furthermore, we denote

$L^p(I; X) :=$ space of strongly measurable functions on the closed interval I, with values in the Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \left(\int_I \|\varphi\|_X^p \mathrm{d}t\right)^{1/p}, \quad \text{for } 1 \leqslant p < \infty,$$

$\mathcal{C}(I; X) :=$ space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm,

$L^2_{loc}(I; X) :=$ space of locally square integrable functions on the interval I, with values in the Banach space X, endowed with the usual norm,

$\hookrightarrow\hookrightarrow$ – the compact embedding between spaces.

Following the above notations, we additionally denote

$$L^2_{\widehat{H}} := L^2(-h, 0; \widehat{H}), \quad L^2_{\widehat{V}} := L^2(-h, 0; \widehat{V}),$$
$$E^2_{\widehat{H}} := \widehat{H} \times L^2_{\widehat{H}}, \quad E^2_{\widehat{V}} := \widehat{V} \times L^2_{\widehat{V}}, \quad E^2_{\widehat{H}\times L^2_{\widehat{V}}} := \widehat{H} \times L^2_{\widehat{V}}.$$

The norm $\|\cdot\|_X$ for $X \in \{E^2_{\widehat{H}}, E^2_{\widehat{V}}, E^2_{\widehat{H}\times L^2_{\widehat{V}}}\}$ is defined as

$$\|(w, v)\|_{E^2_{\widehat{H}}} := (\|w\|^2 + \|v\|^2_{L^2_{\widehat{H}}})^{1/2}, \quad \|(w, v)\|_{E^2_{\widehat{V}}} := (\|w\|^2 + \|v\|^2_{L^2_{\widehat{V}}})^{1/2},$$
$$\|(w, v)\|_{E^2_{\widehat{H}\times L^2_{\widehat{V}}}} := (\|w\|^2 + \|v\|^2_{L^2_{\widehat{V}}})^{1/2}.$$

The rest of this paper is organized as follows. In Sect. 2, we make some preliminaries. Section 3 is devoted to proving the existence of pullback attractor for the universe given by a tempered condition. In Sect. 4, we aim at some properties of the pullback attractor obtained in Sect. 3.

2 Preliminaries

In this section, we recall some key results about the non-autonomous micropolar fluid flows and introduce some notations and definitions about pullback attractor. To begin with, we list some useful estimates and properties for the operators A, $B(\cdot)$ and $N(\cdot)$ defined in (1.5), which have been established in works [25,30,34,37] .

Lemma 2.1 1. *The operator A is linear continuous both from $\widehat{V}$ to $\widehat{V}^*$ and from $D(A)$ to $\widehat{H}$, and so is for the operator $N(\cdot)$ from $\widehat{V}$ to $\widehat{H}$, where $D(A) := \widehat{V} \cap \big(H^2(\Omega)\big)^3$.*

2. *The operator $B(\cdot, \cdot)$ is continuous from $V \times \widehat{V}$ to $\widehat{V}^*$. Moreover, for any $u \in V$ and $w \in \widehat{V}$, there holds*

$$\langle B(u, \psi), \varphi\rangle = -\langle B(u, \varphi), \psi\rangle, \ \forall u \in V, \ \forall \psi \in \widehat{V}, \ \forall \varphi \in \widehat{V}. \tag{2.1}$$

Lemma 2.2 1. *There are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw, w\rangle \leqslant \|w\|^2_{\widehat{V}} \leqslant c_2\langle Aw, w\rangle, \ \forall w \in \widehat{V}. \tag{2.2}$$

2. *There exists some positive constant α_0 which depends only on Ω, such that for any $(u, \psi, \varphi) \in V \times \widehat{V} \times \widehat{V}$ there holds*

$$|\langle B(u, \psi), \varphi\rangle| \leqslant \begin{cases} \alpha_0\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\nabla\varphi\|^{\frac{1}{2}}\|\nabla\psi\|, \\ \alpha_0\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\psi\|^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}\|\nabla\varphi\|. \end{cases} \tag{2.3}$$

Moreover, if $(u, \psi, \varphi) \in V \times D(A) \times D(A)$, then

$$|\langle B(u, \psi), A\varphi\rangle| \leqslant \alpha_0 \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \|\nabla \psi\|^{\frac{1}{2}} \|A\psi\|^{\frac{1}{2}} \|A\varphi\|. \tag{2.4}$$

3. *There exists a positive constant $c(\nu_r)$ such that*

$$\|N(\psi)\| \leqslant c(\nu_r)\|\psi\|_{\widehat{V}}, \ \ \forall \psi \in \widehat{V}. \tag{2.5}$$

In addition,

$$-\langle N(\psi), A\psi\rangle \leqslant \frac{1}{4}\|A\psi\|^2 + c^2(\nu_r)\|\psi\|^2_{\widehat{V}}, \ \ \forall \psi \in D(A), \tag{2.6}$$

$$\delta_1\|\psi\|^2_{\widehat{V}} \leqslant \langle A\psi, \psi\rangle + \langle N(\psi), \psi\rangle, \ \ \forall\, \psi \in \widehat{V}, \tag{2.7}$$

where $\delta_1 := \min\{\nu, \bar{\alpha}\}$.

Then, we recall the global well posedness of solutions established in [30].

Assumption 2.1 Assume that $G : \mathbb{R} \times L^2(-h, 0; \widehat{H}) \mapsto (L^2(\Omega))^3$ satisfies:

- (i) For any $\xi \in L^2(-h, 0; \widehat{H})$, the mapping $\mathbb{R} \ni t \mapsto G(t, \xi) \in (L^2(\Omega))^3$ is measurable.
- (ii) $G(\cdot, 0) = (0, 0, 0)$.
- (iii) There exists a constant $L_G > 0$ such that for any $t \in \mathbb{R}$ and any $\xi, \eta \in L^2(-h, 0; \widehat{H})$,

$$\|G(t, \xi) - G(t, \eta)\| \leqslant L_G\|\xi - \eta\|_{L^2(-h,0;\widehat{H})}.$$

- (iv) There exists $C_G \in (0, \delta_1)$ such that, for any $t \geqslant \tau$ and any $w, v \in L^2(\tau-h, t; \widehat{H})$,

$$\int_\tau^t \|G(\theta, w_\theta) - G(\theta, v_\theta)\|^2 \mathrm{d}\theta \leqslant C_G^2 \int_{\tau-h}^t \|w(\theta) - v(\theta)\|^2 \mathrm{d}\theta.$$

Moreover, for any $t \geqslant \tau$, there exists a $\gamma \in (0, 2\delta_1 - 2C_G)$ such that

$$\int_\tau^t e^{\gamma\theta}\|G(\theta, w_\theta)\|^2 \mathrm{d}\theta \leqslant C_G^2 \int_{\tau-h}^t e^{\gamma\theta}\|w(\theta)\|^2 \mathrm{d}\theta, \ \forall\, w \in L^2(\tau - h, t; \widehat{H}).$$

Theorem 2.1 *Assume $F(t, x) \in L^2_{loc}(\mathbb{R}; \widehat{V}^*)$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$, and G satisfies Assumption 2.1. Then, for any $(w^{\text{in}}, \phi^{\text{in}}) \in E^2_{\widehat{H}}$, there is a unique weak solution $w(\cdot) := w(\cdot; \tau, w^{\text{in}}, \phi^{\text{in}})$ of system (1.6), which satisfies*

$$w \in \mathcal{C}([\tau, T]; \widehat{H}) \cap L^2(\tau, T; \widehat{V}) \ \ \text{and} \ \ w' \in L^2(\tau, T; \widehat{V}^*), \ \ \forall T > \tau.$$

Moreover, let $v(\cdot) := v(\cdot; \tau, v^{\text{in}}, \psi^{\text{in}})$ be another weak solution corresponding to the initial value $(v^{\text{in}}, \psi^{\text{in}}) \in E^2_{\widehat{H}}$, then, for all $t \geqslant \tau$, we have

$$\|w(t) - v(t)\|^2 \leqslant r_1\|(w^{\text{in}} - v^{\text{in}}, \phi^{\text{in}} - \psi^{\text{in}})\|^2_{E^2_{\widehat{H}}} \cdot e^{\sigma(w(t))}, \tag{2.8}$$

$$\int_\tau^t \|w(\theta)-v(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta \leqslant \delta_1^{-1} r_1 \|(w^{\mathrm{in}}-v^{\mathrm{in}}, \phi^{\mathrm{in}}-\psi^{\mathrm{in}})\|_{E_{\widehat{H}}^2}^2 \big(1+\sigma(w(t))\cdot e^{\sigma(w(t))}\big), \tag{2.9}$$

where

$$r_1 := \max\{1, 2a_2 C_G^2\}, \ \ \sigma(w(t)) := \int_\tau^t \big(\delta_1^{-1}\alpha_0^2\|w(s)\|_{\widehat{V}}^2 + 2a_1 + 2a_2 C_G^2\big)\mathrm{d}s,$$

where the positive constants a_1, a_2 satisfy $a_1 a_2 \geqslant \frac{1}{4}$.

On the basis of Theorem 2.1, the biparametric map defined by

$$U(t,\tau) : (w^{\mathrm{in}}, \phi^{\mathrm{in}}) \mapsto (w(t;\tau, w^{\mathrm{in}}, \phi^{\mathrm{in}}), w_t(\cdot;\tau, w^{\mathrm{in}}, \phi^{\mathrm{in}})), \ \ \forall t \geqslant \tau, \tag{2.10}$$

generates a continuous process in $E_{\widehat{H}}^2$ and $E_{\widehat{V}}^2$, respectively, which satisfies the following properties:

$$\text{(i)} U(\tau,\tau)(w^{\mathrm{in}}, \phi^{\mathrm{in}}) = (w^{\mathrm{in}}, \phi^{\mathrm{in}}),$$
$$\text{(ii)} U(t,s)U(s,\tau)(w^{\mathrm{in}}, \phi^{\mathrm{in}}) = U(t,\tau)(w^{\mathrm{in}}, \phi^{\mathrm{in}}).$$

Finally, we end this section with some notations and definitions concerning the pullback attractors for non-autonomous dynamical systems in the following. On can refer to [16,27,35].

We denote by X the space $\widehat{H}$ or $\widehat{V}$ and by $\mathcal{P}(X)$ the family of all nonempty subsets of X. A universe $\mathcal{D}(X)$ in $\mathcal{P}(X)$ represents the class of families parameterized in time $\widehat{B}(X) = \{B(t) \mid t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$.

Definition 2.1 A family of sets $\widehat{B}_0 = \{B_0(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ is called pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in X if for any $t \in \mathbb{R}$ and any $\widehat{B} = \{B(t) | t \in \mathbb{R}\} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{B}) \leqslant t$ such that $U(t,\tau)B(\tau) \subseteq B_0(t)$ for all $\tau \leqslant \tau_0(t, \widehat{B})$.

Definition 2.2 The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be pullback $\widehat{B}_0$-asymptotically compact in X if for any $t \in \mathbb{R}$, any sequences $\{\tau_n\} \subseteq (-\infty, t]$ and $\{x_n\} \subseteq X$ satisfying $\tau_n \to -\infty$ as $n \to \infty$ and $x_n \in B_0(\tau_n)$ for all n, the sequence $\{U(t, \tau_n; x_n)\}$ is relatively compact in X. $\{U(t,\tau)\}_{t\geqslant\tau}$ is called pullback $\mathcal{D}$-asymptotically compact in X if it is pullback $\widehat{B}$-asymptotically compact for any $\widehat{B} \in \mathcal{D}$.

Definition 2.3 A family of sets $\widehat{\mathcal{A}}_X = \{\mathcal{A}_X(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ is called a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ on X if it has the following properties:

- Compactness: for any $t \in \mathbb{R}$, $\mathcal{A}_X(t)$ is a nonempty compact subset of X;
- Invariance: $U(t,\tau)\mathcal{A}_X(\tau) = \mathcal{A}_X(t)$, $\forall t \geqslant \tau$;
- Pullback attracting: $\widehat{\mathcal{A}}_X$ is pullback $\mathcal{D}$ attracting in the following sense:

$$\lim_{\tau\to-\infty} \mathrm{dist}_X\left(U(t,\tau)B(\tau), \mathcal{A}_X(t)\right) = 0, \ \forall \widehat{B} = \{B(s) | s \in \mathbb{R}\} \in \mathcal{D}, t \in \mathbb{R},$$

- Minimality: the family of sets $\widehat{\mathcal{A}}_X$ is the minimal in the sense that if $\widehat{O} = \{O(t) | t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ is another family of closed sets such that

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)B(\tau), O(t)) = 0, \quad \text{for any } \widehat{B} = \{B(t)|\, t\in\mathbb{R}\}\in\mathcal{D},$$

then $\mathcal{A}_X(t)\subseteq O(t)$ for $t\in\mathbb{R}$.

Remark 2.1 If the process U possesses a pullback $\mathcal{D}$-absorbing set $\widehat{B}_0$ and is pullback $\widehat{B}_0$-asymptotically compact in X, we can construct the pullback attractor by the standard method introduced by García-Luengo et al. [16, Proposition 9] and Marín-Rubio and Real in [27, Theorem 18]

3 Existence of Pullback Attractor for the Universe $\mathcal{D}_\gamma$

In the section, we investigate the pullback attractor for the universe $\mathcal{D}_\gamma$ given by a tempered condition in space $E^2_{\widehat{H}}$. To begin with, in order to construct the universe $\mathcal{D}_\gamma$, we give some useful energy estimates.

Lemma 3.1 *For any $t\geqslant\tau$, assume that $F\in L^2(\tau,t;\widehat{V}^*)$ and G satisfies Assumption 2.1. Then, for any $(w^{\text{in}},\phi^{\text{in}})\in E^2_{\widehat{H}}$, we have*

$$\|w(t)\|^2+\beta e^{-\gamma t}\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta \leqslant(1+C_G)e^{-\gamma(t-\tau)}\|(w^{\text{in}},\phi^{\text{in}})\|^2_{E^2_{\widehat{H}}}+\alpha^{-1}e^{-\gamma t}\int_\tau^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta, \tag{3.1}$$

where $\alpha\in(0,2\delta_1-2C_G-\gamma)$, $\beta:=2\delta_1-2C_G-\gamma-\alpha>0$.

Proof Let us denote $w(\cdot)=w(\cdot;\tau,w^{\text{in}},\phi^{\text{in}})$. Testing $(1.6)_1$ by $w(t)$, we obtain from (2.1) and (2.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|^2_{\widehat{V}}\leqslant\langle F(t),w(t)\rangle+(G(t,w_t),w(t)),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t}\|w(t)\|^2)-\gamma e^{\gamma t}\|w(t)\|^2+2\delta_1 e^{\gamma t}\|w(t)\|^2_{\widehat{V}} \leqslant 2e^{\gamma t}\langle F(t),w(t)\rangle+2e^{\gamma t}(G(t,w_t),w(t)).$$

Let $\tau\leqslant\theta\leqslant t$. Replacing the time variable t in the above inequality with θ, then integrating it with respect to θ over $[\tau,t]$ gives

$$e^{\gamma t}\|w(t)\|^2+(2\delta_1-\gamma)\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta \leqslant e^{\gamma\tau}\|w^{\text{in}}\|^2+2\int_\tau^t e^{\gamma\theta}\langle F(\theta),w(\theta)\rangle\mathrm{d}\theta+2\int_\tau^t e^{\gamma\theta}(G(\theta,w_\theta),w(\theta))\mathrm{d}\theta. \tag{3.2}$$

By the Young's inequality and Assumption 2.1, we deduce

$$
\begin{aligned}
2\int_\tau^t e^{\gamma\theta}(G(\theta,w_\theta),w(\theta))\mathrm{d}\theta &\leqslant 2\int_\tau^t e^{\gamma\theta}\|G(\theta,w_\theta\|\|w(\theta)\|\mathrm{d}\theta\\
&\leqslant 2\left(\int_\tau^t e^{\gamma\theta}\|G(\theta,w_\theta\|^2\mathrm{d}\theta\right)^{\frac12}\left(\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2\mathrm{d}\theta\right)^{\frac12}\\
&\leqslant C_G\int_{\tau-h}^{\tau} e^{\gamma\theta}\|w(\theta)\|^2\mathrm{d}\theta+2C_G\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2\mathrm{d}\theta\\
&\leqslant C_G e^{\gamma\tau}\|\phi^{\mathrm{in}}\|^2_{L^2_{\widehat{H}}}+2C_G\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2\mathrm{d}\theta,
\end{aligned}\tag{3.3}
$$

and

$$
\begin{aligned}
2\int_\tau^t e^{\gamma\theta}\langle F(\theta),w(\theta)\rangle\mathrm{d}\theta \leqslant& 2\int_\tau^t e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}*}\|w(\theta)\|_{\widehat{V}}\mathrm{d}\theta\\
\leqslant&\alpha^{-1}\int_\tau^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}*}\mathrm{d}\theta+\alpha\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta,
\end{aligned}\tag{3.4}
$$

where $\alpha\in(0,2\delta_1-\gamma-2C_G)$. Substituting (3.3) and (3.4) into (3.2), yields (3.1). This completes the proof. □

As a consequence of Lemma 3.1, we immediately have

Proposition 3.1 *Under the conditions of Lemma 3.1, for any $t\geqslant T+\tau$, $T>0$ and $(w^{\mathrm{in}},\phi^{\mathrm{in}})\in E^2_{\widehat{H}}$, it holds that*

$$
\begin{aligned}
&\int_{t-T}^t\|w(\theta;\tau,w^{\mathrm{in}},\phi^{\mathrm{in}})\|^2_{\widehat{V}}\mathrm{d}\theta\\
\leqslant&\beta^{-1}(1+C_G)e^{-\gamma(t-T-\tau)}\|(w^{\mathrm{in}},\phi^{\mathrm{in}})\|^2_{E^2_{\widehat{H}}}+(\alpha\beta)^{-1}e^{-\gamma(t-T)}\int_\tau^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}*}\mathrm{d}\theta,
\end{aligned}\tag{3.5}
$$

where α and β come from (3.1).

Proof For $t\geqslant T+\tau$, we have

$$
\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta\geqslant\int_{t-T}^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta\geqslant e^{\gamma(t-T)}\int_{t-T}^t\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta \tag{3.6}
$$

From (3.1), it follows that

$$
\beta\int_\tau^t e^{\gamma\theta}\|w(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta\leqslant(1+C_G)e^{\gamma\tau}\|(w^{\mathrm{in}},\phi^{\mathrm{in}})\|^2_{E^2_{\widehat{H}}}+\alpha^{-1}\int_\tau^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}*}\mathrm{d}\theta,
$$

which together with (3.6) implies (3.5). The proof is complete. □

Now, we can construct the universe $\mathcal{D}_\gamma$ in the following.

Definition 3.1 (*Definition of universe $\mathcal{D}_\gamma$*)
Set

$$\mathcal{R}_\gamma := \{\rho(t) : \mathbb{R} \mapsto \mathbb{R}_+ \mid \lim_{t\to-\infty} e^{\gamma t}\rho^2(t) = 0\}.$$

We denote by $\mathcal{D}_\gamma$ the class of all families $\widehat{D} = \{D(t) \mid t \in \mathbb{R}\} \subseteq \mathcal{P}(E^2_{\widehat{H}})$ such that

$$D(t) \subseteq \bar{B}_{E^2_{\widehat{H}}}(0, \rho_{\widehat{D}}(t)), \text{ for some } \rho_{\widehat{D}}(t) \in \mathcal{R}_\gamma,$$

where $\bar{B}_{E^2_{\widehat{H}}}(0, \rho_{\widehat{D}}(t))$ represents the closed ball in $E^2_{\widehat{H}}$ centered at zero with radius $\rho_{\widehat{D}}(t)$.

3.1 Pullback $\mathcal{D}_\gamma$-Absorbing Set

In this subsection, we prove existence of the pullback $\mathcal{D}_\gamma$-absorbing set.

Assumption 3.1 Assume that $F(t,x) \in L^2_{loc}(\mathbb{R}; \widehat{V}^*)$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$, and

$$\int_{-\infty}^{t} e^{\gamma\theta}\|F(\theta,x)\|^2_{\widehat{V}^*}\mathrm{d}\theta < +\infty, \quad \sup_{t\in\mathbb{R}}\int_{t}^{t+1}\|F(\theta,x)\|^2\mathrm{d}\theta < +\infty. \tag{3.7}$$

Lemma 3.2 (Pullback $\mathcal{D}_\gamma$-absorbing set)
Assume that Assumptions 2.1 and 3.1 hold. Then the family $\widehat{B} := \{B(t) \mid t \in \mathbb{R}\}$ with

$$B(t) := \left\{(\varphi,\psi) \in E^2_{\widehat{H}\times L^2_{\widehat{V}}} \mid \|(\varphi,\psi)\|_{E^2_{\widehat{H}\times L^2_{\widehat{V}}}} \leqslant \mathcal{R}_1(t),\ \|\psi'(s)\|_{L^2_{\widehat{V}^*}} \leqslant \mathcal{R}_2(t)\right\} \tag{3.8}$$

is a pullback $\mathcal{D}_\gamma$-absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$, where

$$\mathcal{R}_1^2(t) := 1 + (1+\beta^{-1}e^{\gamma h})\alpha^{-1}e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta, \tag{3.9}$$

$$\begin{aligned}\mathcal{R}_2^2(t) :=& 4\eta^2\beta^{-1}\Big[2 + \lambda_1^{-1}C_G^2 e^{\gamma h} + \alpha^{-1}e^{-\gamma(t-h)}\int_{-\infty}^{t} e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta\Big]\\ &\cdot\Big[1 + \alpha^{-1}e^{-\gamma(t-h)}\int_{-\infty}^{t} e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta\Big] + 4\eta^2\int_{t-h}^{t}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,\end{aligned} \tag{3.10}$$

where β comes from (3.1), $\alpha \in (0, 2\delta_1 - \gamma - 2C_G)$ and $\eta := \max\{1, \alpha_0, \lambda_1^{-\frac{1}{2}}, c_1^{-1} + c(\nu_r)\lambda_1^{-\frac{1}{2}}\}$.

Proof By (3.7) and (3.9), it is clear that $\lim_{t\to-\infty} e^{\gamma t}\mathcal{R}_1^2(t)=0$. Consequently,

$$\begin{aligned}B(t)\subset\{(\varphi,\psi)\in E^2_{\widehat{H}}\mid \|(\varphi,\psi)\|^2_{E^2_{\widehat{H}}}\\ \leqslant \rho^2(t),\ \lim_{t\to-\infty}\rho^2(t)=0\},\ \text{and therefore } \widehat{B}\in\mathcal{D}_\gamma.\end{aligned}$$

Taking $T=h$ in Proposition 3.1, and for any $t\geqslant\tau+h$, we have

$$\begin{aligned}&\|w_t(\cdot;\tau,w^{\rm in},\phi^{\rm in})\|^2_{L^2_{\widehat{V}}}\\ &\leqslant\beta^{-1}(1+C_G)e^{-\gamma(t-h-\tau)}\|(w^{\rm in},\phi^{\rm in})\|^2_{E^2_{\widehat{H}}}+(\alpha\beta)^{-1}e^{-\gamma(t-h)}\int_{-\infty}^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,\end{aligned}$$

which together with (3.1) gives

$$\begin{aligned}&\|U(t,\tau)(w^{\rm in},\phi^{\rm in})\|^2_{E_{\widehat{H}\times L^2_{\widehat{V}}}}=\|w(t;\tau,w^{\rm in},\phi^{\rm in})\|^2+\|w_t(\cdot;\tau,w^{\rm in},\phi^{\rm in})\|^2_{L^2_{\widehat{V}}}\\ &\quad\leqslant(1+\beta^{-1}e^{\gamma h})(1+C_G)e^{-\gamma(t-\tau)}\|(w^{\rm in},\phi^{\rm in})\|^2_{E^2_{\widehat{H}}}\\ &\quad+(1+\beta^{-1}e^{\gamma h})\alpha^{-1}e^{-\gamma t}\int_{-\infty}^t e^{\gamma\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,\end{aligned}\tag{3.11}$$

In addition, it follows from (1.2), $(1.6)_1$, (2.2), (2.3) and (2.5) that

$$\begin{aligned}|\langle w'(\theta),v\rangle|\leqslant&|\langle Aw(\theta),v\rangle|+|\langle B(u(\theta),w(\theta)),v\rangle|+|\langle N(w(\theta)),v\rangle|\\ &+|\langle F(\theta),v\rangle|+|\langle G(\theta,w_\theta),v\rangle|\\ \leqslant&\big(c^{-1}\|w(\theta)\|_{\widehat{V}}+\alpha_0\|u(\theta)\|^{\frac12}\|\nabla u(\theta)\|^{\frac12}\|w(\theta)\|^{\frac12}\|\nabla w(\theta)\|^{\frac12}\\ &+c(\nu_r)\lambda_1^{-\frac12}\|w(\theta)\|_{\widehat{V}}\\ &+\|F(\theta)\|_{\widehat{V}^*}+\lambda_1^{-\frac12}\|G(\theta,w_\theta)\|\big)\|v\|_{\widehat{V}},\quad\forall\, v\in\widehat{V}.\end{aligned}\tag{3.12}$$

Since $\|u(\theta)\|\leqslant\|w(\theta)\|$ and $\|\nabla u(\theta)\|\leqslant\|\nabla w(\theta)\|\leqslant\|w(\theta)\|_{\widehat{V}}$, (3.12) implies

$$\|w'(\theta)\|_{\widehat{V}^*}\leqslant\eta\big(\|w(\theta)\|_{\widehat{V}}+\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}+\|F(\theta)\|_{\widehat{V}^*}+\|G(\theta,w_\theta)\|\big),$$

where $\eta:=\max\{1,\alpha_0,\lambda_1^{-\frac12},c_1^{-1}+c(\nu_r)\lambda_1^{-\frac12}\}$. Integrating the above inequality and using the Cauchy inequality yield

$$\begin{aligned}\int_{t-h}^t\|w'(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta\leqslant&4\eta^2\int_{t-h}^t\big(\|w(\theta)\|^2_{\widehat{V}}+\|w(\theta)\|^2\|w(\theta)\|^2_{\widehat{V}}\\ &+\|F(\theta)\|^2_{\widehat{V}^*}+\|G(\theta,w_\theta)\|^2\big)\mathrm{d}\theta.\end{aligned}\tag{3.13}$$

Under Assumption 2.1, it holds that

$$\int_{t-h}^{t} \|G(\theta, w_\theta)\|^2 \mathrm{d}\theta \leqslant C_G^2 \int_{t-2h}^{t} \|w(\theta)\|^2 \mathrm{d}\theta \leqslant \lambda_1^{-1} C_G^2 \int_{t-2h}^{t} \|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta. \quad (3.14)$$

From (3.1), we have

$$\begin{aligned}\sup_{\theta\in[t-h,t]} \|w(\theta)\|^2 \leqslant&(1 + C_G)e^{-\gamma(t-h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ \alpha^{-1} e^{-\gamma(t-h)} \int_{-\infty}^{t} e^{\gamma s} \|F(s)\|_{\widehat{V}^*}^2 \mathrm{d}s. \end{aligned} \quad (3.15)$$

Taking $T = 2h$ in (3.5) yields

$$\begin{aligned}\int_{t-2h}^{t} \|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta \leqslant&\beta^{-1}(1 + C_G)e^{-\gamma(t-2h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ (\alpha\beta)^{-1} e^{-\gamma(t-2h)} \int_{-\infty}^{t} e^{\gamma\theta} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta. \end{aligned} \quad (3.16)$$

Now, taking (3.13)–(3.16) into account and writing

$$\begin{aligned}M(t, \tau, w^{\text{in}}, \phi^{\text{in}}) :=&(1 + C_G)e^{-\gamma(t-h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ \alpha^{-1} e^{-\gamma(t-h)} \int_{-\infty}^{t} e^{\gamma\theta} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta,\end{aligned}$$

we obtain

$$\begin{aligned}\|w_t'(s)\|_{L^2_{\widehat{V}^*}} =& \int_{t-h}^{t} \|w'(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta \\ \leqslant&4\eta^2\big[1 + (1 + C_G)e^{-\gamma(t-h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ \alpha^{-1} e^{-\gamma(t-h)} \int_{-\infty}^{t} e^{\gamma s} \|F(s)\|_{\widehat{V}^*}^2 \mathrm{d}s\big] \\ &\times \big[\beta^{-1}(1 + C_G)e^{-\gamma(t-h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ (\alpha\beta)^{-1} e^{-\gamma(t-h)} \int_{-\infty}^{t} e^{\gamma\theta} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta\big] \\ &+ 4\eta^2\lambda_1^{-1} C_G^2 \big[\beta^{-1}(1 + C_G)e^{-\gamma(t-2h-\tau)} \|(w^{\text{in}}, \phi^{\text{in}})\|_{E_{\widehat{H}}^2}^2 \\ &+ (\alpha\beta)^{-1} e^{-\gamma(t-2h)} \int_{-\infty}^{t} e^{\gamma\theta} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta\big] \\ &+ 4\eta^2 \int_{t-h}^{t} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta \\ =&4\eta^2\beta^{-1}\big[1 + \lambda_1^{-1} C_G^2 e^{\gamma h} + M(t, \tau, w^{\text{in}}, \phi^{\text{in}})\big] M(t, \tau, w^{\text{in}}, \phi^{\text{in}}) \\ &+ 4\eta^2 \int_{t-h}^{t} \|F(\theta)\|_{\widehat{V}^*}^2 \mathrm{d}\theta. \end{aligned} \quad (3.17)$$

Therefore, from (3.11) and (3.17), we conclude that the family $\widehat{B}$ given by (3.8) is a pullback $\mathcal{D}_\gamma$-absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$. This completes the proof. □

3.2 Pullback $\mathcal{D}_\gamma$-Asymptotic Compactness

This subsection is to prove the pullback $\mathcal{D}_\gamma$-asymptotic compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$. Since the lack of compactness for Sobolev imbedding in unbounded domains, we first establish the tail estimate with respect to the unbounded domains.

Lemma 3.3 (Tail estimate of unbounded domains)
Assume that Assumptions 2.1 and 3.1 hold. Then, for any $\epsilon > 0$, $t \in \mathbb{R}$ and $\widehat{D} = \{D(t) \mid t \in \mathbb{R}\} \in \mathcal{P}(E^2_{\widehat{H}})$, there exist $l_0 := l_0(\epsilon, t, \widehat{D}) > 0$ and $\tau_0 := \tau_0(\epsilon, t, \widehat{D}) < t$ such that, for any $l \geqslant l_0$, $\tau \leqslant \tau_0$ and $(w^{\text{in}}, \phi^{\text{in}}) \in D(\tau)$, there holds

$$\|w(t;\tau, w^{\text{in}}, \phi^{\text{in}})\|_{L^2(\Omega\backslash\Omega_l)} \leqslant \epsilon, \tag{3.18}$$

where $\Omega_l := \{x \in \Omega \mid |x| < l\}$.

Proof Let the truncation function $\chi(\cdot) \in \mathcal{C}^2(\mathbb{R}^2)$, $\chi(x) \in [0,1]$ satisfies for some constant c_0

$$\chi(x) = \begin{cases} 0, & |x| \leqslant 1,\\ 1, & |x| \geqslant 2, \end{cases} \qquad \|\nabla\chi(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leqslant c_0, \quad \|D^2\chi(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leqslant c_0.$$

In particular, set $\chi_l(x) = \chi(\frac{x}{l})$ with $l \geqslant 1$, we have

$$\|\nabla\chi_l(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leqslant \frac{c_0}{l}, \quad \|D^2\chi_l(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leqslant \frac{c_0}{l^2}. \tag{3.19}$$

Taking the inner product of $(1.6)_1$ yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_l w\|^2 + \langle A(\chi_l w), \chi_l w\rangle - (\nu+\nu_r)\int_\Omega |u\nabla\chi_l|^2\mathrm{d}x - \bar{\alpha}\int_\Omega |\omega\nabla\chi_l|^2\mathrm{d}x\\ &\quad + ((u\cdot\nabla)w, \chi_l^2 w) + \langle N(\chi_l w), \chi_l w\rangle + (\nabla p, \chi_l^2 u)\\ =&\langle F(t,x), \chi_l^2 w\rangle + (G(t,w_t), \chi_l^2 w).\end{aligned} \tag{3.20}$$

It follows from (3.19) and the Hölder inequality that

$$(\nu+\nu_r)\int_\Omega |u\nabla\chi_l|^2\mathrm{d}x \leqslant (\nu+\nu_r)\|\nabla\chi_l\|^2_{\mathbb{L}^\infty(\Omega)}\|u\|^2 \leqslant c_0^2(\nu+\nu_r)l^{-2}\|u\|^2. \tag{3.21}$$

Similarly, it holds that

$$\bar{\alpha}\int_\Omega |\omega\nabla\chi_l|^2\mathrm{d}x \leqslant \bar{\alpha}\|\nabla\chi_l\|^2_{\mathbb{L}^\infty(\Omega)}\|\omega\|^2 \leqslant c_0^2\bar{\alpha}l^{-2}\|\omega\|^2. \tag{3.22}$$

Using integrating by parts and the fact $\nabla\cdot u = 0$, we obtain

$$
\begin{aligned}
((u\cdot\nabla)w,\chi_l^2 w) &= \sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial u_j}{\partial x_i}\chi_l^2 u_j\mathrm{d}x+\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial\omega}{\partial x_i}\chi_l^2\omega\mathrm{d}x\\
&=-\sum_{i,j=1}^{2}\left(\int_\Omega u_iu_j\chi_l^2\frac{\partial u_j}{\partial x_i}\mathrm{d}x+2\int_\Omega u_iu_j^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x\right)\\
&\quad-\sum_{i=1}^{2}\left(\int_\Omega u_i\omega\chi_l^2\frac{\partial\omega}{\partial x_i}\mathrm{d}x+2\int_\Omega u_i\omega^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x\right)\\
&=-((u\cdot\nabla)w,\chi_l^2 w)-2\sum_{i,j=1}^{2}\int_\Omega u_iu_j^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x\\
&\quad-2\sum_{i=1}^{2}\int_\Omega u_i\omega^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x,
\end{aligned}
$$

which together with (3.19), the Hölder inequality, the Gagliardo–Nirenberg inequality and the Young's inequality yield

$$
\begin{aligned}
&|((u\cdot\nabla)w,\chi_l^2 w)|=\Big|\sum_{i,j=1}^{2}\int_\Omega u_iu_j^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x+\sum_{i=1}^{2}\int_\Omega u_i\omega^2\chi_l\frac{\partial\chi_l}{\partial x_i}\mathrm{d}x\Big|\\
&\leqslant\|\nabla\chi_l\|_{\mathbb{L}^\infty(\Omega)}\|u\|\|w\|_{\mathbb{L}^4(\Omega)}^2\leqslant c_0l^{-1}\|u\|\|w\|\|w\|_{\widehat{V}}\leqslant\frac{c_0}{l}(\|w\|^4+\|w\|_{\widehat{V}}^2). \quad (3.23)
\end{aligned}
$$

From (3.19) and the fact $\nabla\cdot u=0$, we also have

$$
\begin{aligned}
|(\nabla p,\chi_l^2u)|=&\Big|\sum_{i=1}^{2}\int_\Omega\frac{\partial p}{\partial x_i}\chi_l^2u_i\mathrm{d}x\Big|=\Big|\sum_{i=1}^{2}\int_\Omega 2p\chi_l\frac{\partial\chi_l}{\partial x_i}u_i\mathrm{d}x\Big|\\
&\leqslant 2\|p\|\|\nabla\chi_l\|_{\mathbb{L}^\infty(\Omega)}\|\chi_lu\|\leqslant 2c_0l^{-1}\|p\|\|\chi_lu\|. \quad (3.24)
\end{aligned}
$$

Taking (1.2), (2.7), (3.20)–(3.24) and Lemma 3.2 into account, we deduce that there exists τ_1 such that, for any $\tau\leqslant\tau_1$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_lw\|^2+\delta_1\|\chi_lw\|_{\widehat{V}}^2\leqslant\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_lw\|^2+\langle A(\chi_lw),\chi_lw\rangle+\langle N(\chi_lw),\chi_lw\rangle\\
&\leqslant\frac{c_0^2(\nu+\nu_r)}{l^2}\|u\|^2+\frac{c_0^2\bar{\alpha}}{l^2}\|\omega\|^2+\frac{c_0}{l}(\|w\|^4+\|w\|_{\widehat{V}}^2)+\|\chi_lF\|_{\widehat{V}^*}\|\chi_lw\|_{\widehat{V}}\\
&\quad+(G(t,w_t),\chi_l^2w)+\frac{2c_0}{l}\|p\|\|\chi_lu\|\\
&\leqslant\frac{c_0^2\cdot\max\{\nu+\nu_l,\bar{\alpha}\}}{l^2}\|w\|^2+\frac{c_0}{l}\mathcal{R}_1^4+\frac{c_0}{l}\|w\|_{\widehat{V}}^2+\frac{1}{4\beta_1}\|\chi_lF\|_{\widehat{V}^*}^2+\beta_1\|\chi_lw\|_{\widehat{V}}^2\\
&\quad+(G(t,w_t),\chi_l^2w)+\frac{c_0}{l}(\|p\|^2+\|\chi_lu\|^2),
\end{aligned}
$$

where the constant $\beta_1 \in (0, \frac{2\delta_1 - 2C_G - \gamma}{2}]$. Hence, there exists constant $c_3 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_l w\|^2 + 2(\delta_1 - \beta_1)\|\chi_l w\|^2 &\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\|\chi_l w\|^2 + 2(\delta_1 - \beta_1)\|\chi_l w\|^2_{\widehat{V}} \\ &\leqslant \frac{c_3}{l^2}\mathcal{R}_1^2 + \frac{2c_0}{l}\mathcal{R}_1^4 + \frac{2c_0}{l}\|w\|^2_{\widehat{V}} \\ &+ \frac{1}{2\beta_1}\|\chi_l F\|^2_{\widehat{V}^*} + 2(G(t, w_t), \chi_l^2 w) + \frac{2c_0}{l}(\|p\|^2 + \|\chi_l u\|^2).\end{aligned}$$

Further, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t}\|\chi_l w(t)\|^2) + [2(\delta_1 - \beta_1) - \gamma]e^{\gamma t}\|\chi_l w(t)\|^2 \\ &\leqslant \frac{c_3}{l^2}e^{\gamma t}\mathcal{R}_1^2(t) + \frac{2c_0}{l}e^{\gamma t}\mathcal{R}_1^4(t) + \frac{2c_0}{l}e^{\gamma t}\|w(t)\|^2_{\widehat{V}} \\ &+ \frac{1}{2\beta_1}e^{\gamma t}\|\chi_l F\|^2_{\widehat{V}^*} + 2e^{\gamma t}(G(t, w_t), \chi_l^2 w(t)) \\ &+ \frac{2c_0}{l}e^{\gamma t}(\|p\|^2 + \|\chi_l w(t)\|^2).\end{aligned}$$

Integrating the above inequality yields

$$\begin{aligned}&e^{\gamma t}\|\chi_l w(t)\|^2 + [2(\delta_1 - \beta_1) - \gamma]\int_\tau^t e^{\gamma s}\|\chi_l w(s)\|^2 \mathrm{d}s \\ &\leqslant e^{\gamma\tau}\|\chi_l w^{\mathrm{in}}\|^2 + \frac{c_3}{l^2}\int_\tau^t e^{\gamma s}\mathcal{R}_1^2(s)\mathrm{d}s + \frac{2c_0}{l}\int_\tau^t e^{\gamma s}\mathcal{R}_1^4(s)\mathrm{d}s \\ &+ \frac{2c_0}{l}\int_\tau^t e^{\gamma s}\|w(s)\|^2_{\widehat{V}}\mathrm{d}s \\ &+ \frac{1}{2\beta_1}\int_\tau^t e^{\gamma s}\|\chi_l F(s)\|^2_{\widehat{V}^*}\mathrm{d}s + 2\int_\tau^t e^{\gamma s}(G(s, w_s), \chi_l^2 w(s))\mathrm{d}s \\ &+ \frac{2c_0}{l}\int_\tau^t e^{\gamma s}\|p\|^2\mathrm{d}s + \frac{2c_0}{l}\int_\tau^t e^{\gamma s}\|\chi_l w(s)\|^2\mathrm{d}s. \end{aligned}\tag{3.25}$$

Similar to (3.3), we have

$$2\int_\tau^t e^{\gamma s}(G(s, w_s), \chi_l^2 w(s))\mathrm{d}s \leqslant C_G e^{\gamma\tau}\|\chi_l \phi^{\mathrm{in}}\|^2_{L^2_{\widehat{H}}} + 2C_G\int_\tau^t e^{\gamma s}\|\chi_l w(s)\|^2\mathrm{d}s. \tag{3.26}$$

Inserting (3.26) into (3.25), noting that $2(\delta_1 - \beta_1 - C_G) - \gamma \geqslant 0$, we deduce that

$$\begin{aligned}\|\chi_l w(t)\|^2 \leqslant &e^{\gamma(\tau - t)}\|\chi_l w^{\mathrm{in}}\|^2 + C_G e^{\gamma(\tau - t)}\|\chi_l \phi^{\mathrm{in}}\|^2_{L^2_{\widehat{H}}} \\ &+ \left(\frac{c_3}{l^2} + \frac{2c_0}{l}\right)e^{-\gamma t}\int_\tau^t e^{\gamma s}\mathcal{R}_1^2(s)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \mathcal{R}_1^4(s) \mathrm{d}s + \frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|w(s)\|_{\widehat{V}}^2 \mathrm{d}s \\
&+ \frac{l}{2\beta_1} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|\chi_l F\|_{\widehat{V}^*}^2 \mathrm{d}s + \frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|p\|^2 \mathrm{d}s. \qquad (3.27)
\end{aligned}
$$

For any $\epsilon > 0$, there exists a $\tau_2 := \tau_2(\epsilon, t, \widehat{D})$ such that

$$
e^{\gamma(\tau - t)} \|\chi_l w^{\text{in}}\|^2 + C_G e^{\gamma(\tau - t)} \|\chi_l \phi^{\text{in}}\|_{L^2_{\widehat{H}}}^2 \leqslant \frac{\epsilon}{6} \text{ for any } \tau \leqslant \tau_1. \qquad (3.28)
$$

Note that (3.7) implies, see [32],

$$
\lim_{l \to \infty} \int_{-\infty}^t e^{\gamma \theta} \|F(\theta, x)\|_{\widehat{V}^*(\Omega \backslash \Omega_l)}^2 \mathrm{d}\theta = 0, \quad \forall t \in \mathbb{R}. \qquad (3.29)
$$

Then, under Assumption 3.1, taking Lemma 3.1, Lemma 3.2 and (3.29) into account, we conclude that there exists $l_1 := l_1(\epsilon, t, \widehat{D})$ such that for any $l \geqslant l_1$,

$$
\begin{cases}
\left(\frac{c_3}{l^2} + \frac{2c_0}{l}\right) e^{-\gamma t} \int_\tau^t e^{\gamma s} \mathcal{R}_1^2(s) \mathrm{d}s \leqslant \frac{\epsilon}{6}, \quad \frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \mathcal{R}_1^4(s) \mathrm{d}s \leqslant \frac{\epsilon}{6}, \\
\frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|w(s)\|_{\widehat{V}}^2 \mathrm{d}s \leqslant \frac{\epsilon}{6},
\end{cases} \qquad (3.30)
$$

and

$$
\frac{1}{2\beta_1} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|\chi_l F\|_{\widehat{V}^*} \mathrm{d}s = \frac{1}{2\beta_1} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|F\|_{\widehat{V}^*(\Omega \backslash \Omega_l)} \mathrm{d}s \leqslant \frac{\epsilon}{6}. \qquad (3.31)
$$

It follows from $(1.3)_1$ that $\nabla p \in L^2_{loc}(\tau, +\infty; \mathbb{H}^{-1}(\Omega))$, which implies $p \in L^2_{loc}(\tau, +\infty; L^2(\Omega))$,

$$
\int_\tau^t e^{\gamma s} \|p\|^2 \mathrm{d}s \leqslant c \int_\tau^t e^{\gamma s} \|w(s)\|_{\widehat{V}}^2 \mathrm{d}s, \quad \text{where } c \text{ is a positive constant.}
$$

Consequently, we deduce that there exists $l_2 := l_2(\epsilon, t, \widehat{D})$ such that for any $l \geqslant l_2$, it holds that

$$
\frac{2c_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|p\|^2 \mathrm{d}s \leqslant \frac{2cc_0}{l} e^{-\gamma t} \int_\tau^t e^{\gamma s} \|w(s)\|_{\widehat{V}}^2 \mathrm{d}s \leqslant \frac{\epsilon}{6}. \qquad (3.32)
$$

Substituting (3.28)–(3.32) into (3.27), we immediately obtain (3.18) with $\tau_0 = \min\{\tau_1, \tau_2\}$ and $l_0 = \max\{l_1, l_2\}$. This completes the proof. □

In order to prove the pullback asymptotic compactness, we need to improve the regularity of solutions.

Lemma 3.4 (see [29]) *Let Y_0, Y be two Banach spaces such that Y_0 is reflexive, and the inclusion $Y_0 \subset Y$ is continuous. Assume that $\{w_n\}$ is a bounded sequence in $L^\infty(t_0, T; Y_0)$ such that $w_n \rightharpoonup w$ weakly in $L^p(t_0, T; Y_0)$ for some $p \in [1, +\infty)$ and $w \in \mathcal{C}(t_0, T; Y)$. Then $w(t) \in Y_0$ and*

$$\|w(t)\|_{Y_0} \leqslant \liminf_{n\to\infty} \|w_n\|_{L^\infty(t_0,T;Y_0)}, \ \forall t \in [t_0, T].$$

Lemma 3.5 (Regularity estimate) *Assume that Assumptions 2.1 and 3.1 hold, then, for any $t \in \mathbb{R}$ and $\widehat{D} = \{D(s) \mid s \in \mathbb{R}\} \in \mathcal{D}_\gamma$, there exists a $\tau^*(\widehat{D}, t)$ such that, for any $\tau \leqslant \tau^*(\widehat{D}, t)$, the weak solution $w(t)$ with initial data $(w_0^{\text{in}}, \phi_0^{\text{in}}) \subset D(\tau)$ is bounded in $\widehat{V}$.*

Proof We consider the Galerkin approximate solutions. For each integer $n \geqslant 1$, we denote by

$$w_n(t) = w_n(t; \tau_n, w_0^n, \phi_0^n) := \sum_{i=1}^n \xi_{ni}(t)e_i, \quad w_{nt}(\cdot) = w_n(t + \cdot; \tau_n, w_0^n, \phi_0^n), \tag{3.33}$$

the Galerkin approximation of the solution $w(t)$ of system (1.6), where $\xi_{ni}(t)$ is the solution of the following Cauchy problem of ODEs:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(w_n(t), e_i) + \langle Aw_n(t) + B(u_n, w_n) + N(w_n(t)), e_i\rangle \\ \quad = \langle F(t), e_i\rangle + (G(t, w_{nt}), e_i), \\ (w_n(\tau), e_i) = (w_0^n, e_i), \ (w_n(s), e_i) \\ \quad = (\phi_0^n, e_i), \quad s \in (\tau - h, \tau), \ i = 1, 2, \ldots, n, \end{cases} \tag{3.34}$$

where $\{e_i : i \geqslant 1\} \subseteq D(A)$, which forms a Hilbert basis of $\widehat{V}$ and is orthonormal in $\widehat{H}$. Multiplying equation $(3.34)_1$ by $A\xi_{ni}(t)$ and summing them for $i = 1$ to n, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n(t), w_n(t)\rangle + \|Aw_n(t)\|^2 + \langle B(u_n(t), w_n(t)), Aw_n(t)\rangle \\ &\quad + \langle N(w_n(t)), Aw_n(t)\rangle \\ &\quad = (F(t), Aw_n(t)) + (G(t, w_{nt}), Aw_n(t)). \end{aligned} \tag{3.35}$$

From (2.4) and the facts $\|u_n\|^2 \leqslant \|w_n\|^2, \ \ \|\nabla u_n\|^2 \leqslant \|w_n\|^2_{\widehat{V}}$, and using the Young's inequality, we deduce that

$$\begin{aligned} -\langle B(u_n, w_n), Aw_n\rangle &\leqslant |\langle B(u_n, w_n), Aw_n\rangle| \leqslant \alpha_0\|u_n\|^{\frac{1}{2}}\|\nabla u_n\|^{\frac{1}{2}}\|\nabla w_n\|^{\frac{1}{2}}\|Aw_n\|^{\frac{3}{2}} \\ &\leqslant \frac{1}{4}\|Aw_n\|^2 + 4^3\alpha_0^4\|w_n\|^2\|w_n\|^4_{\widehat{V}}, \end{aligned}$$

which together with (2.6), (3.35) and Assumption 2.1 implies

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n, w_n\rangle &= -\|Aw_n\|^2 + (F, Aw_n) + (G(t, w_{nt}), Aw_n) \\
&\quad - \langle B(u_n, w_n), Aw_n\rangle - \langle N(w_n), Aw_n\rangle \\
&\leqslant -\|Aw_n\|^2 + \frac{1}{4}\|Aw_n\|^2 + \|F(t)\|^2 + \|G(t, w_{nt})\|^2 + \frac{1}{4}\|Aw_n\|^2 \\
&\quad + \frac{1}{4}\|Aw_n\|^2 + 4^3\alpha_0^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4 \\
&\quad + c^2(\nu_r)\|w_n\|_{\widehat{V}}^2 + \frac{1}{4}\|Aw_n\|^2 \\
&= \|F(t)\|^2 + \|G(t, w_{nt})\|^2 + \|w_n\|_{\widehat{V}}^2\big(4^3\alpha_0^4\|w_n\|^2\|w_n\|_{\widehat{V}}^2 + c^2(\nu_r)\big).
\end{aligned}
$$

Further, from (2.2) and the above inequality, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n(t), w_n(t)\rangle \\
&\leqslant 2\|F(t)\|^2 + 2\|G(t, w_{nt})\|^2 + \langle Aw_n(t), w_n(t)\rangle\big(2^7c_2\alpha_0^4\|w_n(t)\|^2\|w_n(t)\|_{\widehat{V}}^2 \\
&\quad + 2c_2c^2(\nu_r)\big).
\end{aligned} \tag{3.36}
$$

Let us set

$$
\begin{aligned}
&H_n(\theta) := \langle Aw_n(\theta), w_n(\theta)\rangle,\ \ I_n(\theta) := 2(\|F(\theta)\|^2 + \|G(\theta, w_{n\theta})\|^2), \\
&K_n(\theta) := 2^7c_2\alpha_0^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r).
\end{aligned}
$$

Replacing the variable t with θ in (3.36), we get

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}H_n(\theta) \leqslant K_n(\theta)H_n(\theta) + I_n(\theta). \tag{3.37}
$$

Using the Gronwall inequality to (3.37), for all $\tau \leqslant t-h \leqslant s \leqslant t$, we have

$$
H_n(t) \leqslant \big(H_n(s) + \int_{t-h}^t I_n(\theta)\mathrm{d}\theta\big)\cdot \exp\left\{\int_{t-h}^t K_n(\theta)\mathrm{d}\theta\right\}. \tag{3.38}
$$

Integrating (3.38) from $s = t-h$ to $s = t$, we obtain that

$$
hH_n(t) \leqslant \left(\int_{t-h}^t H_n(s)\mathrm{d}s + h\int_{t-h}^t I_n(\theta)\mathrm{d}\theta\right)\cdot \exp\left\{\int_{t-h}^t K_n(\theta)\mathrm{d}\theta\right\}. \tag{3.39}
$$

In addition, it follows from (2.2), Lemma 3.2 and Assumption 2.1 that

$$
\begin{aligned}
&\int_{t-h}^t H_n(s)\mathrm{d}s + h\int_{t-h}^t I_n(\theta)\mathrm{d}\theta = \int_{t-h}^t \langle Aw_n(s), w_n(s)\rangle\mathrm{d}s \\
&\quad + h\int_{t-h}^t 2(\|F(\theta)\|^2 + \|G(\theta, w_{n\theta})\|^2)\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c_1{}^{-1}\int_{t-h}^{t}\|w_n(s)\|_{\widehat{V}}^2\mathrm{d}s+2h\int_{t-h}^{t}\|F(\theta)\|^2\mathrm{d}\theta+2hC_G^2\int_{t-2h}^{t}\|w(\theta)\|^2\mathrm{d}\theta\\
&\leqslant c_1^{-1}\mathcal{R}_1^2(t)+4h^2C_G^2\mathcal{R}_1^2(t)+2h\int_{t-h}^{t}\|F(\theta)\|^2\mathrm{d}\theta\leqslant c_4\big(\mathcal{R}_1^2(t)\\
&+\int_{t-h}^{t}\|F(\theta)\|^2\mathrm{d}\theta\big),
\end{aligned}
$$

where $c_4:=\max\{2h, c_1^{-1}+4h^2C_G^2\}$. From (3.1) and Lemma 3.2, it holds that

$$
\begin{aligned}
\int_{t-h}^{t}K_n(\theta)\mathrm{d}\theta&=\int_{t-h}^{t}(2^7c_2\alpha_0^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2+2c_2c^2(\nu_r))\mathrm{d}\theta\\
&\leqslant 2^7c_2\alpha_0^4\sup_{\theta\in[t-h,t]}\|w_n(\theta)\|^2\int_{t-h}^{t}\|w_n(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta+2c_2hc^2(\nu_r)\\
&\leqslant 2^7c_2\alpha_0^4\big[(1+C_G)e^{\gamma(h+\tau-t)}\|(w_0^n,\phi_0^n)\|_{E_{\widehat{H}}^2}^2\\
&+\alpha^{-1}e^{\gamma(h-t)}\int_{-\infty}^{t}e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\big]\mathcal{R}_1^2(t)+2c_2hc^2(\nu_r).
\end{aligned}
$$

With the aid of (2.2), substituting the above two inequalities into (3.39), yields

$$
\begin{aligned}
\|w_n(t)\|_{\widehat{V}}^2&\leqslant c_2H_n(t)\\
&\leqslant c_2c_4h^{-1}\big(\mathcal{R}_1^2(t)+\int_{t-h}^{t}\|F(\theta)\|^2\mathrm{d}\theta\big)\\
&\times\exp\big\{c_5\big(e^{\gamma(\tau-t)}\|(w_0^n,\phi_0^n)\|_{E_{\widehat{H}}^2}^2\\
&+e^{-\gamma t}\int_{-\infty}^{t}e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\big)\mathcal{R}_1^2(t)+2c_2hc^2(\nu_r)\big\},
\end{aligned}
\tag{3.40}
$$

where $c_5:=2^7c_2\alpha_0^4e^{\gamma h}\cdot\max\{(1+C_G),\alpha^{-1}\}$. Under Assumption 3.1, it is clear that there exists a constant $\bar{M}$ such that

$$
\int_{t-h}^{t}\|F(\theta)\|^2\mathrm{d}\theta\leqslant\bar{M},\quad\forall\, t\in\mathbb{R},
$$

which together with (3.40), Lemma 3.4 implies the boundedness of $\|w(t;\tau,w_0^{\text{in}},\phi_0^{\text{in}})\|_{\widehat{V}}$ for all $\tau\leqslant\tau^*(\widehat{D},t)$. The proof is complete. □

On the basis of the above results, we are ready to prove the pullback $\mathcal{D}_\gamma$-asymptotic compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$.

Lemma 3.6 (pullback$\mathcal{D}_\gamma$-asymptotic compactness)
Under the conditions of Lemma 3.3, the process $\{U(t,\tau)\}_{t\geqslant\tau}$ generated by (2.10) is pullback $\mathcal{D}_\gamma$-asymptotically compact in $\widehat{H}$.

Proof For any fixed $t \in \mathbb{R}$, and family $\widehat{D} = \{D(s) \mid s \in \mathbb{R}\} \in \mathcal{D}_\gamma$, any sequences $\{\tau_n\} \subseteq (-\infty, t]$ satisfying $\tau_n \to -\infty$ as $n \to +\infty$ and $\{(w_n^{\text{in}}, \phi_n^{\text{in}})\} \subset D(\tau_n)$. It suffice to show the sequence $\{(w^n(t), w_t^n(\cdot))\}_{n \geqslant 1}$ defined by

$$(w^n(t), w_t^n(\cdot)) := U(t, \tau_n)(w_n^{\text{in}}, \phi_n^{\text{in}}) = (w(t; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}}), w_t(\cdot; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}}))$$

is relatively compact in $E^2_{\widehat{H}}$.

Step 1 (the sequence $\{w^n(t)\}_{n\geqslant 1}$ is relatively compact in $\widehat{H}$)

In fact, by Lemma 3.2, there exists a time $\tau_1 := \tau_1(\widehat{D}, t) < t$ such that, for any $\tau \leqslant \tau_1$, $U(t, \tau)D(\tau) \subset B(t)$. Moreover, (3.8)–(3.10) implies $B(t)$ is uniformly bounded with respect to t. Consequently, $D(t)$ is uniformly bounded in $E^2_{\widehat{H}\times L^2_{\widehat{V}}}$ with respect to t. Since $E^2_{\widehat{H}\times L^2_{\widehat{V}}}$ is a reflexive Banach space, we can extract a subsequence (denoting by the same symbol) $\{(w_n^{\text{in}}, \phi_n^{\text{in}})\}_{n\geqslant 1}$ and some $(w, \phi) \in E^2_{\widehat{H}\times L^2_{\widehat{V}}}$ such that

$$U(t, \tau_n)(w_n^{\text{in}}, \phi_n^{\text{in}}) \rightharpoonup (w, \phi) \text{ weakly in } E^2_{\widehat{H}\times L^2_{\widehat{V}}} \text{ as } n \to \infty, \tag{3.41}$$

which implies

$$w^n(t) \rightharpoonup w(t) \text{ weakly in } \widehat{H} \text{ as } n \to \infty. \tag{3.42}$$

Moreover, from Lemma 3.3, we conclude that, for any $\epsilon > 0$, there exist $\tau_3 := \tau_3(\epsilon, t, \widehat{D})$ and $l_3 := l_3(\epsilon, t, \widehat{D}) > 0$ such that

$$\|w^n(t)\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} = \|w^n(t; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}})\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} \leqslant \frac{\epsilon}{3}, \quad \forall \tau_n \leqslant \tau_3,\ l \geqslant l_3. \tag{3.43}$$

Observe that, for any fixed $t \in \mathbb{R}$, $w(t) \in \widehat{H}$ is fixed. Hence, for the above $\epsilon > 0$, there exists $l_4 > 0$ such that

$$\|w(t)\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} \leqslant \frac{\epsilon}{3}, \quad \forall \tau_n \leqslant \tau_3,\ l \geqslant l_4. \tag{3.44}$$

Now, we define the restrictions of w^n and w in Ω_l, respectively, as

$$w^n(t)\big|_{\Omega_l} := \begin{cases} w^n(t), & x \in \Omega_l, \\ 0, & x \in \Omega\backslash\Omega_l, \end{cases} \qquad w(t)\big|_{\Omega_l} := \begin{cases} w(t), & x \in \Omega_l, \\ 0, & x \in \Omega\backslash\Omega_l. \end{cases}$$

It follows from Lemma 3.5 that, for any $l > 0$, the sequence $\{w^n(t)\big|_{\Omega_l}\}_{n\geqslant 1}$ is bounded in $\widehat{V}(\Omega_l)$. Since $\widehat{V}(\Omega_l) \hookrightarrow\hookrightarrow \widehat{H}(\Omega_l)$, there exists a subsequence (denoting by the same symbol) $\{w^n(t)\big|_{\Omega_l}\}_{n\geqslant 1}$ satisfying

$$\|w^n(t) - w(t)\|_{\widehat{H}(\Omega_l)} \to 0, \quad as \quad n \to \infty, \tag{3.45}$$

which combine with (3.43) and (3.44) implies that there exists a $N_0 \in \mathbb{N}$ such that, for any $n \geqslant N_0$,

$$\begin{aligned}\|w^n(t) - w(t)\|_{\widehat{H}} =& \|w^n(t) - w(t)\|_{\widehat{H}(\Omega_l)} + \|w^n(t) - w(t)\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} \\ \leqslant& \|w^n(t) - w(t)\|_{\widehat{H}(\Omega_l)} + \|w^n(t)\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} \\ &+ \|w(t)\|_{\mathbb{L}^2(\Omega\backslash\Omega_l)} \leqslant \epsilon. \end{aligned} \tag{3.46}$$

Therefore, the sequence $\{w^n(t)\}_{n\geqslant 1}$ is relatively compact in $\widehat{H}$.

Step 2 (the sequence $\{w_t^n(\cdot)\}_{n\geqslant 1}$ is relatively compact in $L^2_{\widehat{H}}$)

Let us denote $\{\theta_j\}_{j\geqslant 0}$ the sequence of all rational numbers from the interval $[-h, 0]$. From the above argument, we deduce that there exists a subsequence (denoting by the same symbol) $\{(w_n^{\text{in}}, \phi_n^{\text{in}})\}_{n\geqslant 1}$ such that for each j there exists a $w^j \in \widehat{H}$ satisfying

$$w(t+\theta_j; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}}) \to w^j \ \text{ strongly in } \widehat{H} \text{ as } n \to \infty. \tag{3.47}$$

Then for any $t_1, t_2 \in [t-h, t]$ with $t_1 < t_2$, we have

$$\begin{aligned} w^n(t_2) - w^n(t_1) =& \int_{t_1}^{t_2} (w^n)'(s)\mathrm{d}s \\ =& \int_{t_1}^{t_2} \big[-Aw^n(s) - B(u^n(s), w^n(s)) - N(w^n(s)) \\ &+ F(s) + G(s, w_s^n)\big]\mathrm{d}s. \end{aligned}$$

Hence, from (2.2), (2.3) and (2.5), it follows that

$$\begin{aligned} &\|w^n(t_2) - w^n(t_1)\|_{\widehat{V}^*} \\ &\leqslant \int_{t_1}^{t_2} \big(\|Aw^n(s)\|_{\widehat{V}^*} + \|B(u^n(s), w^n(s))\|_{\widehat{V}^*} + \|N(w^n(s))\|_{\widehat{V}^*} + \|F(s)\|_{\widehat{V}^*} \\ &+ \|G(s, w_s^n)\|_{\widehat{V}^*}\big)\mathrm{d}s \\ &\leqslant c\int_{t_1}^{t_2} \big(\|w^n(s)\|_{\widehat{V}} + \|w^n(s)\|\|w^n(s)\|_{\widehat{V}} + \|F(s)\|_{\widehat{V}^*} + \|G(s, w_s^n)\|\big)\mathrm{d}s. \end{aligned} \tag{3.48}$$

Moreover, applying the Cauchy inequality and Assumption 2.1, we have

$$\begin{aligned} \int_{t_1}^{t_2} \|G(s, w_s^n)\|\mathrm{d}s \leqslant& C_G(t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1-h}^{t_2} \|w^n(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}} \\ \leqslant& C_G(t_2 - t_1)^{\frac{1}{2}} \left(\int_{t-2h}^{t} \|w^n(s)\|_{\widehat{V}}^2\mathrm{d}s\right)^{\frac{1}{2}}. \end{aligned} \tag{3.49}$$

Similar to (3.49), it also holds that

$$\int_{t_1}^{t_2}\|F(s)\|_{\widehat{V}^*}\mathrm{d}s \leqslant (t_2-t_1)^{\frac{1}{2}}\left(\int_{t-h}^{t}\|F(s)\|_{\widehat{V}^*}^2\mathrm{d}s\right)^{\frac{1}{2}}, \tag{3.50}$$

$$\int_{t_1}^{t_2}\|w^n(s)\|_{\widehat{V}}\mathrm{d}s \leqslant (t_2-t_1)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|w^n(s)\|_{\widehat{V}}^2\mathrm{d}s\right)^{\frac{1}{2}}, \tag{3.51}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\|w^n(s)\|\|w^n(s)\|_{\widehat{V}}\mathrm{d}s &\leqslant \sup_{s\in[t-h,t]}\|w^n(s)\|\int_{t_1}^{t_2}\|w^n(s)\|_{\widehat{V}}\mathrm{d}s\\ &\leqslant (t_2-t_1)^{\frac{1}{2}}\sup_{s\in[t-h,t]}\|w^n(s)\|\left(\int_{t_1}^{t_2}\|w^n(s)\|_{\widehat{V}}^2\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned} \tag{3.52}$$

(3.48)–(3.52) imply

$$\begin{aligned}\|w^n(t_2)-w^n(t_1)\|_{\widehat{V}^*} \leqslant c(t_2-t_1)^{\frac{1}{2}}\Bigg[&\left(1+\sup_{s\in[t-h,t]}\|w^n(s)\|\right)\\ &\times\left(\int_{t-2h}^{t}\|w^n(s)\|_{\widehat{V}}^2\mathrm{d}s\right)^{\frac{1}{2}}+\left(\int_{t-h}^{t}\|F(s)\|_{\widehat{V}^*}^2\mathrm{d}s\right)^{\frac{1}{2}}\Bigg].\end{aligned} \tag{3.53}$$

It follows from (3.1) that for all $s\in[t-h,t]$,

$$\begin{aligned}\|w^n(s)\|^2 &\leqslant (1+C_G)e^{-\gamma(s-\tau_n)}\|(w_n^{\mathrm{in}},\phi_n^{\mathrm{in}})\|_{E_{\widehat{H}}^2}^2+\alpha^{-1}e^{-\gamma s}\int_{\tau_n}^{s}e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\\ &\leqslant (1+C_G)e^{\gamma(h+\tau_n-t)}\|(w_n^{\mathrm{in}},\phi_n^{\mathrm{in}})\|_{E_{\widehat{H}}^2}^2\\ &\quad+\alpha^{-1}e^{\gamma(h-t)}\int_{-\infty}^{t}e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta.\end{aligned} \tag{3.54}$$

In addition, (3.5) gives

$$\begin{aligned}\int_{t-2h}^{t}\|w^n(s)\|_{\widehat{V}}^2\mathrm{d}s \leqslant &\beta^{-1}(1+C_G)e^{-\gamma(t-2h-\tau_n)}\|(w_n^{\mathrm{in}},\phi_n^{\mathrm{in}})\|_{E_{\widehat{H}}^2}^2\\ &+(\alpha\beta)^{-1}e^{-\gamma(t-2h)}\int_{-\infty}^{t}e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta.\end{aligned} \tag{3.55}$$

Since $e^{\gamma\tau_n}\|(w_n^{\mathrm{in}},\phi_n^{\mathrm{in}})\|_{E_{\widehat{H}}^2}^2$ is bounded, taking Assumption 3.1 and (3.53)–(3.55) into account, we conclude that

$$\text{the sequence } \{w^n(\cdot)\}_{n\geqslant 1} \text{ is equicontinuous in } \mathcal{C}([t-h,t];\widehat{V}^*).$$

Next, observe that for any $r \in [t-h, t]\backslash\mathbb{Q}$, where $\mathbb{Q}$ represents the set of all rational number,

$$\|w^n(r) - w^m(r)\|_{\widehat{V}^*} \leqslant \|w^n(r) - w^n(t+\theta_j)\|_{\widehat{V}^*} + \|w^n(t+\theta_j) - w^m(t+\theta_j)\|_{\widehat{V}^*} + \|w^m(t+\theta_j) - w^m(r)\|_{\widehat{V}^*}, \quad \forall j \geqslant 1.$$

On the one hand, by the equicontinuity of $\{w^n(s)\}_{n\geqslant 1}$, there exist a subsequence $\{\theta_{j_k}\} \subset \{\theta_j\}$ such that

$$\|w^n(r) - w^n(t+\theta_{j_k})\|_{\widehat{V}^*} \to 0 \ \text{ as } \ k \to \infty,$$
$$\|w^m(t+\theta_{j_k}) - w^m(r)\|_{\widehat{V}^*} \to 0 \ \text{ as } \ k \to \infty.$$

On the other hand, (3.47) implies

$$\|w^n(t+\theta_{j_k}) - w^m(t+\theta_{j_k})\|_{\widehat{V}^*} \to 0 \text{ as } n,\ m \to \infty.$$

Therefore, for any $\theta \in [t-h, t]\backslash\mathbb{Q}$,

$$\text{the sequence } \{w^n(r)\}_{n\geqslant 1} \text{ is a Cauchy sequence of } \widehat{V}^*.$$

Thus, for each $r \in [t-h, t]$, there exists some $v(\theta) \in \widehat{V}^*$ such that

$$w^n(\theta) \to v(\theta) \text{ strongly in } \widehat{V}^* \text{ as } n \to \infty.$$

Based on the continuous injection of $\widehat{H}$ into $\widehat{V}^*$ and (3.54), we conclude that the sequence $\{w^n(\cdot)\}$ is bounded in $\mathcal{C}([t-h, t]; \widehat{V}^*)$. Then, applying the Lebesgue dominated convergence theorem, we obtain

$$w^n(\cdot) \to v(\cdot) \text{ strongly in } L^2(t-h, t; \widehat{V}^*) \text{ as } n \to \infty.$$

So

$$w^n(t+\cdot) =: w_t^n(\cdot; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}}) \to v_t(\cdot) := v(t+\cdot) \text{ strongly in } L^2_{\widehat{V}^*} \text{ as } n \to \infty. \tag{3.56}$$

From (3.41) and the uniqueness of limit, (3.56) implies

$$w_t^n(\cdot; \tau_n, w_n^{\text{in}}, \phi_n^{\text{in}}) \to \phi \text{ strongly in } L^2_{\widehat{V}^*} \text{ as } n \to \infty. \tag{3.57}$$

Further, we conclude that

$$\int_{-h}^0 \|w_t^n(s) - \phi(s)\|^2 \mathrm{d}s = \int_{-h}^0 \langle w_t^n(s) - \phi(s), w_t^n(s) - \phi(s)\rangle \mathrm{d}s \leqslant \|w_t^n(s) - \phi(s)\|_{L^2_{\widehat{V}^*}} \|w_t^n(s) - \phi(s)\|_{L^2_{\widehat{V}}} \to 0 \ \text{ as } \ n \to \infty,$$

which together with (3.46) gives the pullback $\mathcal{D}_\gamma$-asymptotic compactness. □

3.3 Existence of Pullback Attractor for the Universe $\mathcal{D}_\gamma$

Theorem 3.1 *Assume that Assumptions 2.1 and 3.1 hold, then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in (2.10) has a unique pullback $\mathcal{D}_\gamma$-attractor $\widehat{\mathcal{A}}_{\mathcal{D}_\gamma}=\{\mathcal{A}_{\mathcal{D}_\gamma}(t)\mid t\in\mathbb{R}\}$ for the universe $\mathcal{D}_\gamma$.*

Proof Define

$$\mathcal{A}_{\mathcal{D}_\gamma}(t)=\bigcap_{\tau_0\leqslant t}\overline{\bigcup_{\tau\leqslant\tau_0}U(t,\tau)D(\tau)},\quad \widehat{D}=\{D(t)\mid t\in\mathbb{R}\}\in\mathcal{D}_\gamma. \tag{3.58}$$

According to [16, Proposition 9] or [27, Theorem 18], Lemmas 3.2 and 3.6 imply $\widehat{\mathcal{A}}_{\mathcal{D}_\gamma}=\{\mathcal{A}_{\mathcal{D}_\gamma}(t)\mid t\in\mathbb{R}\}$ in (3.58) is the unique pullback attractor for the universe $\mathcal{D}_\gamma$. □

4 Some Properties of Pullback Attractor for the Universe $\mathcal{D}_\gamma$

In this section, we conclude some properties of pullback attractor for the universe $\mathcal{D}_\gamma$. The first property is that pullback attractor for the universe $\mathcal{D}_\gamma$ is consistent with that for the universe of fixed bounded sets. The other property is the tempered behavior.

4.1 Consistency with Pullback Attractor for the Universe of Fixed Bounded Sets

Let us denote $\mathcal{D}_F$ the class of all families

$$\widehat{D}_F=\left\{D_F(t)=D\mid t\in\mathbb{R},\ D \text{ is some bounded set in } E^2_{\widehat{H}}\right\}.$$

It is clear that $\mathcal{D}_F\subset\mathcal{D}_\gamma$. Then we consider the universe $\mathcal{D}_F$ in $\mathcal{P}(E^2_{\widehat{H}})$.

Theorem 4.1 *Under Assumptions 2.1 and 3.1, the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in (2.10) has a unique pullback $\mathcal{D}_F$-attractor $\widehat{\mathcal{A}}_{\mathcal{D}_F}=\{\mathcal{A}_{\mathcal{D}_F}(t)\mid t\in\mathbb{R}\}$, Moreover,*

$$\mathcal{A}_{\mathcal{D}_F}(t)=\mathcal{A}_{\mathcal{D}_\gamma}(t),\ \forall t\in\mathbb{R}. \tag{4.1}$$

Proof The existence of pullback $\mathcal{D}_F$-attractor $\widehat{\mathcal{A}}_{\mathcal{D}_F}$ is as a consequence of Theorem 3.1. Under Assumption 3.1, (4.1) follows from [27, Proposition 23]. □

Remark 4.1 By the pullback attracting property of the pullback attractor $\widehat{\mathcal{A}}_{\mathcal{D}_\gamma}$ and (4.1), we can check that, for any $\widehat{D}=\{D(t)\mid t\in\mathbb{R}\}\in\mathcal{D}_\gamma$, there holds

$$\lim_{\tau\to-\infty}\mathrm{dist}_{E^2_{\widehat{H}}}(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_F}(t))=\lim_{\tau\to-\infty}\mathrm{dist}_{E^2_{\widehat{H}}}(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\gamma}(t))=0,$$

which implies that $\widehat{\mathcal{A}}_{\mathcal{D}_F}$ not only attracts any bounded sets but also attracts some tempered sets in the pullback sense.

4.2 Tempered Behavior of the Pullback Attractor

Theorem 4.2 *Under the conditions of Assumptions 2.1 and 3.1, it holds that*

$$\lim_{t\to-\infty}\left(e^{\gamma t}\sup_{(w,\phi)\in\mathcal{A}_{\mathcal{D}_\gamma}(t)}\|(w,\phi)\|_{E^2_{\widehat{H}}}\right)=0, \tag{4.2}$$

$$\lim_{t\to-\infty}\left(e^{\gamma t}\sup_{(w,\phi)\in\mathcal{A}_{\mathcal{D}_\gamma}(t)}\|(w,\phi)\|_{E^2_{\widehat{V}}}\right)=0. \tag{4.3}$$

Proof By Definition 3.1 of universe $\mathcal{D}_\gamma$, we have

$$\widehat{\mathcal{A}}_{\mathcal{D}_\gamma}\in\mathcal{D}_\gamma.$$

Thus, (4.2) holds.

Moreover, (4.3) is a consequence of Lemmas 3.2, 3.5 and Assumption 3.1. □

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